Technische
Universität
Braunschweig

Introduction to PDEs and Numerical Methods

Lecture 2:

Analytical solution of ODEs and PDEs

Dr. Noemi Friedman, 24.10. 2018.

Overview of the course

- Introduction (definition of PDEs, classification, basic math, modelling: introductory examples of PDEs)

- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)

- Numerical solutions of PDEs:

  - Finite difference method

  - Finite element method

Overview of this lecture

- Classification of PDEs, some examples of PDEs, boundary conditions
- Solving linear systems – solving linear PDEs
- Eigenvalues, eigenfunctions, solving linear equations with spectral method – spectral method for PDEs
- Fourier series
- Separation of variables – analytical solution of the heat equation

Classification of PDEs

- Constant/variable coefficients
- Stationary/instationary (not time dependent/time dependent)
- Linear/nonlinear

  linearity condition: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$
- order

  order of the highest derivative
- homogeneous/inhomogenous

  inhomogeneous: additive terms which do not depend on unknown function

  homogenous: $u = 0$ is a solution of the equation
- elliptic/parabolic/hyperbolic (only for second order PDEs)

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower order derivatives} = 0$$

  - $AC - B^2 = 0$ parabolic
  - $AC - B^2 < 0$ hyperbolic
  - $AC - B^2 > 0$ elliptic

Classification of PDEs, examples of PDEs

	Wave equation $u_{tt} - \boldsymbol{c}^2 u_{xx} = 0$	**Laplace equation** $u_{xx} + u_{yy} = 0$	**Heat equation** $u_t - u_{xx} = 0$
Order	2	2	2
Constant coefficient?	yes	yes	yes
Linear?	yes	yes	yes
Homogenous?	yes	yes	yes
Class	$A=1, B=0, C=-\boldsymbol{c}^2$ $AC - B^2 = -\boldsymbol{c}^2$ Hyperbolic	$A=1, B=0, C=1$ $AC - B^2 = 1$ Elliptic	$A=-1, B=0, C=0$ $AC - B^2 = 0$ Parabolic

Classification of PDEs, examples of PDEs

$$L_2(u(t,x)) = \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2}$$

The L_2 operator is hyperbolic if $\alpha > 0$

Linearity:

$$L_2(\beta u_1 + \gamma u_2) =$$

$$= \frac{\partial^2(\beta u_1 + \gamma u_2)}{\partial t^2} - \alpha \frac{\partial^2(\beta u_1 + \gamma u_2)}{\partial x^2} =$$

$$\beta \frac{\partial^2(u_1)}{\partial t^2} + \gamma \frac{\partial^2(u_2)}{\partial t^2} - \alpha \left(\beta \frac{\partial^2(u_1)}{\partial x^2} - \gamma \frac{\partial^2(u_2)}{\partial x^2} \right) =$$

$$= \beta \left(\frac{\partial^2(u_1)}{\partial t^2} - \alpha \frac{\partial^2(u_1)}{\partial x^2} \right) + \gamma \left(\frac{\partial^2(u_2)}{\partial t^2} - \alpha \frac{\partial^2(u_2)}{\partial x^2} \right) =$$

$$= \beta L_2(u_1) + \gamma L_2(u_2)$$

Classification of PDEs, examples of PDEs

$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = -\frac{q}{D}$ order: 4; linear, stationary

$\frac{\partial u}{\partial t} - 4u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 0$ order: 2; nonlinear, instationary

$\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2 u}{\partial x^2})^2 = f$ order: 2; nonlinear, instationary

