Technische Universität Braunschweig

Introduction to PDEs and Numerical Methods
Lecture 13.
The finite element method: assembling the matrices, isoparametric mapping, FEM in higher dimension

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RECAP: How to solve PDE with FEM with nodal basis, piecewise linear shape functions

Finite Element method with piecewise linear functions in 1D, hom DBC

1) Weak formulation of the PDE, definition of the ‚energy' inner product (the bilinear functional, a) and and the linear functional (F)

$$a(u, v) = F(v)$$

2) Define approximating subspace by definition of a mesh (nodes 0,1,..N, with coordinates, elements) and setup the hat functions on them

$$\Phi_i(x) = \mathrm{N_i(x)} = \begin{cases} \dfrac{x - x_{i-1}}{l} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} + x}{l} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad i = 1..N-1$$

3) Compute the elements of the stiffness matrix (Grammian) – evaluation of integrals

$$K_{ij} = a\big(N_i(x), N_j(x)\big) = \big\langle N_i(x), N_j(x) \big\rangle_E \quad i, j = 1..N-1$$

4) Compute the elements of the vector of the right hand side – evaluation of integrals

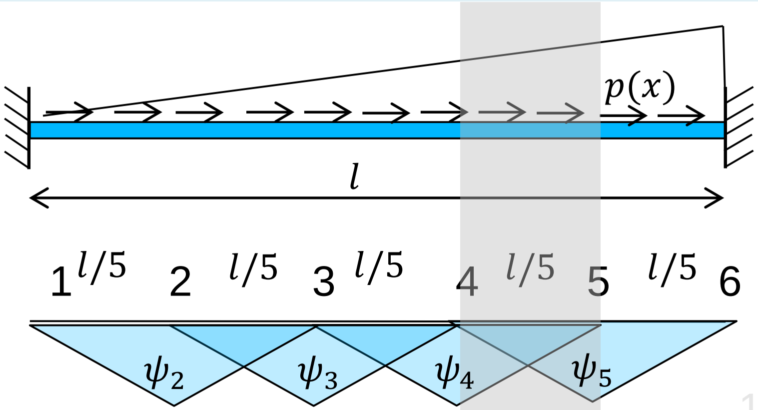
$$f_i = F(N_i),\ i = 1..N-1$$

5) Solve the system of equations: $\mathbf{Ku} = \mathbf{f}$

for $\mathbf{u}$, which gives the solution at the nodes.

The solution in between the nodes can be calculated from: $u(\mathrm{x}) \approx \sum_{i=1}^{N} u_i\, N_i(\mathrm{x})$

Recap:
1D Example with linear nodal basis



instead:
Compute stiffness matrix elementwisely and then assemble

$$K_4{}^e(1,1) = EA\int_{\Omega_4} \frac{\partial\psi_4(x)}{\partial x}\frac{\partial\psi_4(x)}{\partial x}dx$$

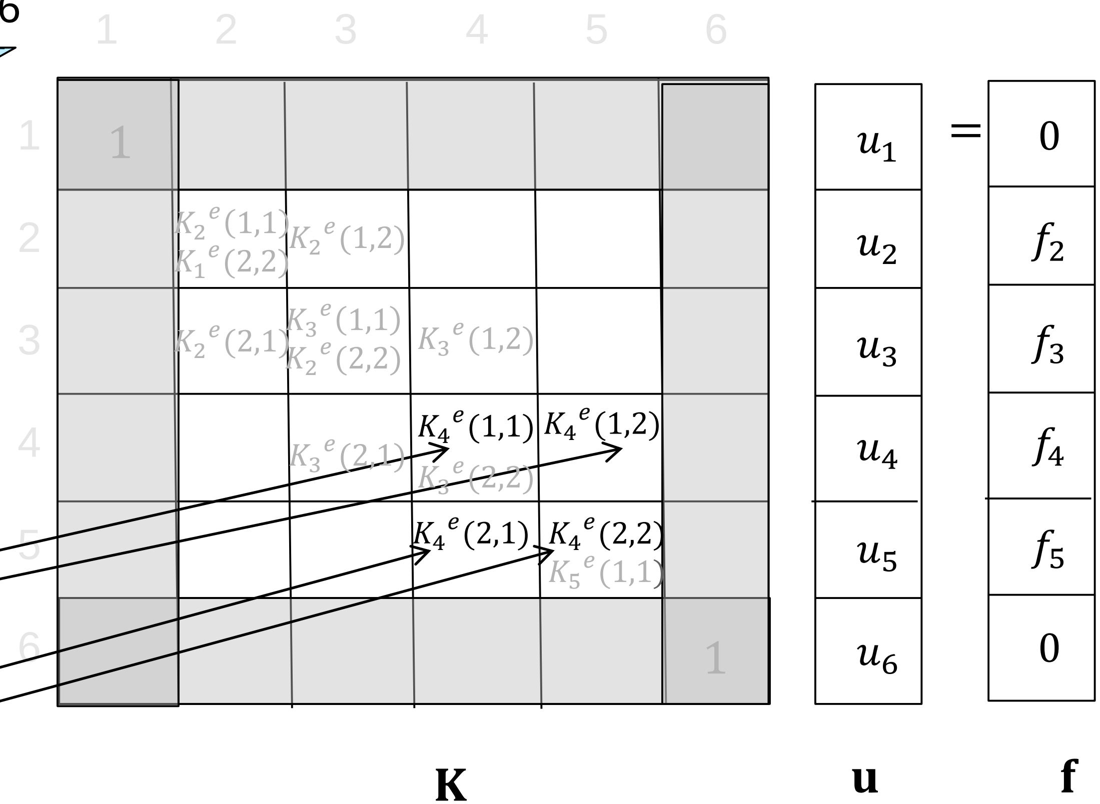
$$K_4{}^e(1,2) = EA\int_{\Omega_4} \frac{\partial\psi_4(x)}{\partial x}\frac{\partial\psi_5(x)}{\partial x}dx$$

