Technische Universität Braunschweig

Partitioned Methods for Multifield Problems

Rang, 21.6.2016

Generalised-α method

Consider

$$\dot{\mathbf{u}} = \mathbf{f}(t, \mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0.$$

generalised-α method:

$$\begin{aligned}
\dot{\mathbf{u}}_{n+\alpha_m} &= \mathbf{f}(t_{n+\alpha_f}, \mathbf{u}_{n+\alpha_f}), \\
\mathbf{u}_{n+1} &= \mathbf{u}_n + \tau\dot{\mathbf{u}}_n + \tau\gamma(\dot{\mathbf{u}}_{n+1} - \dot{\mathbf{u}}_n), \\
\dot{\mathbf{u}}_{n+\alpha_m} &= \dot{\mathbf{u}}_n + \alpha_m(\dot{\mathbf{u}}_{n+1} - \dot{\mathbf{u}}_n), \\
\mathbf{u}_{n+\alpha_f} &= \mathbf{u}_n + \alpha_f(\mathbf{u}_{n+1} - \mathbf{u}_n).
\end{aligned}$$

Formulation as onestep method

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \tau\left(1 - \frac{\gamma}{\alpha_m}\right)\dot{\mathbf{u}}_n + \frac{\tau\gamma}{\alpha_m}\mathbf{f}_{n+\alpha_f},$$

$$\dot{\mathbf{u}}_{n+1} = \frac{1}{\tau\gamma}\left(\mathbf{u}_{n+1} - \mathbf{u}_n - \tau(1-\gamma)\dot{\mathbf{u}}_n\right),$$

if $\alpha_m \neq 0$.

- $\dot{\mathbf{u}}_0$ can be computed from the ODE.
- Consistency order 2, if $\alpha_m = \alpha_f$ and $\gamma = 1/2$.
- zero-stability and convergency, if $\alpha_m > 1/2$.

Formulation as multistep method

Formulation:

$$\mathbf{u}_{n+1} = \frac{2\alpha_m - 1}{\alpha_m}\mathbf{u}_n - \frac{\alpha_m - 1}{\alpha_m}\mathbf{u}_{n-1} + \frac{\tau(1-\gamma)}{\alpha_m}\mathbf{f}_{n-1+\alpha_f} + \frac{\tau\gamma}{\alpha_m}\mathbf{f}_{n+\alpha_f}.$$

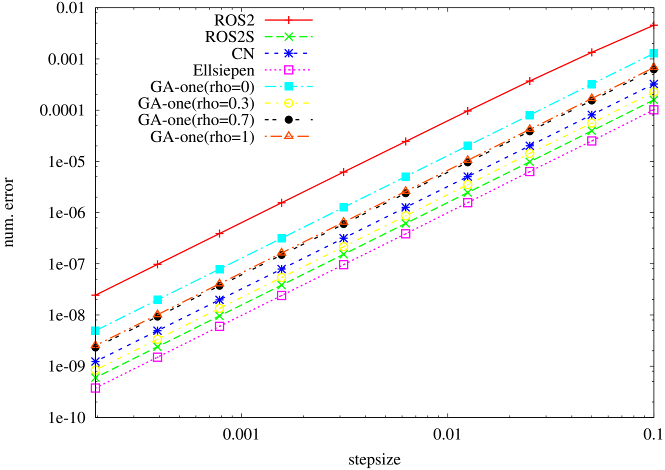
- second order consistency:

$$\gamma = \frac{1}{2} - \alpha_f + \alpha_m, \tag{1}$$

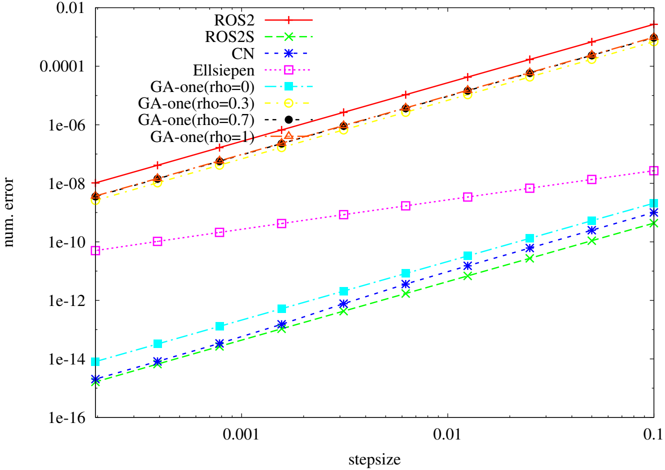
- generalised-α method is convergent of order 2 if $\alpha_m > 1/2$ and (1) hold.
- setting due to stability reasons:

$$\alpha_f = \gamma = \frac{1}{1+\rho_\infty}, \qquad \alpha_m = \frac{3-\rho_\infty}{2(1+\rho_\infty)}.$$

Example of Prothero and Robinson ($\lambda = -1$)