Introductory example: heat flow in a bar

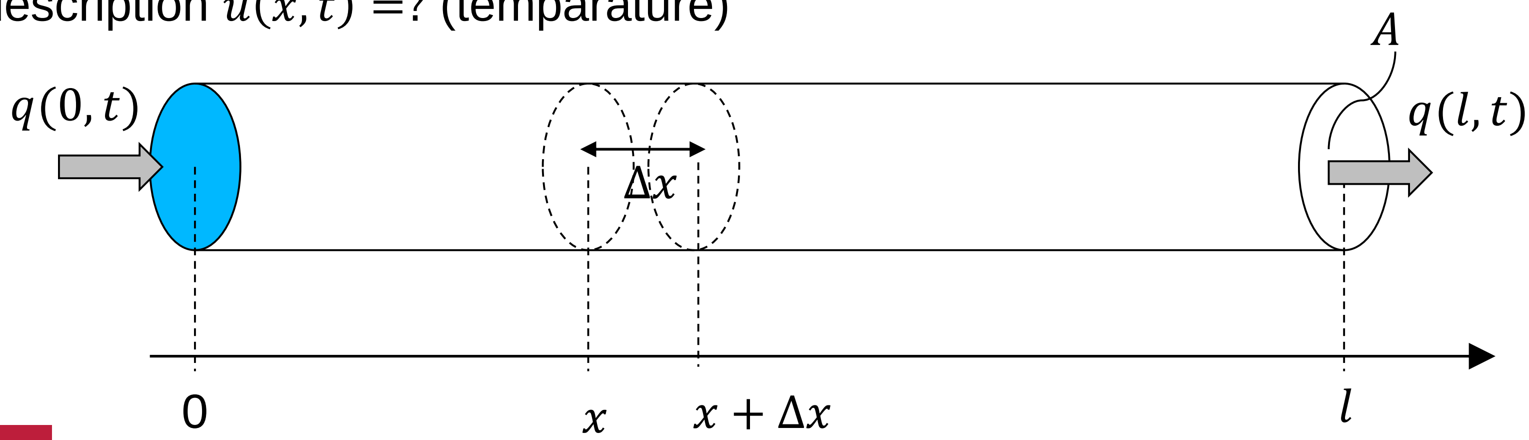
- Basic assumptions:
- Uniform cross-section
- Temperature varies only in the longitudinal direction
- Relationship between heat energy and temperature is linear

  $c\left[\frac{J}{gK}\right]$: specific heat capacity ⟺ Energy $[J]$

  $c\ J$ energy is required to raise the tempreture by 1K of 1g material
- Homogenous material properties along the bar (ρ and c are constants along the bar)

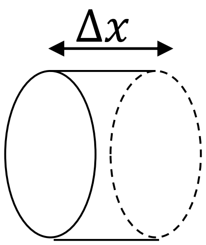
  $\rho[\frac{g}{m^3}]$: density of the material of the bar

- Problem description $u(x,t) =$? (temparature)



Introductory example: heat flow in a bar

Total energy in the bar section of length Δx is:

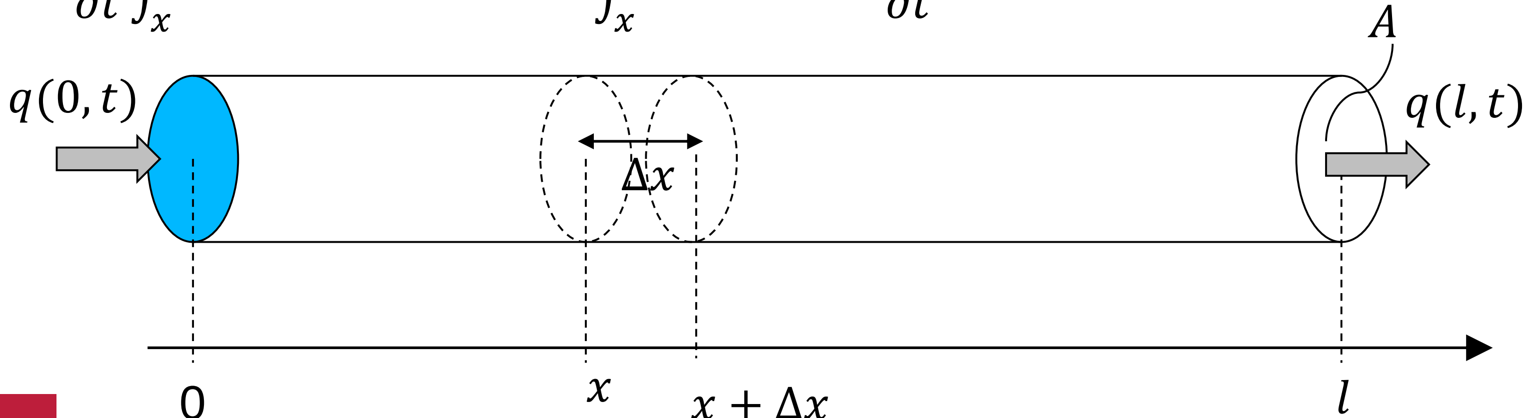


$$E_{tot} = E_0 + \int_x^{x+\Delta x} A\rho c(u(s,t) - T_0))ds =$$

$$E_0 + \int_x^{x+\Delta x} A\rho c u(s,t)ds - \underbrace{\int_x^{x+\Delta x} A\rho c T_0 ds}_{A\rho c T_0 ds}$$