$$K_4{}^e(2,1) = EA\int_{\Omega_4} \frac{\partial\psi_5(x)}{\partial x}\frac{\partial\psi_4(x)}{\partial x}dx$$

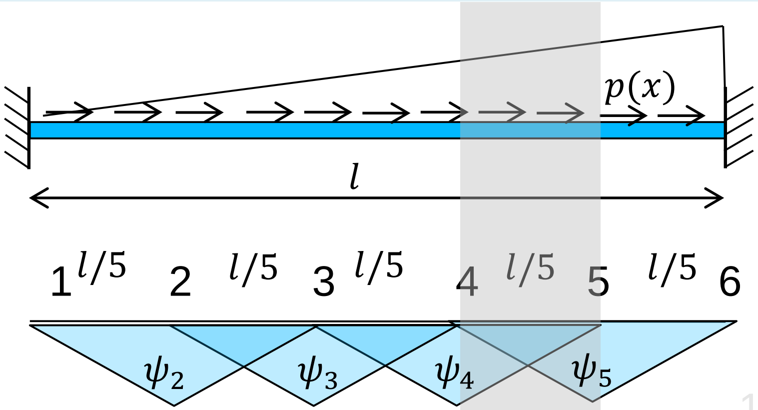
$$K_4{}^e(2,2) = EA\int_{\Omega_4} \frac{\partial\psi_5(x)}{\partial x}\frac{\partial\psi_5(x)}{\partial x}dx$$

$$K_4{}^e = \begin{array}{c|cc} & 4 & 5 \\ \hline 4 & K_4{}^e(1,1) & K_4{}^e(1,2) \\ 5 & K_4{}^e(2,1) & K_4{}^e(2,2) \end{array}$$

Global stiffness matrix



Recap:
1D Example with linear nodal basis



$$f_4{}^e(1) = \int_{\Omega_4} p(x)\psi_4(x)dx$$

$$f_4{}^e(2) = \int_{\Omega_4} p(x)\psi_5(x)dx$$

instead:
Compute stiffness matrix elementwisely and then assemble

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & & & & & \\ 2 & & K_2{}^e(1,1)\; K_1{}^e(2,2) & K_2{}^e(1,2) & & & \\ 3 & & K_2{}^e(2,1) & K_3{}^e(1,1)\; K_2{}^e(2,2) & K_3{}^e(1,2) & & \\ 4 & & & K_3{}^e(2,1) & K_4{}^e(1,1)\; K_3{}^e(2,2) & K_4{}^e(1,2) & \\ 5 & & & & K_4{}^e(2,1) & K_4{}^e(2,2)\; K_5{}^e(1,1) & \\ 6 & & & & & & 1 \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ f_1{}^e(2)\; f_2{}^e(1) \\ f_2{}^e(2)\; f_3{}^e(1) \\ f_3{}^e(2)\; f_4{}^e(1) \\ f_4{}^e(2)\; f_5{}^e(1) \\ 0 \end{bmatrix}$$

$$\mathbf{K} \qquad \mathbf{u} \qquad \mathbf{f}$$

$$f_4{}^e = \begin{array}{c} 4 \\ 5 \end{array} \begin{bmatrix} f_4{}^e(1) \\ f_4{}^e(2) \end{bmatrix}$$

The same but elementwisely: How to solve PDE with FEM with nodal basis, piecewise linear shape functions

Finite Element method with piecewise linear functions in 1D, hom DBC

1) Weak formulation of the PDE, definition of the ‚energy' inner product (the bilinear functional, a) and and the linear functional (l)

$$a(u, v) = l(v)$$

2) a.) Define reference element, define maping between global and local coordinate systems

$$\xi(x) \Longleftrightarrow x(\xi)$$

b.) Define reference linear shape functions $\mathrm{N}_1(\xi) = 1 - \xi$ $\quad \mathrm{N}_2(\xi) = \xi$