Example of Prothero and Robinson ($\lambda = -1.0E+6$)



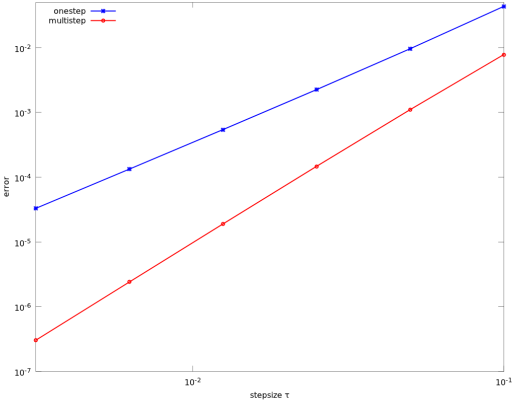
Comparision

- Let $\alpha_f = 1, \quad \alpha_m = 5/6, \quad \gamma = 1/3 \to$ 3rd order for multistep version
- Let $\tau = 1/(10 \cdot 2^k)$ with $k = 0, \ldots, 5$
- Consider the second order ODE

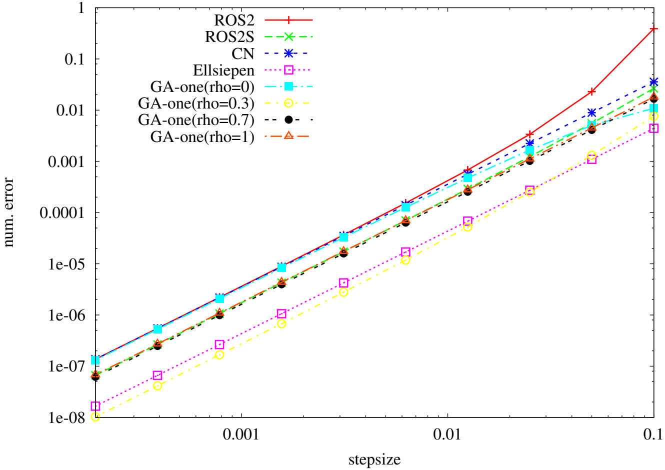
$$\ddot{y}_i = -\frac{y_i}{(y_1^2 + y_2^2)^{3/2}}, \quad i = 1, 2.$$

  The initial conditions are given by $\mathbf{u}_0 = \left(0, \sqrt{\frac{1+e}{1-e}}, 1-e, 0\right)^\top$, where $e = 1/2$.
- time interval: $[0, 10]$

Kepler’s problem



Comparision with other 2nd order metods



Formulation as onestep method

Consider the second order ODE

$$\ddot{\mathbf{u}} = \mathbf{f}(t, \mathbf{u}, \dot{\mathbf{u}}), \quad \mathbf{u}(0) = \mathbf{u}_0, \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0. \tag{2}$$

generalised-α method:

$$\begin{aligned}
\mathbf{u}_{n+\alpha_f} &= \alpha_f \mathbf{u}_{n+1} + (1-\alpha_f)\mathbf{u}_n, \\
\dot{\mathbf{u}}_{n+\alpha_f} &= \alpha_f \dot{\mathbf{u}}_{n+1} + (1-\alpha_f)\dot{\mathbf{u}}_n, \\
\ddot{\mathbf{u}}_{n+\alpha_m} &= \alpha_m \ddot{\mathbf{u}}_{n+1} + (1-\alpha_m)\ddot{\mathbf{u}}_n, \\
\mathbf{u}_{n+1} &= \mathbf{u}_n + \tau\dot{\mathbf{u}}_n + \tau^2 \left[\left(\frac{1}{2} - \beta \right) \ddot{\mathbf{u}}_n + \beta \ddot{\mathbf{u}}_{n+1} \right], \\
\dot{\mathbf{u}}_{n+1} &= \dot{\mathbf{u}}_n + \tau \left[(1-\gamma)\ddot{\mathbf{u}}_n + \gamma \ddot{\mathbf{u}}_{n+1} \right], \\
\ddot{\mathbf{u}}_{n+\alpha_m} &= \mathbf{f}(t_{n+\alpha_f}, \alpha_f \mathbf{u}_{n+1} + (1-\alpha_f)\mathbf{u}_n, \alpha_f \dot{\mathbf{u}}_{n+1} + (1-\alpha_f)\dot{\mathbf{u}}_n),
\end{aligned}$$

where $t_{n+\alpha_f} = t_n + \tau\alpha_f$.