But I'm only interested in:

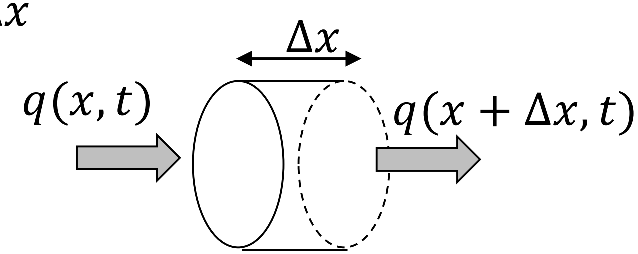
$$\frac{\partial}{\partial t}\int_x^{x+\Delta x} A\rho c u(s,t)ds = \int_x^{x+\Delta x} A\rho c \frac{\partial u(s,t)}{\partial t} ds$$



Introductory example: heat flow in a bar

Change of heat energy by time in the section of length Δx

$$\frac{\partial}{\partial t}\int_x^{x+\Delta x} A\rho c u(s,t)ds = \int_x^{x+\Delta x} A\rho c\frac{\partial u(s,t)}{\partial t}ds$$



Change of energy by time in the section of length Δx from heat flux:

$q(x,t)\left[\frac{J}{m^2 s}\right]$: heat flux

$$Aq(x+\Delta x,t) - Aq(x,t) = \int_x^{x+\Delta x} A\frac{\partial q(s,t)}{\partial x}ds$$

Fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

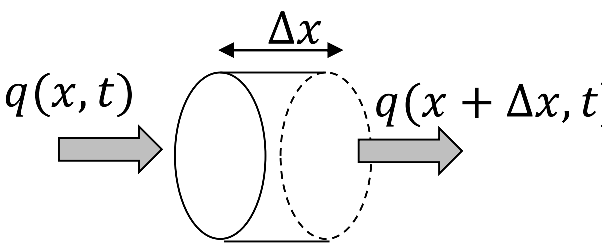
Conservation of energy:

$$\int_x^{x+\Delta x} A\frac{\partial q(s,t)}{\partial x}ds = -\int_x^{x+\Delta x} A\rho c\frac{\partial u(s,t)}{\partial t}ds$$

$$\Rightarrow \int_x^{x+\Delta x} A\frac{\partial q(s,t)}{\partial x} + A\rho c\frac{\partial u(s,t)}{\partial t}ds = 0$$

Introductory example: heat flow in a bar

$$\int_x^{x+\Delta x} A\frac{\partial q(s,t)}{\partial x} + A\rho c\frac{\partial u(s,t)}{\partial t}\, ds = 0$$



$$\frac{\partial q(x,t)}{\partial x} + \rho c\frac{\partial u(x,t)}{\partial t} = 0 \qquad 0 < x < l$$

$\frac{\partial u(x,t)}{\partial x}$: change of temperature with increasing x

Assumption: Fourier's law of heat conduction: $q(x,t) = -\kappa\frac{\partial u(x,t)}{\partial x}$

κ: thermal conductivity

Heat equation $$\boxed{\rho c\frac{\partial u(x,t)}{\partial t} - \kappa\frac{\partial^2 u(x,t)}{\partial x^2} = 0}$$ (for homogenous material properties)

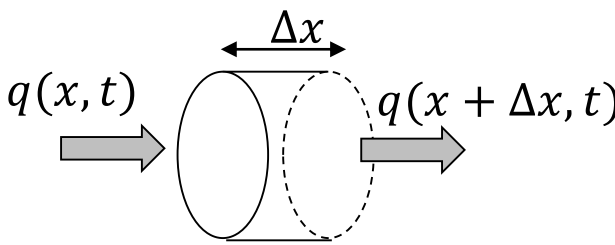
$$\rho(x)c(x)\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\kappa(x)\frac{\partial u(x,t)}{\partial x} = 0$$ (for inhomogenous material properties)

Introductory example: heat flow in a bar

The heat equation with source or sink (inhomogenous heat equation)