3) Compute the ‚element stiffness' matrix – evaluation of integrals

$$K_{ij} = a\big(N_i(x), N_j(x)\big) = \langle N_i(x), N_j(x)\rangle_E \quad i, j = 1..2$$

$$K^e = \begin{bmatrix} K_4{}^e(1,1) & K_4{}^e(1,2) \\ K_4{}^e(2,1) & K_4{}^e(2,2) \end{bmatrix}$$

4) Compute the right hand side elementwisely $\big(f_{ie} = l(N_i)\big), i = 1,2$

$$f_4{}^e = \begin{bmatrix} f_4{}^e(1) \\ f_4{}^e(2) \end{bmatrix}$$

5) Compile, global stiffness' matrix

6) Solve the system of equations: $\mathbf{Ku} = \mathbf{f}$

for $\mathbf{u}$, which gives the solution at the nodes.

The solution in between the nodes can be calculated from:

$$u(\mathrm{x}) \approx \sum_{i=1}^{N} u_i \, N_i(\mathrm{x})$$

Local/global coordinate system 1D

Idea:
coordinate transformation to have unit length elements ➡ element stiffnes matrix is the same for each element

$$\xi = \frac{x - x_i}{x_{i+1} - x_i} \Longrightarrow \xi = [0,1] \qquad \frac{\partial \xi}{\partial x} = \frac{1}{l^e}$$

$$K_4{}^e(k,l) = EA \int_{\Omega_4} \frac{\partial \psi_4(x)}{x} \frac{\partial \psi_5(x)}{\partial x} dx \qquad \begin{array}{l} i,j \in [4,5] \\ k,l \in [1,2] \end{array}$$

$$K_4{}^e(k,l) = EA \int_{\Omega_4} \frac{\partial N_k(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\frac{1}{l^e}} \frac{\partial N_l(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\frac{1}{l^e}} dx = \frac{EA}{(l_4{}^e)^2} \int_{\Omega_4} \frac{\partial N_k(\xi)}{\partial \xi} \frac{\partial N_l(\xi)}{\partial \xi} dx$$

$$K_4{}^e(k,l) = \frac{EA}{(l^e)^2} \int_0^1 \frac{\partial N_k(\xi)}{\partial \xi} \frac{\partial N_l(\xi)}{\partial \xi} \underbrace{\left| \frac{dx(\xi)}{d\xi} \right|}_{l^e} d\xi = \frac{EA}{l^e} \int_0^1 \frac{\partial N_k(\xi)}{\partial \xi} \frac{\partial N_l(\xi)}{\partial \xi} d\xi$$

Local/global coordinate system 1D

$$f_4{}^e(l) = \begin{bmatrix} \int_{\Omega_4} p(x)\psi_4(x)dx \\ \int_{\Omega_4} p(x)\psi_5(x)dx \end{bmatrix} = \int_0^1 p(\xi)N_l(\xi)\left|\frac{dx(\xi)}{d\xi}\right| d\xi = l^{\,e}\int_0^1 p(\xi)N_l(\xi)d\xi$$

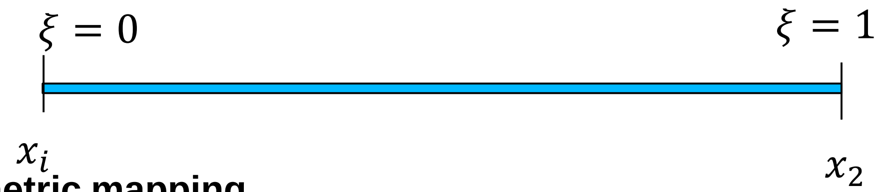
$$l \in [1,2]$$

Local/ coordinate system, isoparametric mapping 1D

$x \Rightarrow \xi = [0,1]$

local coordinate

global coordinate



coordinate transformation using the **ansatzfunctions** $\Rightarrow$ **isoparametric mapping**

functions of lower order: **subparametric**
functions of higher order: **superparametric**

Shape functions: $N_1(\xi) = 1 - \xi$

$$N_2(\xi) = \xi$$

Transformation from local to global coordinates:

$$x(\xi) = x_i N_1(\xi) + x_{i+1} N_2(\xi) = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} \Rightarrow$$

$$\frac{dx}{d\xi} = x_i \frac{dN_1(\xi)}{d\xi} + x_{i+1} \frac{dN_2(\xi)}{d\xi}$$

$$\frac{dx}{d\xi} = \begin{bmatrix} \frac{dN_1(\xi)}{d\xi} & \frac{dN_2(\xi)}{d\xi} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix}$$