Formulation as onestep method

Abbreviation:

$$\mathbf{f}_{n+\alpha_f} := \mathbf{f}(t_{n+\alpha_f}, \alpha_f \mathbf{u}_{n+1} + (1-\alpha_f)\mathbf{u}_n, \alpha_f \dot{\mathbf{u}}_{n+1} + (1-\alpha_f)\dot{\mathbf{u}}_n).$$

Then

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \tau \dot{\mathbf{u}}_n + \tau^2 \left[\left(\frac{1}{2} - \frac{\beta}{\alpha_m} \right) \ddot{\mathbf{u}}_n + \frac{\beta}{\alpha_m} \mathbf{f}_{n+\alpha_f} \right],$$

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \tau \left[\left(1 - \frac{\gamma}{\alpha_m} \right) \ddot{\mathbf{u}}_n + \frac{\gamma}{\alpha_m} \mathbf{f}_{n+\alpha_f} \right].$$

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\alpha_m} \left[\ddot{\mathbf{u}}_{n+\alpha_m} - (1-\alpha_m)\ddot{\mathbf{u}}_n \right] = \frac{1}{\alpha_m} \left[\mathbf{f}_{n+\alpha_f} - (1-\alpha_m)\ddot{\mathbf{u}}_n \right].$$

Formulation as multistep method

Consider

$$M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + \mathbf{S}(\mathbf{u}) = \mathbf{F}(t), \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}_0 = \mathbf{v}_0.$$

Then

$$M\ddot{\mathbf{u}}_{n+\alpha_m} =$$
$$\mathbf{F}(t_{n+\alpha_f}) - S(\alpha_f \mathbf{u}_{n+1} + (1-\alpha_f)\mathbf{u}_n) - C(\alpha_f \dot{\mathbf{u}}_{n+1} + (1-\alpha_f)\dot{\mathbf{u}}_n).$$

The formulation as multistep method

Method:

$$\sum_{j=0}^{3} [M\alpha_j + \tau C\gamma_j]\mathbf{u}_{n+j} + \tau^2 \sum_{j=0}^{2} \delta_j [\mathbf{S}_{n+j+\alpha_f} - \mathbf{F}(t_{n+j+\alpha_f})] = 0,$$

where

$$\begin{aligned}
&\alpha_0 = 1 - \alpha_m, \quad \alpha_1 = 3\alpha_m - 2, \quad \alpha_2 = 1 - 3\alpha_m, \quad \alpha_3 = \alpha_m, \\
&\gamma_0 = (1 - \alpha_f)(\gamma - 1), \quad \gamma_1 = 1 - 2\alpha_f - 2\gamma + 3\gamma\alpha_f, \gamma_2 = \alpha_f + \gamma - 3\gamma\alpha_f, \quad \gamma \\
&\delta_0 = \frac{1}{2} + \beta - \gamma, \quad \delta_1 = \frac{1}{2} - 2\beta + \gamma, \quad \delta_2 = \beta
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}_{n+j-\alpha_f} &= \mathbf{F}(\alpha_f t_{n+j+1} + (1 - \alpha_f)t_{n+j}) = \mathbf{F}(t_{n+j} + \alpha_f \tau) \\
\mathbf{S}_{n+j+\alpha_f} &= \alpha_f \mathbf{S}(\mathbf{u}_{n+j+1}) + (1 - \alpha_f)\mathbf{S}(\mathbf{u}_{n+j}).
\end{aligned}$$

Properties

- consistency order 2: $\gamma = \dfrac{1}{2} + \alpha_m - \alpha_f$.
- zero-stable and convergence: $\alpha_m \geqslant 1/2$, $\alpha_f \leqslant 1/2$ and $\gamma \leqslant 1/2$.
- for stability reasons:

$$\beta = \frac{(1 + \alpha_m - \alpha_f)^2}{4}, \alpha_f = \frac{1}{1 + \rho_\infty}, \alpha_m = \frac{2 - \rho_\infty}{1 + \rho_\infty}$$

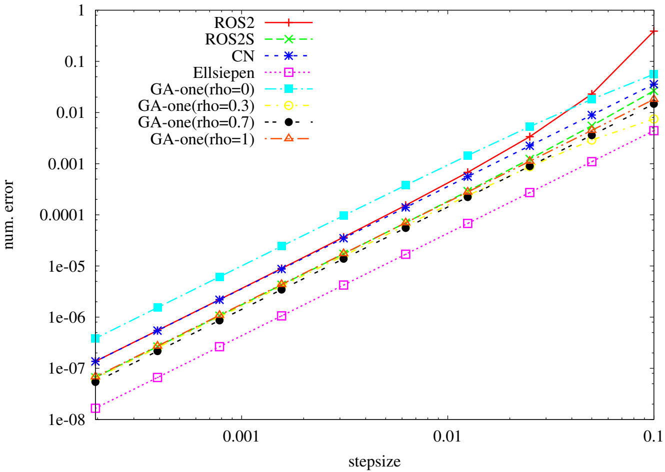
Comparision

- Let $\tau = 1/(10 \cdot 2^k)$ with $k = 0, \dots, 5$
- Consider the second order ODE

$$\ddot{y}_i = -\frac{y_i}{(y_1^2 + y_2^2)^{3/2}}, \quad i = 1, 2.$$

  The initial conditions are given by $\mathbf{u}_0 = \left(0, \sqrt{\frac{1+e}{1-e}}, 1-e, 0\right)^\top$, where $e = 1/2$.
- time interval: $[0, 10]$