$f(s,t)$: *heat energy per unit time, per unit volume*

$$\int_x^{x+\Delta x} A\frac{\partial q(s,t)}{\partial x} + A\rho c\frac{\partial u(s,t)}{\partial t}\,ds = \int_x^{x+\Delta x} Af(s,t)\,ds$$



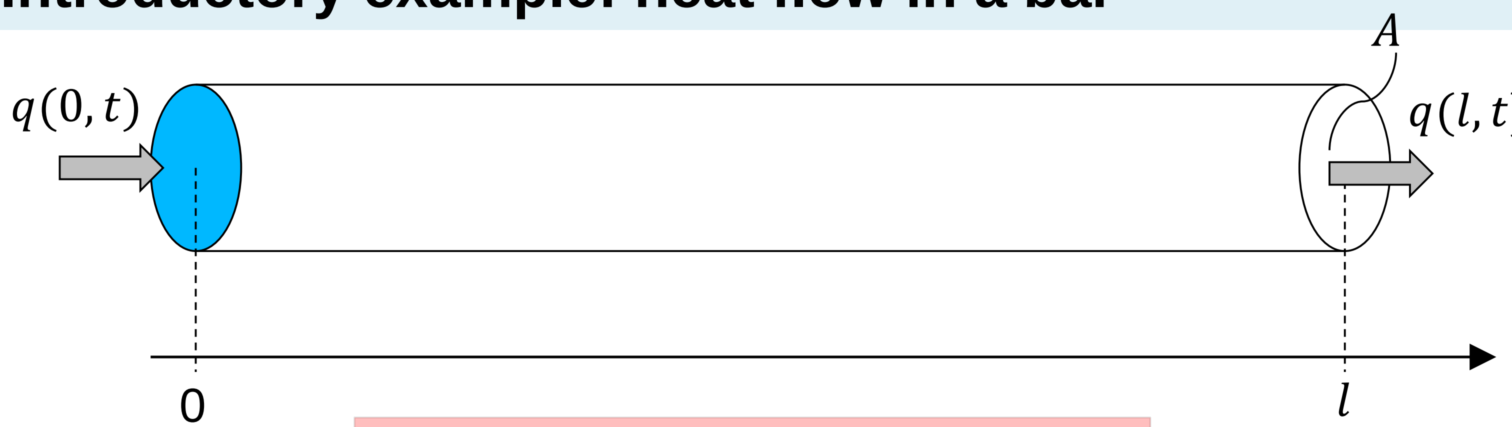
$$\frac{\partial q(x,t)}{\partial x} + \rho c\frac{\partial u(x,t)}{\partial t} = f(x,t) \quad 0 < x < l$$

Heat equation

$$\boxed{\rho c\frac{\partial u(x,t)}{\partial t} - \kappa\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)}$$ (for homogenous material properties)

$$\rho(x)c(x)\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\kappa(x)\frac{\partial u(x,t)}{\partial x} = f(x,t)$$ (for inhomogenous material properties)

Introductory example: heat flow in a bar



$$\rho c \frac{\partial u(x,t)}{\partial t} - \kappa \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)$$

Boundary conditions:

a) Perfect isolation at the end (flux across the boundaries is zero):

$$-\kappa \frac{\partial u(0,t)}{\partial x} = 0 \qquad -\kappa \frac{\partial u(l,t)}{\partial x} = 0 \qquad \forall t$$

b) Perfect thermal contact:

$$u(0,t) = 0 \qquad u(l,t) = 0 \qquad \forall t$$

Examples of PDEs and their classification

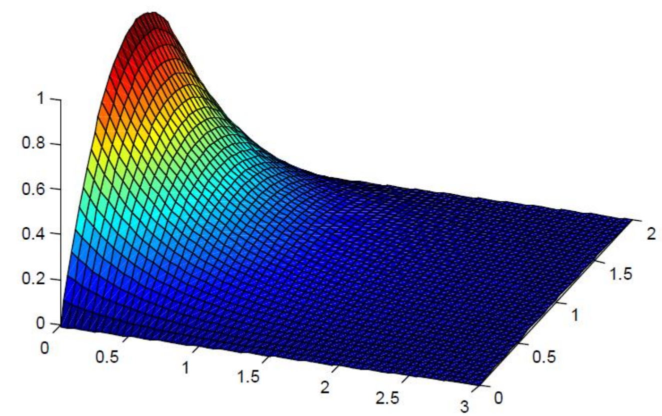
Transport equation: $u_t + cu_x = 0$

Heat equation: $\boxed{\rho c u_t - \kappa u_{xx} = f}$ $\rho(x)c(x)u_t - \frac{\partial}{\partial x}(\kappa(x)u_x) = f \Rightarrow \rho c u_t - \kappa \Delta u = f$
(Diffusion/membrane equation/
electrical conduction problem $\rightarrow$ temperature: electric potential, heat flux: electric current)