Stiffness matrix with isoparametric elements:

$$K_4{}^e(k,l) = EA \int_{\Omega_4} \frac{\partial \psi_i(x)}{x} \frac{\partial \psi_j(x)}{\partial x} dx = EA \int_{\Omega_4} \frac{\partial N_k(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\left(\frac{dx}{d\xi}\right)^{-1}} \frac{\partial N_l(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\left(\frac{dx}{d\xi}\right)^{-1}} dx$$

$$i,j \in [4,5]$$

$$k,l \in [1,2]$$

$$K_4{}^e(k,l) = EA \int_0^1 \frac{\partial N_k(\xi)}{\partial \xi} \left(\frac{dx}{d\xi}\right)^{-1} \frac{\partial N_l(\xi)}{\partial \xi} \left(\frac{dx}{d\xi}\right)^{-1} \left| \frac{dx(\xi)}{d\xi} \right| d\xi$$

The same but differently: Local/ coordinate system, isoparametric mapping 1D

$$[-1\ldots 1] \;\Rightarrow\; [x_i \ldots x_{i+1}]$$

coordinate transformation
using the **ansatzfunctions** ⇒ **isoparametric mapping**

functions of lower order: **subparametric**
functions of higher order: **superparametric**

Basis functions:

$$N_1(\xi) = \frac{1}{2}(1-\xi)$$
$$N_2(\xi) = \frac{1}{2}(1+\xi)$$

Transformation from local to global coordinates:

$$x_{glob}(\xi) = x_i N_1(\xi) + x_{i+1} N_2(\xi) = [N_1(\xi) \quad N_2(\xi)] \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} \Rightarrow \frac{\partial}{\partial \xi} x_{glob}(\xi) = \begin{bmatrix} \underbrace{\frac{\partial}{\partial \xi} N_1(\xi)}_{-1/2} & \underbrace{\frac{\partial}{\partial \xi} N_2(\xi)}_{+1/2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \frac{1}{2l_e}$$

Stiffness matrix with isoparametric elements:

$$K_{ij} = EA \int_{-1}^{1} \left(\underbrace{\left(\frac{\partial x_{glob}}{\partial \xi} \right)^{-1}}_{2l_e} \underbrace{\frac{\partial N_j}{\partial \xi}}_{\pm 1/2} \cdot \underbrace{\left(\frac{\partial x_{glob}}{\partial \xi} \right)^{-1}}_{2l_e} \underbrace{\frac{\partial N_i}{\partial \xi}}_{\pm 1/2} \right) \underbrace{\left| \frac{dx_{glob}(\xi)}{d\xi} \right|}_{1/2l_e} d\xi \qquad i, j \in [1,2]$$

$$\Rightarrow \boxed{K_4{}^e = \frac{EA}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

local coordinate

global coordinate

$\xi = -1$ $\quad \xi = 0$ $\quad \xi = 1$

x_i $\quad x_2$

FROM STRONG FORM TO WEAK FORM in higher dimension
Steps of formulating the weak form (recipe)

$$\boxed{Lu(\mathbf{x}) = f(\mathbf{x})}$$

1.) Multiply by test function φ and integrate

$$\langle Lu, v\rangle - \langle p, v\rangle = 0 \qquad \forall v \in V$$

$$\int Lu(\mathbf{x})v(\mathbf{x})d\mathbf{x} - \int f(\mathbf{x})v(\mathbf{x})d\mathbf{x} = 0$$

2.) Reduce order of $\langle Lu, \varphi\rangle$ by using Green's theoreem (generalized integration by parts)

$$\int_\Omega v\Delta u d\Omega = -\int_\Omega \nabla v \cdot \nabla u d\Omega + \int_{\partial\Omega} v\frac{\partial u}{\partial n}d\Gamma$$

3.) Apply boundary conditions

FROM STRONG FORM TO WEAK FORM
Recap: differential operators (grad, div, curl), Green's theorem

Green's identity similar to integration by part in multiple dimensions

1) rewrite equation with the product rule in multiple dimensions

$$\nabla \cdot (v\, \nabla u) = \nabla v \cdot \nabla u + v\, \Delta u$$

2) integrate both sides over the domain Ω (bounded by $\partial\Omega$)

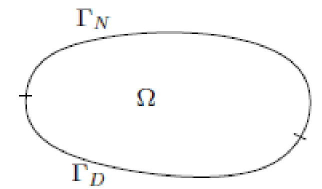
$$\int_\Omega \nabla \cdot (v\, \nabla u)\, d\Omega = \int_\Omega \nabla v \cdot \nabla u\, d\Omega + \int_\Omega v\, \Delta u\, d\Omega$$

3) apply divergence theorem

$$\int_\Omega \nabla \cdot (v\, \nabla u)\, d\Omega = \int_\Omega div(v\, \nabla u)\, d\Omega = \int_{\partial\Omega} (v\, \nabla u) \cdot n\, d\partial\Omega$$

$$\boxed{-\int_\Omega v\, \Delta u\, d\Omega = \int_\Omega \nabla v \cdot \nabla u\, d\Omega - \int_{\partial\Omega} (v\, \nabla u) \cdot n\, d\partial\Omega}$$

Recap: Multidimensional stationary heat equation with inhomogeneous Dirichlet and Neumann BC.