Comparision with other 2nd order metods



Adaptivity

- Idea: use PI-controller from Gustafsson et. al. (1988)
- suggest a new timestep size with solutions of order p and $p-1$.
- use approximation of generalised-α method as a second order approximation.
- use as the second solution with order 1 backward Euler method.
- next timestep size τ_{n+1}:

$$\tau_{n+1} = \rho_{adapt} \frac{\tau_n^2}{\tau_{n-1}} \left(\frac{TOL \cdot r_{n-1}}{r_n^2} \right)^{1/2},$$

where $\rho_{adapt} \in (0, 1]$ is a safety factor, $TOL > 0$ is a given tolerance, and $r_{n+1} := \|\mathbf{u}_{n+1} - \hat{\mathbf{u}}_{n+1}\|$.

Adaptivity for onestep versions

- Compute $(\mathbf{u}_{n+1}, \dot{\mathbf{u}}_{n+1})^\top$ with generalised-α method.
- Compute second solution with the backward Euler method and use $\dot{\mathbf{u}}_{n+1}$ as approximation for $\mathbf{f}(t_{n+1}, \mathbf{u}_{n+1})$, i. e. $\hat{\mathbf{u}}_{n+1} = \mathbf{u}_n + \tau_n \dot{\mathbf{u}}_{n+1}$.
- Compute the numerical error with r_{n+1} and approximate the new timestep length τ_{n+1}.
- If the numerical error is smaller than the given tolerance the timestep is accepted, otherwise it is rejected and has to be recomputed with the new timestep length τ_{n+1}.

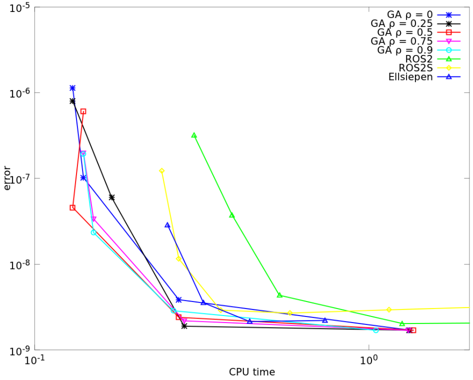
The chemical reaction E5

Let $A = 7,89 \times 10^{-10}$, $B = 1,1 \times 10^{7}$, $C = 1,13 \times 10^{3}$, and $M = 10^{6}$. Consider

$$\begin{aligned}
\dot{u}_1 &= -Au_1 - Bu_1 u_3, \\
\dot{u}_2 &= Au_1 - MCu_2 u_3, \\
\dot{u}_3 &= Au_1 - Bu_1 u_3 - MCu_2 u_3 + Cu_4, \\
\dot{u}_4 &= Bu_1 u_3 - Cu_4
\end{aligned}$$

with the initial conditions $u_1(0) = 1,76 \times 10^{-3}$ and $u_i(0) = 0$, $i \in \{2, 3, 4\}$.
The equations should be solved in the time interval $[0, 10^{13}]$.

The chemical reaction E5



Kepler's problem

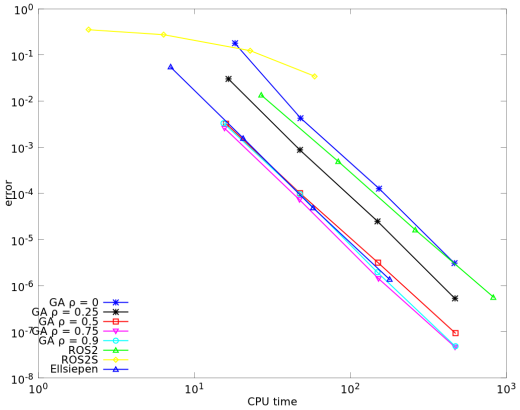
Consider the second order ODE

$$\ddot{y}_i = -\frac{y_i}{(y_1^2 + y_2^2)^{3/2}}, \quad i = 1, 2.$$

The initial conditions are given by $\mathbf{u}_0 = \left(0, \sqrt{\frac{1+e}{1-e}}, 1-e, 0\right)^\top$,

where $e \in [0, 1)$ is a given parameter. In our numerical example we choose $e = 1/2$. We solve the problem in the interval $[0, 20000]$

Kepler's problem



Model problem

Damped spring mass system:

$$(1-\alpha)\ddot{u}^s + \omega^2 u^s = -\alpha\ddot{u}^f - 2\xi\omega\dot{u}^f$$

with

$$\alpha = \frac{m^f}{m}, \quad 1-\alpha = \frac{m^s}{m}, \quad m^f + m^s = m,$$
$$\omega = \sqrt{k/m}, \quad \xi = \frac{c}{2\sqrt{km}},$$

Initial conditions:

$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0.$$

Model problem

- m ... total mass of the system
- m^f ... mass of the domain f
- m^s ... mass of the domain s
- k ... spring stiffness
- c ... a damping coefficient
- ω ... natural frequency of the system
- ξ ... a dimensionless damping ration

Analytical solution

Consider the case $u := u^f = u^s$. Then

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = 0.$$

- u ... displacement of the total mass
- $\dot{u}$ ... velocity
- $\ddot{u}$ ... acceleration.