Steady state of the heat equation: $\lim_{t\to\infty} \frac{\partial}{\partial t} u = 0$

$\boxed{-\kappa u_{xx} = f}$ $\Rightarrow$ $-\kappa \Delta u = f$ $-\kappa u_{xx} = 0$ $\Rightarrow$ $-\Delta u = 0$ (Laplace equation)

(Poisson's equation)



Hanging bar: $\boxed{\rho u_{tt} - k u_{xx} = f}$
(the wawe equation : vibrating string)

Steady state of the hanging bar: $-k u_{xx} = f$

Examples of PDEs and their classification

Gauss's law: $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ ε_0: permittivity of free space, E electrical field

Gauss's law for magnetism: $\nabla \cdot B = 0$ B: magnetic field

Faraday' law of induction: $\nabla \times B = -B_t$ B_t: magnetic flux

Burger's equation: $u_t + uu_x = \nu u_{xx}$ ν viscosity/diffusion coefficient

(fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow)

Inviscid Burger's equation: $u_t + uu_x = 0$

Plate equation: $u_{xxxx} + u_{xxyy} + u_{yyyy} = 0$

Analytical solution of the 2D Laplace equation

$$
\begin{array}{llll}
(a) & \text{The wave equation:} & u_{tt} - c^2\, u_{xx} = 0, & \textit{hyperbolic}, \\
(b) & \text{The heat equation:} & u_t - \gamma\, u_{xx} = 0, & \textit{parabolic}, \\
(c) & \text{Laplace's equation:} & u_{xx} + u_{yy} = 0, & \textit{elliptic}.
\end{array}
$$

$$
\begin{aligned}
\Delta u = u_{xx} + u_{yy} &= 0, \qquad (x,y) \in \Omega \\
u(0,y) = u(1,y) &= 0, \qquad y \in [0,1] \\
u(x,0) &= 0, \qquad x \in [0,1] \\
u(x,1) &= -x^2 + x, \qquad x \in (0,1)
\end{aligned}
$$

Analytical solution of the 2D Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0, \qquad (x,y) \in \Omega$$

- Solve by seperation of variables

- ansatz + seperation:

$$u(x,y) = v(x)\, w(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)\, w(y), \qquad \frac{\partial^2 u}{\partial y^2} = v(x)\, w''(y)$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v''(x)\, w(y) + v(x)\, w''(y) = 0$$

$$v''(x)\, w(y) = -\, v(x)\, w''(y)$$

$$\frac{v''(x)}{v(x)} = -\,\frac{w''(y)}{w(y)} = \lambda$$

Analytical solution of the 2D Laplace equation

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda$$ seperation constant

$v'' - \lambda v = 0,$ $w'' + \lambda w = 0$ eigenvalue-eigenfunction problem: $Lu = \lambda u$

$w'' + \lambda w = 0$

a) Let's assume $\boldsymbol{\lambda = 0}$

$$\omega''(y) = 0$$

Any linear equation would satisfy this equation:

$$\omega(y) = C_1 y + C_2$$

b) Let's assume $\boldsymbol{\lambda \neq 0}$

Assuming the solution in the form:

$$w(y) = e^{\sigma y}$$

$\sigma^2 e^{\sigma y} + \lambda e^{\sigma y} = 0$ *(characterisitic equation)*

$\sigma^2 = -\lambda$ $\quad\Rightarrow\quad$ $\sigma = \pm\sqrt{-\lambda}$

The solution is the linear combination of the two possible solutions

$$w(y) = C_3 e^{\sqrt{-\lambda} y} + C_4 e^{-\sqrt{-\lambda} y}$$

C_3, C_4 are arbitrary constants

Eigenvalue problem – Poisson equation with hom. D.B.C

Let's assume $\lambda > 0$

$\sigma = \pm i\sqrt{\lambda}$

The solution:

$$w(y) = C_3 e^{i\sqrt{\lambda}y} + C_4 e^{-i\sqrt{\lambda}y} =$$

$$C_3\{\cos(\sqrt{\lambda}y) + i\sin(\sqrt{\lambda}y)\} + C_4\{\cos(\sqrt{\lambda}y) - i\sin(\sqrt{\lambda}y)\} =$$

$$\underbrace{(C_3 + C_4)}_{=A}\cos(\sqrt{\lambda}y) + \underbrace{(C_3 - C_4)i}_{=B}\sin(\sqrt{\lambda}y)$$

where A, B are new constants with

$C_3 + C_4 = \mathrm{A}$ $\quad$ $(C_3 - C_4)i = \mathrm{B}$

$w(y) = A\cos(\sqrt{\lambda}y) + B\sin(\sqrt{\lambda}y)$

Introducing : $\lambda = \omega^2$

$$\boxed{w(y) = A\cos(\omega y) + B\sin(\omega y)}$$

Let's assume $\lambda < 0$

$w(y) = C_3 e^{\sqrt{-\lambda}y} + C_4 e^{-\sqrt{-\lambda}y}$

Introducing : $\lambda = -\omega^2$

$$\boxed{w(y) = C_3 e^{\omega y} + C_4 e^{-\omega y}}$$

λ: separation constant, **eigenvalue** of $\mathrm{L} = \frac{\partial}{\partial x^2}$

ω: **eigenfrequency** of $\mathrm{L} = \frac{\partial}{\partial x^2}$

Analytical solution of the 2D Laplace equation

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda \quad \Rightarrow \quad \text{seperation constant}$$

$$v'' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x,y) = v(x)\,w(y)$
$\lambda = -\omega^2 < 0$	$\cos\omega x,\ \sin\omega x$	$e^{-\omega y},\ e^{\omega y},$	$e^{\omega y}\cos\omega x,\quad e^{\omega y}\sin\omega x,$ $e^{-\omega y}\cos\omega x,\quad e^{-\omega y}\sin\omega x$
$\lambda = 0$	$1,\ x$	$1,\ y$	$1,\ x,\ y,\ xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x},\ e^{\omega x}$	$\cos\omega y,\ \sin\omega y$	$e^{\omega x}\cos\omega y,\quad e^{\omega x}\sin\omega y,$ $e^{-\omega x}\cos\omega y,\quad e^{-\omega x}\sin\omega y$

Analytical solution of the 2D Laplace equation

$$v'' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x,y) = v(x)\,w(y)$
$\lambda = -\omega^2 < 0$	$\cos\omega x,\ \sin\omega x$	$e^{-\omega y},\ e^{\omega y},$	$e^{\omega y}\cos\omega x,\quad e^{\omega y}\sin\omega x,$ $e^{-\omega y}\cos\omega x,\ e^{-\omega y}\sin\omega x$
$\lambda = 0$	$1,\ x$	$1,\ y$	$1,\ x,\ y,\ xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x},\ e^{\omega x}$	$\cos\omega y,\ \sin\omega y$	$e^{\omega x}\cos\omega y,\quad e^{\omega x}\sin\omega y,$ $e^{-\omega x}\cos\omega y,\ e^{-\omega x}\sin\omega y$

$$\boxed{u(0,y) = u(1,y) = 0, \qquad y \in [0,1]}$$

$$u(x,y) = v(x)\,w(y)$$

$$\Rightarrow \quad u(0,y) = v(0) = v(1) = 0$$

Analytical solution of the 2D Laplace equation

$$v(x) = \begin{cases} A\cos\omega x + B\sin\omega x & \lambda < 0 \\ C_1 x + C_2 b & \lambda = 0 \\ C_3 e^{-\omega x} + C_4 e^{-\omega x} & \lambda > 0 \end{cases}$$

BC: $v(0) = 0$ $\Rightarrow$ $v(0) = \begin{cases} A\cos\omega x + 0 \\ 0 + C_2 \\ C_3 e^{-\omega\cdot 0} + C_4 e^{-\omega\cdot 0} \end{cases} = 0 \longrightarrow$ $A = 0$, $C_2 = 0$, $-C_3 = C_4 := \frac{C}{2}$

$$v(x) = \begin{cases} B\sin\omega x & \lambda < 0 \\ C_1 x & \lambda = 0 \\ C\dfrac{e^{\omega x} - e^{-\omega x}}{2} = C\sinh x & \lambda > 0 \end{cases}$$