Strong form: $\boxed{Lu(\mathbf{x}) = f(\mathbf{x})}$

Example:

$$\begin{cases} -\Delta u(\mathbf{x}) = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = h & \text{on } \Gamma_N, \end{cases}$$

convert to homogeneous problem:

$$u = \omega + \hat{u}$$

ω:known function,$\omega = g$ on Γ_D

$\hat{u}$:new function that we look for

$$V = \{v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \Gamma_D\}$$

1.) Multiply by test function v and integrate

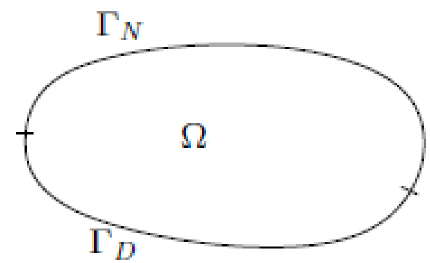
$$\int -\Delta u(\mathbf{x}) v(\mathbf{x}) d\Omega - \int f(\mathbf{x}) v(\mathbf{x}) d\Omega = 0$$

2.) Reduce order of $\langle Lu, v\rangle$ by using divergence theorem

$$\int -\Delta u(\mathbf{x}) v(\mathbf{x}) d\Omega = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial n} v(\mathbf{x}) d\Gamma$$

Recap: Multidimensional stationary heat equation with inhomogeneous Dirichlet and Neumann BC.

$$\int -\Delta u(\mathbf{x})v(\mathbf{x})d\Omega = \int_\Omega \nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\Omega - \int_{\partial\Omega}\frac{\partial u}{\partial n}v(\mathbf{x})d\Gamma$$



3.) Apply boundary conditions

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}v(\mathbf{x})d\Gamma = \int_{\Gamma_N}\underbrace{\frac{\partial u}{\partial n}}_{h}v(\mathbf{x})d\Gamma + \int_{\Gamma_D}\frac{\partial u}{\partial n}\underbrace{v(\mathbf{x})}_{0}d\Gamma = \int_{\Gamma_N}hv(\mathbf{x})d\Gamma$$

$$\int_\Omega \nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\Omega = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\Omega + \int_{\Gamma_N}hv(\mathbf{x})d\Gamma$$

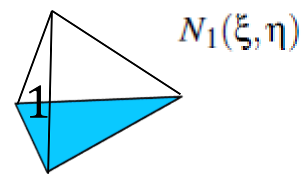
$$\int_\Omega \nabla\left(\omega(\mathbf{x})+\hat{u}(\mathbf{x})\right)\cdot\nabla v(\mathbf{x})d\Omega = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\Omega + \int_{\Gamma_N}hv(\mathbf{x})d\Gamma$$

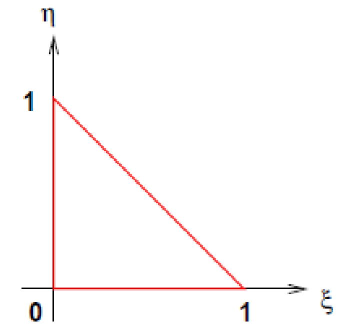
$$\int_\Omega \nabla\hat{u}(\mathbf{x})\cdot\nabla v(\mathbf{x})d\Omega = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\Omega + \underbrace{\int_{\Gamma_N}hv(\mathbf{x})d\Gamma}_{\text{from natural/Neumann BC}} - \underbrace{\int_\Omega \nabla\omega(\mathbf{x})\cdot\nabla v(\mathbf{x})d\Omega}_{\text{from essential/Dirichlet BC}}$$

Isoparametric linear mapping 2D triangular elements

Basis functions:

$$\begin{aligned} N_1(\xi,\eta) &= \xi \\ N_2(\xi,\eta) &= \eta \\ N_3(\xi,\eta) &= 1-\xi-\eta \end{aligned}$$





Transformation from local to global coordinates:

$$\begin{pmatrix} x_{glob} \\ y_{glob} \end{pmatrix}(\xi,\eta) = N_1(\xi,\eta)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + N_2(\xi,\eta)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + N_3(\xi,\eta)\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\begin{bmatrix} x_{glob}(\xi,\eta) \\ y_{glob}(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) & \\ & N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

Stiffness matrix:

$$\mathbf{K}_{ij} = \int_{\Omega_{elm}} \begin{pmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} d\Omega_{elm} \qquad i,j \in [1,2,3]$$