Analytical solution

- Ansatz
$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$
where $\lambda_1, \lambda_2 \in \mathbb{C}$.
- System is undercritically damped, i. e. $\xi < 1$.
- λ_1 and λ_2 can be determined by solving the quadratic problem
$$\lambda^2 + 2\xi\omega\lambda + \omega^2 = 0,$$
- solution:
$$\lambda_{1,2} = -\xi\omega \pm i\omega_D,$$
where $\omega_D = \omega\sqrt{1 - \xi^2}$.

Analytical solution

Then we have

$$u(t) = \exp(-\xi\omega t)\left[c_1 \exp(-i\omega_D t) + c_2 \exp(i\omega_D t)\right].$$

First initial condition:

$$u_0 = u(0) = c_1 + c_2.$$

Second initial condition:

$$\dot{u}_0 = \dot{u}(0) = -\xi\omega u_0 + i\omega_D(c_2 - c_1).$$

Analytical solution

$$u(t) = \exp(-\xi\omega t)\left[u_0 \cos(\omega_D t) + \frac{\dot{u}_0 + \xi\omega u_0}{\omega} \sin(\omega_D t)\right].$$

Numerical solution

Idea:

- Discretise LHS with the generalised-α method for second order problems
- Discretise RHS with the generalised-α method for first order problems.

Onestep formulation:

$$
\begin{aligned}
(1-\alpha)\ddot{u}^s_{n+\alpha^s_m} + \omega^2 u^s_{n+\alpha^s_f} &= -\alpha\ddot{u}^f_{n+\alpha^f_m} - \xi\omega\dot{u}^f_{n+\alpha^f_f} \\
\dot{u}^s_{n+1} = \dot{u}^f_{n+1} &=: \dot{u}^I_{n+1} \\
\dot{u}^s_n = \dot{u}^f_n &=: \dot{u}^I_n.
\end{aligned}
$$

$\dot{u}^I_{n+1}$ and $\dot{u}^I_n$ represent the interface velocities at time t_{n+1} and t_n

Numerical solution

Insert formulas for $\ddot{u}^s_{n+\alpha^s_m}$, $\ddot{u}^f_{n+\alpha^f_m}$, and $\dot{u}^f_{n+\alpha^f_f}$:

$$
\begin{aligned}
&(1-\alpha)\left[\alpha^s_m \ddot{u}^s_{n+1} + (1-\alpha^s_m)\ddot{u}^s_n\right] + \omega^2\left[\alpha^s_f u^s_{n+1} + (1-\alpha^s_f)u^s_n\right] \\
&\quad = -\alpha\left[\alpha^f_m \ddot{u}^f_{n+1} + (1-\alpha^f_m)\ddot{u}^f_n\right] - 2\xi\omega\left[\alpha^f_f \dot{u}^f_{n+1} + (1-\alpha^f_f)\dot{u}^f_n\right].
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
u^s_{n+1} &= u^s_n + \tau\dot{u}^s_n + \tau^2\left[\left(\frac{1}{2}-\beta^s\right)\ddot{u}^s_n + \beta_s\ddot{u}^s_{n+1}\right], \\
\dot{u}^f_{n+1} &= \dot{u}^f_n + \tau\left[\gamma^f\ddot{u}^f_{n+1} + (1-\gamma^f)\ddot{u}^f_n\right], \\
\dot{u}^s_{n+1} &= \dot{u}^s_n + \tau\left[\gamma^s\ddot{u}^s_{n+1} + (1-\gamma^s)\ddot{u}^s_n\right].
\end{aligned}
$$

Numerical solution

Then $A_1 \mathbf{v}_{n+1} = A_2 \mathbf{v}_n$, where

$$A_1 = \begin{pmatrix} \omega^2 \alpha_f^s & 2\frac{\xi\omega}{\tau}\alpha_f^f & \frac{(1-\alpha)\alpha_m^s}{\tau^2} & \frac{\alpha\alpha_m^f}{\tau^2} \\ 1 & 0 & -\beta^s & 0 \\ 0 & \frac{1}{\tau} & 0 & -\frac{\gamma^f}{\tau} \\ 0 & \frac{1}{\tau} & -\frac{\gamma^s}{\tau} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\omega^2(1-\alpha_f^s) & -2\frac{\xi\omega}{\tau}(1-\alpha_f^f) & .\frac{(1-\alpha)(1-\alpha_m^s)}{\tau^2} & -\frac{\alpha(1-\alpha_m^f)}{\tau^2} \\ 1 & 1 & \frac{1}{2}-\beta^s & 0 \\ 0 & \frac{1}{\tau} & 0 & \frac{1-\gamma^f}{\tau} \\ 0 & \frac{1}{\tau} & \frac{1-\gamma^s}{\tau} & 0 \end{pmatrix}$$

$$\mathbf{v}_n = (u_n^s, \tau\dot{u}_n^l, \tau^2\ddot{u}_n^s, \tau^2\ddot{u}_n^f)^\top.$$