BC: $v(1) = 0$ $\Rightarrow$ $v(1) = \begin{cases} B\sin\omega \\ C_1 \\ C\underbrace{\sinh(1)}_{\neq 0} \end{cases} = 0 \longrightarrow$ $\omega_k = k\pi, k = 1, 2, 3, ..$; $C_1 = 0$, $C = 0$ $\Big\}$ leads to $v(x) = 0$ (trivial solution)

Analytical solution of the 2D Laplace equation

$$v'' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x,y) = v(x)\,w(y)$
$\lambda = -\omega^2 < 0$	$\cancel{\cos\omega x}$, $\sin\omega x$	$e^{-\omega y}$, $e^{\omega y}$,	$\cancel{e^{\omega y}\cos\omega x}$, $e^{\omega y}\sin\omega x$, $\cancel{e^{-\omega y}\cos\omega x}$, $e^{-\omega y}\sin\omega x$
$\lambda = 0$	$1,\ x$	$1,\ y$	$1,\ x,\ y,\ xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x}$, $e^{\omega x}$	$\cos\omega y$, $\sin\omega y$	$e^{\omega x}\cos\omega y$, $e^{\omega x}\sin\omega y$, $e^{-\omega x}\cos\omega y$, $e^{-\omega x}\sin\omega y$

$$v_k(x) = B_k \sin(\omega_k x) \qquad \omega_k = k\pi, k = 1, 2, 3, ..$$

Analytical solution of the 2D Laplace equation

$$v'' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x,y) = v(x)\, w(y)$
$\lambda = -\omega^2 < 0$	$\cancel{\cos \omega x}$, $\sin \omega x$	$e^{-\omega y}$, $e^{\omega y}$,	$\cancel{e^{\omega y} \cos \omega x}$, $e^{\omega y} \sin \omega x$, $\cancel{e^{-\omega y} \cos \omega x}$, $e^{-\omega y} \sin \omega x$

$$u(0,y) = u(1,y) = 0, \qquad y \in [0,1] \ \checkmark$$

$$\boxed{u(x,0) = 0, \qquad x \in [0,1]}$$

$$\Rightarrow w(0) = 0 \quad w_k(0) = C_5 e^{-\omega_k \cdot 0} + C_6 e^{-\omega_k \cdot 0}$$

$$w_k(0) = C_{k5} + C_{k6} = 0 \Rightarrow -C_{k5} = C_{k6} := \frac{D_k}{2}$$

$$w_k(y) = D_k \underbrace{\frac{e^{\omega_k y} - e^{-\omega_k y}}{2}}_{\sinh(\omega_k y)} = D_k \sinh(\omega_k y)$$

Analytical solution of the 2D Laplace equation

$$u_k(x,y) = v_k(x)w_k(y) = \sin(\omega_k \mathrm{x})\,\sinh(\omega_k y)$$

$$\boxed{u(x,y) = \sum_{k=1}^{\infty} a_k v_k(x) w_k(y) = \sum_{k=1}^{\infty} a_k \sin(\omega_k \mathrm{x}) \sinh(\omega_k y)}$$

Analytical solution of the 2D Laplace equation

$$u(0,y) = u(1,y) = 0, \qquad y \in [0,1] \checkmark$$

$$u(x,0) = 0, \qquad x \in [0,1] \checkmark$$

$$\boxed{u(x,1) = -x^2 + x, \qquad x \in (0,1)}$$

$$u(x,y) = \sum_{k=1}^{\infty} \boxed{a_k} \sin(\omega_k x) \sinh(\omega_k y) \qquad \omega_k = k\,\pi$$

$$u(x,1) = \sum_{k=1}^{\infty} a_k \sin(\omega_k x) \sinh(\omega_k) = -x^2 + x$$

write the Fourier series of the initial condition : $-x^2 + x = \sum_{k=1}^{\infty} g_k \sin(\omega_k x)$ $\Rightarrow$ g_k

$$\Downarrow$$

$$\boxed{a_k} = \frac{g_k}{\sinh(\omega_k)}$$