Isoparametric linear mapping 2D triangular elements

Stiffness matrix:

$$\mathbf{K}_{ij} = \int_{\Omega_{elm}} \begin{pmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} d\Omega_{elm} \qquad i,j \in [1,2,3]$$

Stiffness matrix with local coordinates:

$$\boldsymbol{K}_{ij} = \int_0^1 \int_0^{1-\eta} \boldsymbol{J}^{-T} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \boldsymbol{J}^{-T} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} |\boldsymbol{J}| d\xi d\eta$$

substitution rule

determinant should not be negative or zero!

$$i,j \in [1,2,3]$$

where:

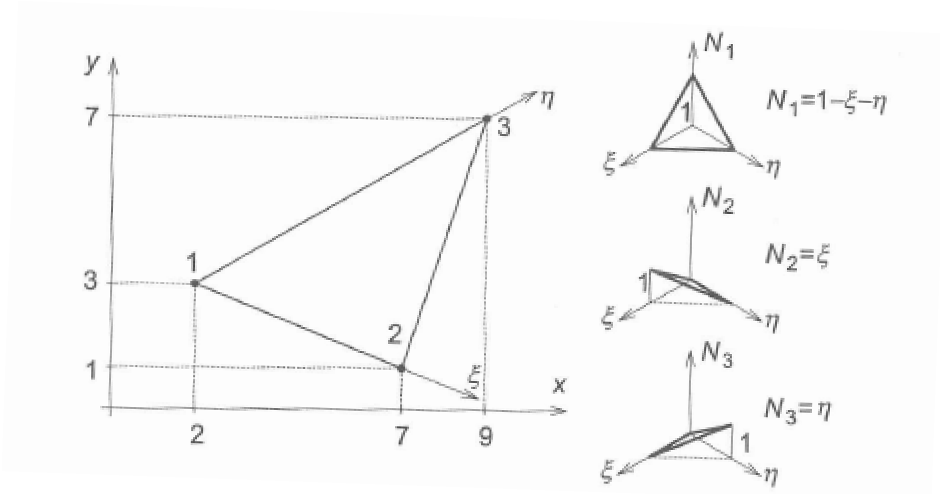
$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_{glob}}{\partial \xi} & \frac{\partial x_{glob}}{\partial \eta} \\ \frac{\partial y_{glob}}{\partial \xi} & \frac{\partial y_{glob}}{\partial \eta} \end{pmatrix}$$

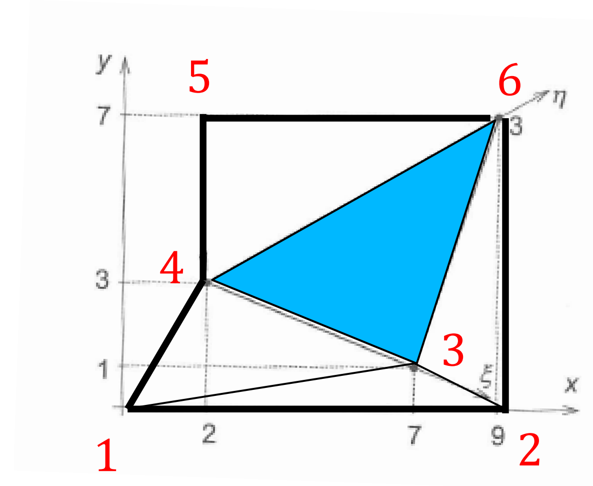
$$\mathbf{J} = \begin{pmatrix} \sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \xi} x_i & \sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \eta} x_i \\ \sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \xi} y_i & \sum_{i=1}^{3} \frac{\partial N_i(\xi,\eta)}{\partial \eta} y_i \end{pmatrix}$$

$$\begin{aligned} \frac{\partial N}{\partial \xi} &= \frac{\partial N}{\partial x}\frac{\partial x_{glob}}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y_{glob}}{\partial \xi} \\ \frac{\partial N}{\partial \eta} &= \frac{\partial N}{\partial x}\frac{\partial x_{glob}}{\partial \eta} + \frac{\partial N}{\partial y}\frac{\partial y_{glob}}{\partial \eta} \end{aligned} \Rightarrow \begin{pmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{pmatrix} = \mathbf{J}^T \begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix}$$

$$\mathbf{J}^{-T} \begin{pmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix}$$