Convergence order

- Write our problem as $\mathbf{v}_{n+1} = A_1^{-1} A_2 \mathbf{v}_n$.
- Expand this expression in a Taylor series

$$u_{n+1}^s = u_n^s + \tau \dot{u}_n^s - \frac{\tau^2}{2[\alpha\alpha_m^f\gamma^s + (1-\alpha)\alpha_m^s\gamma^f]} \left[2\beta^s\alpha\alpha_m^f(\ddot{u}_s - \ddot{u}_f) - \alpha\alpha_m^f\gamma^s\ddot{u}_s\right].$$

Numerical solution with interpolation

- Problem: LHS and RHS are taken at different times.
- Let

$$
\begin{aligned}
F^s_{n+\alpha^s_f} &= (1-\alpha)\ddot{u}^s_{n+\alpha^s_m} + \omega^2 u^s_{n+\alpha^s_f},\\
F^f_{n+\alpha^f_f} &= -\alpha\ddot{u}^f_{n+\alpha^f_m} - 2\xi\omega\dot{u}^f_{n+\alpha^f_f},
\end{aligned}
$$

where

$$
\begin{aligned}
F^s_{n+\alpha^s_f} &= \alpha^s_f F_{n+1} + (1-\alpha^s_f)F_n,\\
F^f_{n+\alpha^f_f} &= \alpha^f_f F_{n+1} + (1-\alpha^f_f)F_n.
\end{aligned}
$$

Numerical solution with interpolation

Then

$$\alpha_f^s F_{n+1} + (1-\alpha_f^s)F_n = (1-\alpha)\ddot{u}^s_{n+\alpha_m^s} + \omega^2 u^s_{n+\alpha_f^s},$$
$$\alpha_f^f F_{n+1} + (1-\alpha_f^f)F_n = -\alpha\ddot{u}^f_{n+\alpha_m^f} - 2\xi\omega\dot{u}^f_{n+\alpha_f^f}.$$

and

$$\begin{aligned}
(1-\alpha)&\left[\alpha_m^s\ddot{u}^s_{n+1} + (1-\alpha_m^s)\ddot{u}^s_n\right] \\
&+\omega^2\left[\alpha_f^s u^s_{n+1} + (1-\alpha_f^s)u^s_n\right] = \alpha_f^s F_{n+1} + (1-\alpha_f^s)F_n, \\
\alpha_f^f F_{n+1} + (1-\alpha_f^f)F_n &= -\alpha\left[\alpha_m^f\ddot{u}^f_{n+1} + (1-\alpha_m^f)\ddot{u}^f_n\right] \\
&\quad - 2\xi\omega\left[\alpha_f^f\dot{u}^f_{n+1} + (1-\alpha_f^f)\dot{u}^f_n\right] \\
u^s_{n+1} &= u^s_n + \tau\dot{u}^s_n + \tau^2\left[\left(\frac{1}{2} - \beta^s\right)\ddot{u}^s_n + \beta_s\ddot{u}^s_{n+1}\right], \\
\dot{u}^f_{n+1} &= \dot{u}^f_n + \tau\left[\gamma^f\ddot{u}^f_{n+1} + (1-\gamma^f)\ddot{u}^f_n\right], \\
\dot{u}^s_{n+1} &= \dot{u}^s_n + \tau\left[\gamma^s\ddot{u}^s_{n+1} + (1-\gamma^s)\ddot{u}^s_n\right].
\end{aligned}$$

Numerical solution with interpolation

Finally $\mathbf{v}_{n+1} = A_1^{-1} A_2 \mathbf{v}_n$ with

$$A_1 = \begin{pmatrix} \omega^2 \alpha_f^s & 0 & \frac{(1-\alpha)\alpha_m^s}{\tau^2} & 0 & -\frac{\alpha_f^s}{\tau^2} \\ 0 & 2\frac{\xi\omega}{\tau}\alpha_f^f & 0 & \frac{\alpha\alpha_m^f}{\tau^2} & \frac{\alpha_f^f}{\tau^2} \\ 1 & 0 & -\beta^s & 0 & 0 \\ 0 & \frac{1}{\tau} & 0 & -\frac{\gamma^f}{\tau} & 0 \\ 0 & \frac{1}{\tau} & -\frac{\gamma^s}{\tau} & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\omega^2(1-\alpha_f^s) & 0 & -\frac{(1-\alpha)(1-\alpha_m^s)}{\tau^2} & 0 & \frac{1-\alpha_f^s}{\tau^2} \\ 0 & -2\frac{\xi\omega}{\tau}(1-\alpha_f^f) & 0 & \frac{\alpha(\alpha_m^f-1)}{\tau^2} & -\frac{1-\alpha_f^f}{\tau^2} \\ 1 & 1 & \frac{1}{2}-\beta^s & 0 & 0 \\ 0 & \frac{1}{\tau} & 0 & \frac{1-\gamma^f}{\tau} & 0 \\ 0 & \frac{1}{\tau} & \frac{1-\gamma^s}{\tau} & 0 & 0 \end{pmatrix},$$

$$\mathbf{v}_n = (u_n^s, \tau\dot{u}_n^l, \tau^2\ddot{u}_n^s, \tau^2\ddot{u}_n^f, \tau^2 F_n)^\top.$$

