# Heuristic Optimality Checks for Noise-Aware Sparse Recovery by $\ell_1$-Minimization

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## Primal Problem

We consider the general class

$$\min_{x\in\mathbb{R}^n} \quad \|x\|_1 \quad \text{s.t.} \quad \|Ax-b\|\leq\delta \qquad \text{(P1)}$$

of $\ell_1$-minimization problems. This class includes some of the most popular approaches for sparse reconstruction from incomplete linear measurements. Therein, $A\in\mathbb{R}^{m\times n}$ has full rank $m$, $\delta\geq 0$ is an estimate of measurement noise, $\|\cdot\|$ is an arbitrary norm and $b\in\mathbb{R}^m$ satisfies $\|b\|>\delta$.

## Dual Problem

With the dual norm $\|\cdot\|_*$ of $\|\cdot\|$, the dual problem of (P1) is

$$\max_{y\in\mathbb{R}^m} \quad -b^\top y-\delta\|y\|_* \quad \text{s.t.} \quad \|A^\top y\|_\infty\leq 1. \qquad \text{(D1)}$$

## Optimality Conditions

A solution $\hat{x}$ is optimal for (P1) if and only if there exists $\hat{y}\in\mathbb{R}^m$ with

$$-A^\top\hat{y}\in\operatorname{Sign}(\hat{x}) \quad \text{and} \quad A\hat{x}\in b+\delta\partial\|\hat{y}\|_*.$$

The tuple $(\hat{x},\hat{y})$ is called a primal-dual optimal pair. A primal-dual optimal pair can also be characterized by the conditions

$$\|\hat{x}\|_1+b^\top\hat{y}+\delta\|\hat{y}\|_*=0 \quad , \quad \|A\hat{x}-b\|=\delta \quad \text{and} \quad \|A^\top\hat{y}\|_\infty=1.$$

## Motivation

- A typical feature of iterative solvers for (P1) is that convergence becomes slow towards the end.
- We propose a heuristic optimality check (HOC) that often allows to "jump" from an iterate to an approximately optimal solution.
- Given a point $x\in\mathbb{R}^n$ with approximate support $S$, the idea of HOC is to construct an optimal point $\hat{x}$ with support $\hat{S}\subseteq S$ exploiting optimality conditions and duality.
- The HOC scheme can be integrated into any solver for (P1).

**Algorithm:** HOC for (P1)

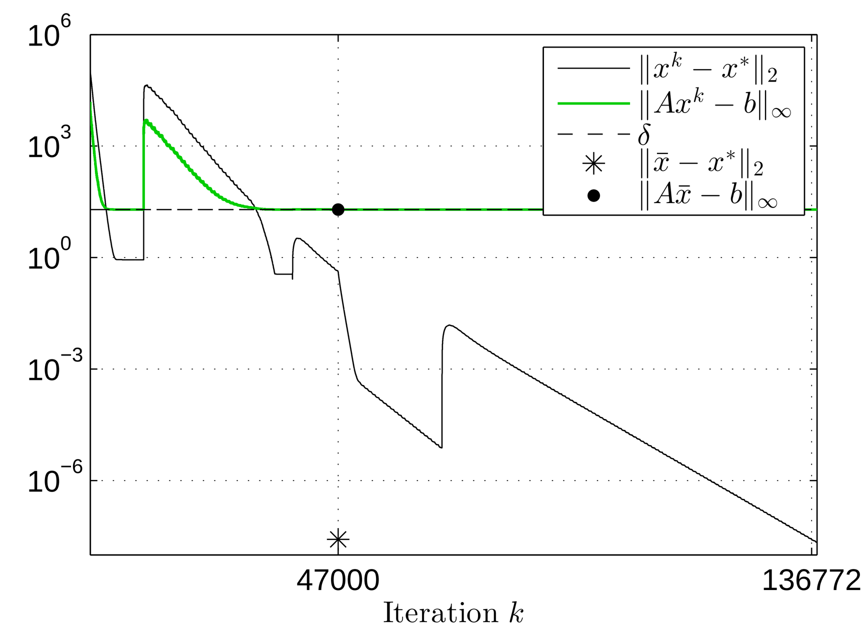
**input:** $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$, $\delta\geq 0$, $x\in\mathbb{R}^n$

1 $S\leftarrow$ approximate support deduced from $x$
2 $\hat{y}\leftarrow$ approximate solution to $-A_S^\top w=\operatorname{sign}(x_S)$
3 **if** $\|A^\top\hat{y}\|_\infty\approx 1$ **then**
4     $\hat{x}_S\leftarrow$ approximate solution to $A_S z\in b+\delta\partial\|\hat{y}\|_*$
5     $\hat{x}_{S^C}\leftarrow 0$
6     **if** $\|A\hat{x}-b\|\approx\delta$ **then**
7         **if** $(\|\hat{x}\|_1+\delta\|\hat{y}\|_*+b^\top\hat{y})/\|\hat{x}\|_1\approx 0$ **then**
8             **return** *approximate primal-dual optimal pair* $(\hat{x},\hat{y})$