Isoparametric linear mapping 2D triangular elements, example



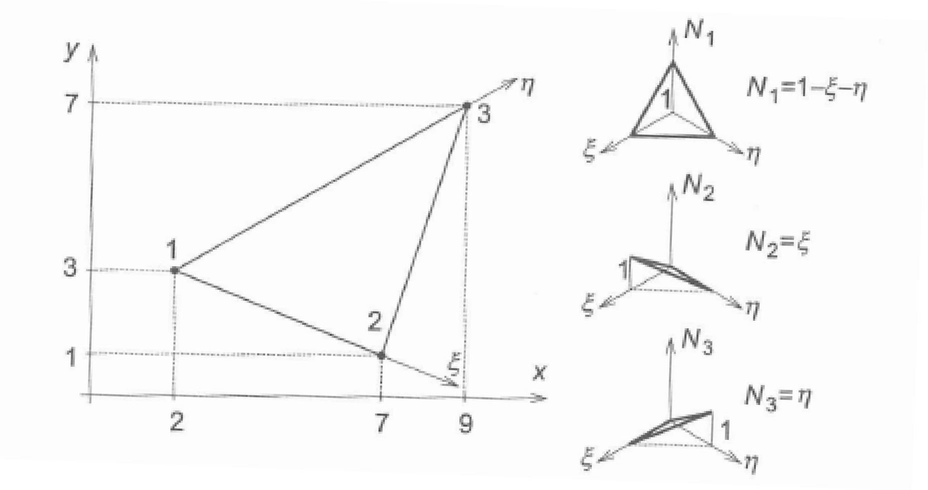


Transformation from local to global coordinates (isoparametric mapping):

$$\begin{bmatrix} x_{glob}(\xi,\eta) \\ y_{glob}(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) & \\ & N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix} = \begin{bmatrix} (1-\xi-\eta) & & \xi & & \eta & \\ & (1-\xi-\eta) & & \xi & & \eta \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \\ 1 \\ 9 \\ 7 \end{bmatrix}$$

Local/ coordinate system, isoparametric mapping 2D triangular elements, example



Stiffness matrix with local coordinates:

$$\boldsymbol{K}_{ij} = \int_0^1 \int_0^{1-\eta} \boldsymbol{J}^{-\boldsymbol{T}} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \boldsymbol{J}^{-\boldsymbol{T}} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} |\boldsymbol{J}| d\xi d\eta \qquad i,j \in [1,2,3]$$

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_{glob}}{\partial \xi} & \frac{\partial x_{glob}}{\partial \eta} \\ \frac{\partial y_{glob}}{\partial \xi} & \frac{\partial y_{glob}}{\partial \eta} \end{pmatrix}$$

Local/ coordinate system, isoparametric mapping 2D triangular elements, example

$$\boldsymbol{J}^{\boldsymbol{T}} = \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} 1-\xi-\eta & \xi & \eta \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 1 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 1 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 7 & 4 \end{bmatrix}.$$

$$\boldsymbol{J} = \begin{bmatrix} \mathbf{5} & \mathbf{7} \\ \mathbf{-2} & \mathbf{4} \end{bmatrix} \qquad |\boldsymbol{J}^{\boldsymbol{T}}| = |\boldsymbol{J}| = 34$$

$$\boldsymbol{J}^{-\boldsymbol{T}} = \frac{\mathbf{1}}{|\boldsymbol{J}^{\boldsymbol{T}}|} \begin{bmatrix} \mathbf{4} & \mathbf{2} \\ \mathbf{-7} & \mathbf{5} \end{bmatrix}$$

$$\boldsymbol{K}_{ij}{}^{e} = \int_0^1 \int_0^{1-\eta} \boldsymbol{J}^{-\boldsymbol{T}} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \boldsymbol{J}^{-\boldsymbol{T}} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} |\boldsymbol{J}| d\xi d\eta$$

$$\boldsymbol{K}_{ij}{}^{e} = \int_0^1 \int_0^{1-\eta} \frac{1}{34} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \frac{1}{34} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} 34 d\xi d\eta$$

$\boldsymbol{K}_{11}{}^{e}$	$\boldsymbol{K}_{12}{}^{e}$	$\boldsymbol{K}_{13}{}^{e}$
$\boldsymbol{K}_{21}{}^{e}$	$\boldsymbol{K}_{22}{}^{e}$	$\boldsymbol{K}_{23}{}^{e}$
$\boldsymbol{K}_{13}{}^{e}$	$\boldsymbol{K}_{23}{}^{e}$	$\boldsymbol{K}_{33}{}^{e}$

$$\boldsymbol{K}_{21}{}^{e} = \int_0^1 \int_0^{1-\eta} \frac{1}{34} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \cdot \frac{1}{34} \begin{bmatrix} 4 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 34 d\xi d\eta = \int_0^1 \int_{-1}^{1-\eta} \frac{1}{34} \begin{bmatrix} -6 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -7 \end{bmatrix} d\xi d\eta$$

$$\boldsymbol{K}_{21}{}^{e} = -1.118 \int_0^1 \int_0^{1-\eta} d\xi d\eta = -1.118 \cdot \frac{1}{2} = -0.559$$

