Exercise sheet 3

Exercise 1 (15 Points)

In the one–dimensional tight binding model, spinless fermions hop from one site of a ring to the other with a hopping amplitude t, see Fig. 1. Assuming also a finite chemical potential μ, the system is described by the Hamiltonian

$$H = -t \sum_{l=0}^{N-1} \left(c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l \right) - \mu \sum_{l=0}^{N-1} c_l^\dagger c_l \, . \tag{1}$$

Here, N is the number of lattice sites, and $c_l^\dagger(c_l)$ are creation(annihilation) operators obeying the usual fermionic anti-commutation relations

$$\{c_l, c_{l'}^\dagger\} = \delta_{ll'} \, , \quad \text{and} \quad \{c_l, c_{l'}\} = 0 = \{c_l^\dagger, c_{l'}^\dagger\} \, . \tag{2}$$

- By going to the reciprocal space diagonalize Hamiltonian (1).
- Calculate the ground state of the system as a function of the chemical potential μ and the bandwidth t.
- Calculate the ground state for $N/2$ fermions.
- Calculate the density–density dynamical susceptibility $\chi_{nn}(q,z)$, defined as*

$$\chi_{nn}(q,z) = -\langle [n(-q), \frac{1}{z-L} n(q)] \rangle \, , \tag{3}$$

  with q the wavevector, $z = \omega + i0^+$ the complex frequency, L the Liouville operator, and n the density operator

$$n = \frac{1}{N} \sum_l c_l^\dagger c_l \, . \tag{4}$$

- Give a graphical representation of the imaginary part of the susceptibility $\chi''_{nn}(q,\omega)$, at zero temperature, as a function of ω, and q, for $N/2$ particles. What do you conclude? What happens for N particles?

Exercise 2 (40 Points)

Consider a one–dimensional tight binding model on a ring with 2 atoms per unit cell. This results to a model where spinless fermions hop from one site to the other with two alternating hopping amplitudes t_1, and t_2, see Fig. 1. The Hamiltonian that describes this system is given by

$$\begin{aligned} H &= -\mu \sum_{l=0}^{N-1} a_{2l}^\dagger a_{2l} + b_{2l+1}^\dagger b_{2l+1} \\ &- t_1 \sum_{l=0}^{N-1} \left(a_{2l}^\dagger b_{2l+1} + b_{2l+1}^\dagger a_{2l} \right) \\ &- t_2 \sum_{l=0}^{N-1} \left(a_{2(l+1)}^\dagger b_{2l+1} + b_{2l+1}^\dagger a_{2(l+1)} \right) \, . \end{aligned} \tag{5}$$

* *Hint:* Think of the action of the Liouville operator on the density operator in the Hamiltonian basis. What about powers of the Liouvillian operator?

Figure 1: (left) A tight binding model on a ring with one atom per unit cell. Fermions hop from site to site with an amplitude t. (right) A tight binding model on a ring with two atoms per unit cell. Fermions hop from site to site with two different hopping amplitudes t_1 and t_2.

Here, $N = N_s/2$ is the number of unit cells, N_s is the number of sites, μ is the chemical potential, and $a_l^\dagger, b_l^\dagger$ (a_l, b_l) are creation(annihilation) operators. a and b operators always anticommute with each other while for each species we have the anticommutation relations

$$\{a_l, a_{l'}^\dagger\} = \delta_{ll'}\,, \quad \text{and} \quad \{a_l, a_{l'}\} = 0 = \{a_l^\dagger, a_{l'}^\dagger\}\,, \tag{6}$$

$$\{b_l, b_{l'}^\dagger\} = \delta_{ll'}\,, \quad \text{and} \quad \{b_l, b_{l'}\} = 0 = \{b_l^\dagger, b_{l'}^\dagger\}\,. \tag{7}$$

- Show that Hamiltonian (5) can be written in a matrix form in reciprocal space as

$$H = \sum_k \Psi_k^\dagger \mathcal{H}_k \Psi_k \tag{8}$$

  where $\Psi_k^\dagger = (a_k^\dagger, b_k^\dagger)$, $\mathcal{H}_k$ some 2×2 matrix and k is the quantized momentum index. Note that the sum in Eq. (8) runs over the first Brillouin zone.

- Diagonalize Hamiltonian (8) and find the energy dispersion relations.

- Calculate the density–density dynamical susceptibilities $\chi_{aa}(q,z), \chi_{bb}(q,z)$ defined as (see footnote)

$$\chi_{aa}(q,z) = -\langle [n_a(-q), \frac{1}{z-L} n_a(q)] \rangle\,, \qquad \chi_{bb}(q,z) = -\langle [n_b(-q), \frac{1}{z-L} n_b(q)] \rangle\,, \tag{9}$$

  with q the wavevector, $z = \omega + i0^+$ the complex frequency, L the Liouville operator, and n_a, n_b the density operators

$$n_a = \frac{1}{N} \sum_{l=0}^{N-1} a_{2l}^\dagger a_{2l}\,, \qquad n_b = \frac{1}{N} \sum_{l=0}^{N-1} b_{2l+1}^\dagger b_{2l+1}\,. \tag{10}$$

- Give a graphical representation of the imaginary part of the susceptibilities $\chi''_{aa}(q,\omega)$, and $\chi''_{bb}(q,\omega)$ for $t_1 = -t_2$, at zero temperature, as a function of ω,and q, for N particles. What do you conclude?