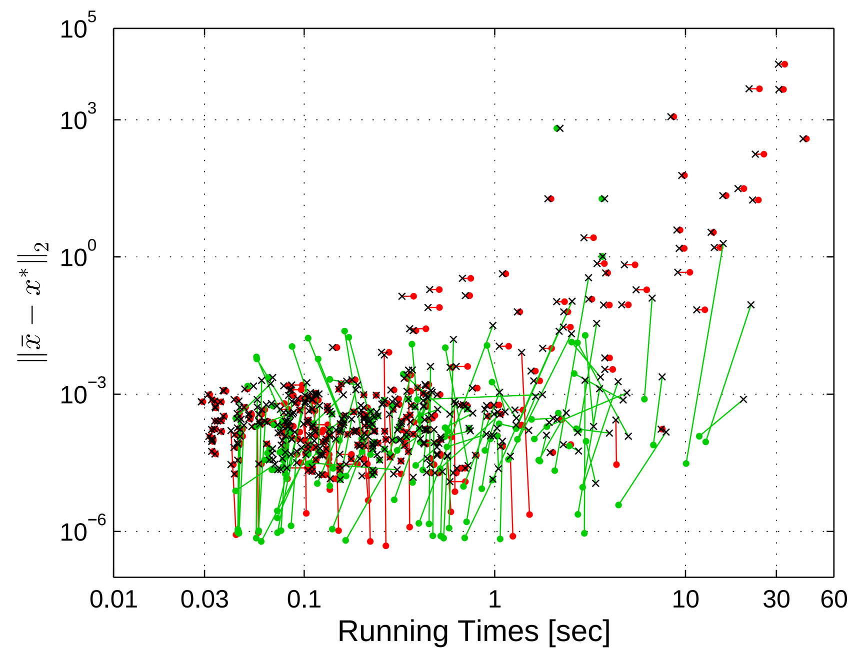
The choice of $\|\cdot\|$ affects Step 4 crucially. We display two examples:

- $\|\cdot\|=\|\cdot\|_\infty \implies \partial\|\hat{y}\|_*=\partial\|\hat{y}\|_1=\operatorname{Sign}(\hat{y})$ (multivalued)
- $\|\cdot\|=\|\cdot\|_2 \implies \partial\|\hat{y}\|_*=\partial\|\hat{y}\|_2=\hat{y}/\|\hat{y}\|_2$ (singlevalued)

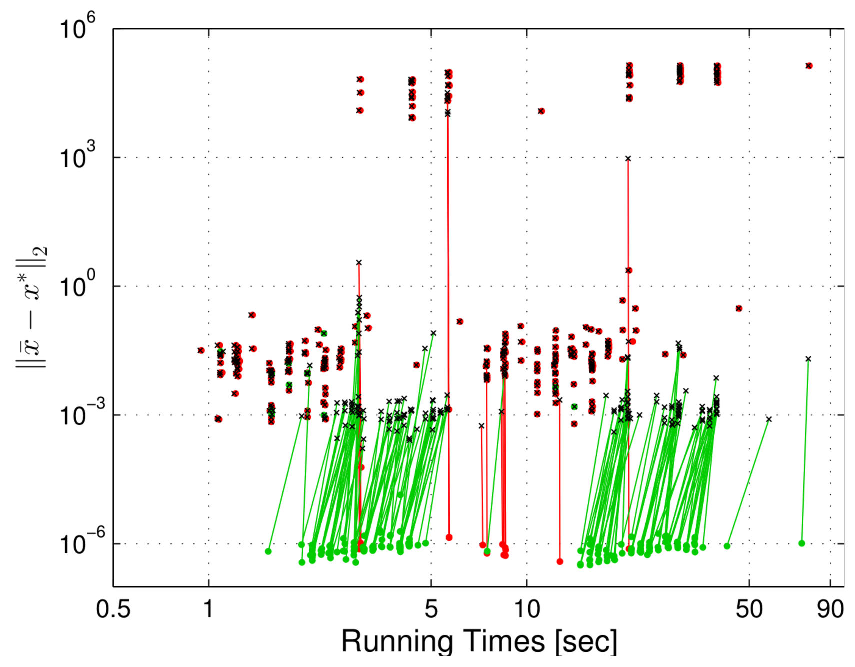
## Numerical Results




Example for HOC efficiency in the incremental subgradient method, applied to (P1) with $\ell_\infty$-constraints.




HOC impact within SPGL1 applied to (P1) with $\ell_2$-constraints. Average $\delta\approx 0.1$.




HOC impact within SolveBP/PDCO applied to $\min\lambda\|x\|_1+\frac{1}{2}\|Ax-b\|_2^2$. Average $\lambda=10$.

Middle and right: 444 test instances with varying solution sparsities and dimensions. Crossmarks represent results without HOC, dots those with HOC. Results for the same instance are connected by lines (green/red: faster/slower with HOC).

## Conclusion

The HOC scheme is empirically demonstrated to allow for early termination and to improve solution speed and / or accuracy for different methods and a large number of instances of (P1).

## Extensions and Future Work

- HOC schemes for the problems $\min\ \lambda\|x\|_1+\frac{1}{2}\|Ax-b\|_2^2$ and $\min\|D^\top x\|_1$ s.t. $\|Ax-b\|\leq\delta$ exist as well.
- Our results can be used to generate test instances for (P1).

SPARS 2015    Cambridge (UK)    05/07 - 09/07/2015

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