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Hand in until Tuesday, May 10, 09:45

Exercise sheet 2

Exercise 1 (10 Points)

Show that for two Hermitian operators $A_\mu^\dagger = A_\mu$ and $A_\nu^\dagger = A_\nu$ the following equality for the Mori product holds

$$\langle A_\mu | A_\nu \rangle = \langle A_\nu | A_\mu \rangle \; . \tag{1}$$

Exercise 2 (30 Points)

Consider a system described by the Hamiltonian

$$H = H_0 + H_1 \; , \quad H_1 = \sum_\nu f_\nu A_\nu \; , \tag{2}$$

where H_0 is the unperturbed Hamiltonian, and H_1 some perturbation.

- Show that the partition function for the perturbed system is given by

  $$Z = Z_0 \langle S(\beta, 0) \rangle_0 \; , \tag{3}$$

  where the index "0" refers to the unperturbed system, and $S(\tau, \tau')$ is the imaginary time evolution operator from τ' to τ in the interaction picture, i.e.

  $$S(\tau, \tau') = e^{H_0 \tau} e^{-H(\tau - \tau')} e^{-H_0 \tau'} \; . \tag{4}$$

- Show that the evolution of the operator S is described by the differential equation

  $$\frac{\partial S(\tau, \tau')}{\partial \tau} = -H_1(\tau) S(\tau, \tau') \; , \quad \text{with} \quad H_1(\tau) = e^{H_0 \tau} H_1 e^{-H_0 \tau} \; . \tag{5}$$

- By integrating Eq. (5) from τ' to τ, and using a recursive procedure, show that S can be written as

  $$S(\beta, 0) = \mathcal{T} \left[\exp - \int_0^\beta d\tau H_1(\tau) \right] \; , \tag{6}$$

  with $\mathcal{T}$ the time ordering operator.* Note that Eq. (6) is merely a symbolic way to write the expansion

  $$S(\tau, \tau') = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\tau'}^{\tau} d\tau_1 \cdots \int_{\tau'}^{\tau} d\tau_n \mathcal{T} \left[H_1(\tau_1) \cdots H_1(\tau_n) \right] \; . \tag{8}$$

- Assuming H_1 is some weak perturbation, show that the free energy can be written up to second order in the perturbation couplings f_ν terms as

  $$F = F_0 + \sum_\nu f_\nu \langle A_\nu \rangle_0 - \frac{1}{2} \sum_{\mu,\nu} f_\mu^* f_\nu \langle A_\mu | A_\nu \rangle_0 + \mathcal{O}(f^3) \; . \tag{9}$$

- Show that the isothermal susceptibility is given by

  $$\chi_{\mu\nu}^T = \langle A_\mu | A_\nu \rangle_0 \; . \tag{10}$$

*The time ordering operator orders the operators on its right in a descending time order. For example, for two bosonic operators A, B we have

$$\mathcal{T} \left[A(\tau_1) B(\tau_2) \right] = \theta(\tau_1 - \tau_2) A(\tau_1) B(\tau_2) + \theta(\tau_2 - \tau_1) B(\tau_2) A(\tau_1) \; . \tag{7}$$

Exercise 3 (5 Points)

Show that if at least one of the operators A_μ, A_ν commutes with the Hamiltonian H, the Mori product reduces to the thermal average (assume that $\langle A_\mu^\dagger \rangle = 0 = \langle A_\nu \rangle$)

$$\langle A_\mu | A_\nu \rangle = \beta \langle A_\mu^\dagger A_\nu \rangle \,. \tag{11}$$