Convergence

- Compute u_{n+1} and expand it in a Taylor series.
- Local error:

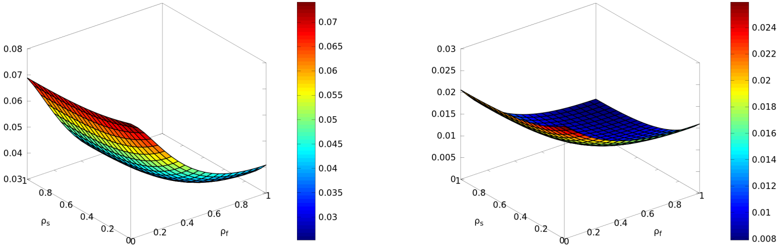
$$\delta_\tau = \tau^2 \frac{\beta^s \gamma^f (\alpha_f^f - \alpha_f^s)}{(1-\alpha)\alpha_m^s \alpha_f^f \gamma^f + \alpha \alpha_f^s \alpha_m^f \gamma^s} + \mathcal{O}(\tau^3).$$

- If the coefficients of the generalised-α method are choosen with ρ_s and ρ_f the error gets large for the setting

$$\rho_s = 1, \quad \rho_f = 0, \quad \alpha = 0.9.$$

Numerical experiment

- Setting 1: $\omega = 1$, $\xi = 0.001$, and $\alpha = 0.5$
- Setting 2: $\omega = 1$, $\xi = 0.01$, and $\alpha = 0.8$.
- stepsize $\tau = \pi/15$



Coupling multistep version

First we have

$$\mathbf{u}_{n+1} = \frac{2\alpha_m - 1}{\alpha_m}\mathbf{u}_n - \frac{\alpha_m - 1}{\alpha_m}\mathbf{u}_{n-1} + \frac{\tau(1-\gamma)}{\alpha_m}\mathbf{f}_{n-1+\alpha_f} + \frac{\tau\gamma}{\alpha_m}\mathbf{f}_{n+\alpha_f}.$$

for the first order problem and

$$\sum_{j=0}^{3}[M\alpha_j + \tau C\gamma_j]\mathbf{u}_{n+j} + \tau^2 \sum_{j=0}^{2}\delta_j[\mathbf{S}_{n+j+\alpha_f} - \mathbf{F}(t_{n+j+\alpha_f})] = 0,$$

for the second order problem.

Coupling multistep version

Then

$$(1-\alpha)\sum_{j=0}^{3}\alpha_j^s \mathbf{u}_{n+j}^s + \tau^2 \sum_{j=0}^{2}\delta_j[\omega^2 \mathbf{u}_{n+j+\alpha_f} - \mathbf{F}(t_{n+j+\alpha_f})] = 0$$

$$\begin{aligned}\alpha_m^f \dot{\mathbf{u}}_{n+3}^f - (2\alpha_m^f - 1)\dot{\mathbf{u}}_{n+2}^f + (\alpha_m^f - 1)\dot{\mathbf{u}}_{n+1}^f \\ +\tau(1-\gamma^f)(2\xi\omega\dot{\mathbf{u}}_{n+1+\alpha_f}^f + \mathbf{f}_{n+1+\alpha_f}) \\ +\tau\gamma(2\xi\omega\dot{\mathbf{u}}_{n+2+\alpha_f}^f + \mathbf{f}_{n+2+\alpha_f}) = 0 \\ \frac{1}{2\tau}(\mathbf{u}_{n+3} - \mathbf{u}_{n+1}) = \dot{\mathbf{u}}_{n+2}\end{aligned}$$

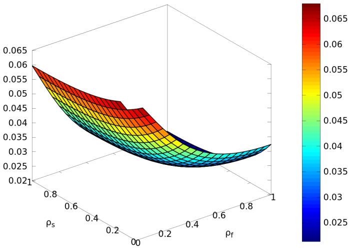
Coupling multistep version

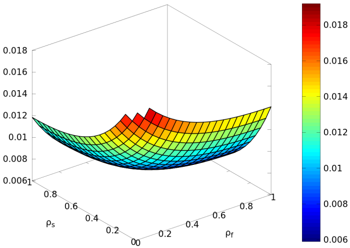
It follows

$$
\begin{aligned}
&[(1-\alpha)\alpha_m^s + \tau^2\beta\omega^2\alpha_f^f]\mathbf{u}_{n+3}^s - \tau^2\beta\omega^2\alpha_f^f\mathbf{F}_{n+3}\\
&= -[(1-\alpha)(1-3\alpha_m) + \tau^2\left(\frac{1}{2}-2\beta+\gamma\right)\omega^2\alpha_f^f + \tau^2\beta\omega^2(1-\alpha_f^f)]\mathbf{u}_{n+2}^s\\
&- [(1-\alpha)(3\alpha_m-2) + \tau^2\left(\frac{1}{2}+\beta-\gamma\right)\omega^2\alpha_f^f + \tau^2\left(\frac{1}{2}-2\beta+\gamma\right)\omega^2(1-\alpha_f^f)]\mathbf{u}_{n+1}^s\\
&- [(1-\alpha)(1-\alpha_m) + \tau^2\left(\frac{1}{2}+\beta-\gamma\right)\omega^2(1-\alpha_f^f)]\mathbf{u}_n^s\\
&+ \tau^2[\left(\frac{1}{2}-2\beta+\gamma\right)\omega^2\alpha_f^f + \beta\omega^2(1-\alpha_f^f)]\mathbf{F}_{n+2}\\
&+ \tau^2[\left(\frac{1}{2}+\beta-\gamma\right)\omega^2\alpha_f^f + \tau^2\left(\frac{1}{2}-2\beta+\gamma\right)\omega^2(1-\alpha_f^f)]\mathbf{F}_{n+1}\\
&+ \tau^2\left(\frac{1}{2}+\beta-\gamma\right)\omega^2(1-\alpha_f^f)\mathbf{F}_n\\
&(\alpha_m^f + \tau\gamma 2\xi\omega\alpha_f^f)\dot{\mathbf{u}}_{n+3}^f + \tau\gamma\alpha_f^f\mathbf{F}_{n+3}\\
&= [(2\alpha_m^f - 1) - \tau\gamma(2\xi\omega(1-\alpha_f^f) - \tau(1-\gamma^f)2\xi\omega\alpha_f^f]\dot{\mathbf{u}}_{n+2}^f\\
&- [(\alpha_m^f - 1) + \tau(1-\gamma^f)2\xi\omega(1-\alpha_f^f)]\dot{\mathbf{u}}_{n+1}^f\\
&- \tau[\gamma(1-\alpha_f^f) + (1-\gamma^f)\alpha_f^f]\mathbf{F}_{n+2} - \tau(1-\gamma^f)(1-\alpha_f^f)\mathbf{F}_{n+1}
\end{aligned}
$$

$$
\dot{\mathbf{u}}_{n+2} = \frac{1}{2\tau}(\mathbf{u}_{n+3} - \mathbf{u}_{n+1})
$$

Numerical results

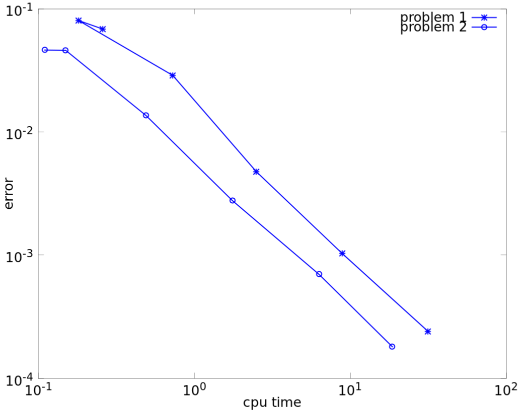




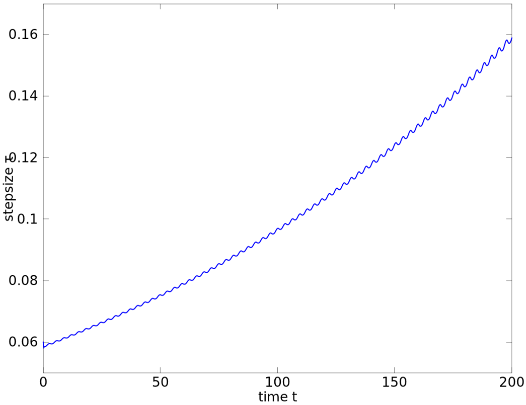
Adaptivity for the partitioned approach

- First compute u^s_{n+1} and $\dot{u}^l_{n+1}$
- Compute $\ddot{u}_{n+1}$ by evaluating the model problem.
- Compute a second solution with the backward Euler method, i. e. $u^s_{n+1} = u^s_n + \tau \dot{u}^l_{n+1}$ and $\dot{u}^l_{n+1} = \dot{u}^l_n + \tau \ddot{u}_{n+1}$.
- Compute the numerical error r_{n+1} and approximate the new timestep length τ_{n+1}.
- If the numerical error is smaller than the given tolerance the timestep is accepted otherwise it is rejected and has to recomputed with the new timestep length τ_{n+1} .

Numerical result: CPU versus error



Numerical result: t versus τ



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