Local/ coordinate system, isoparametric mapping 2D triangular elements, example

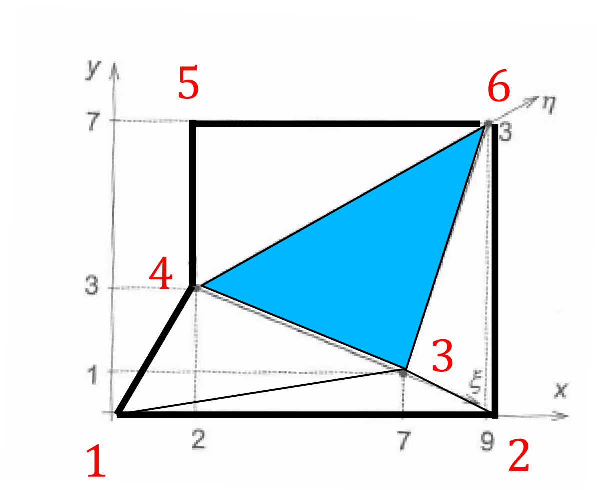
$\boldsymbol{K}_{11}{}^{e}$	$\boldsymbol{K}_{12}{}^{e}$	$\boldsymbol{K}_{13}{}^{e}$
$\boldsymbol{K}_{21}{}^{e}$	$\boldsymbol{K}_{22}{}^{e}$	$\boldsymbol{K}_{23}{}^{e}$
$\boldsymbol{K}_{13}{}^{e}$	$\boldsymbol{K}_{23}{}^{e}$	$\boldsymbol{K}_{33}{}^{e}$

$$\begin{bmatrix} u_1{}^e \\ u_2{}^e \\ u_3{}^e \end{bmatrix} = \begin{bmatrix} f_1{}^e \\ f_2{}^e \\ f_3{}^e \end{bmatrix}$$

$$|\boldsymbol{J}^T| = |\boldsymbol{J}| = 34$$

$$f^e = \begin{bmatrix} \int_{\Omega_e} p(x)N_1(x,y)dx \\ \int_{\Omega_e} p(x)N_2(x,y)dx \\ \int_{\Omega_e} p(x)N_3(x,y)dx \end{bmatrix} = \begin{bmatrix} \int_0^1 \int_0^{1-\eta} p(\xi)N_1(\xi,\eta)\ |\boldsymbol{J}|d\xi d\eta \\ \int_0^1 \int_0^{1-\eta} p(\xi)N_2(\xi,\eta)\ |\boldsymbol{J}|d\xi d\eta \\ \int_0^1 \int_0^{1-\eta} p(\xi)N_3(\xi,\eta)\ |\boldsymbol{J}|d\xi d\eta \end{bmatrix}$$

Local/ coordinate system, isoparametric mapping 2D triangular elements, example



	1	2	3	4	5	6		u	=	f
1								u_1	=	
2								u_2		
3			K_{22}^{e}	K_{21}^{e}		K_{23}^{e}		u_3		$f_2^{\,e}$
4			K_{12}^{e}	K_{11}^{e}		K_{13}^{e}		u_4		$f_1^{\,e}$
5								u_5		
6			K_{32}^{e}	K_{31}^{e}		K_{33}^{e}		u_6		$f_3^{\,e}$

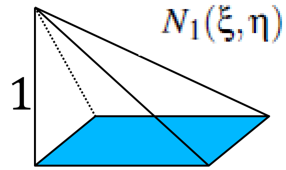
local	1	2	3
global	4	3	6

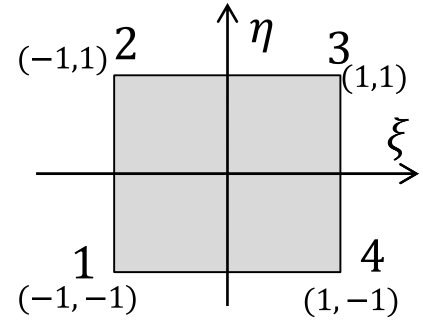
1 4	K_{11}^{e}	K_{12}^{e}	K_{13}^{e}	$u_1^{\,e}$	=	$f_1^{\,e}$	4
2 3	K_{21}^{e}	K_{22}^{e}	K_{23}^{e}	$u_2^{\,e}$		$f_2^{\,e}$	3
3 6	K_{31}^{e}	K_{32}^{e}	K_{33}^{e}	$u_3^{\,e}$		$f_3^{\,e}$	6

Local/ coordinate system, isoparametric mapping 2D quadrilateral elements

Basis functions:

$$
\begin{aligned}
N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta)\\
N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)\\
N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)\\
N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
$$





Transformation from local to global coordinates:

$$
\begin{pmatrix} x_{glob} \\ y_{glob} \end{pmatrix}(\xi,\eta) = N_1(\xi,\eta)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + N_2(\xi,\eta)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + N_3(\xi,\eta)\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} + N_4(\xi,\eta)\begin{pmatrix} x_4 \\ y_4 \end{pmatrix}
$$

$$
\begin{bmatrix} x_{glob}(\xi,\eta) \\ y_{glob}(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) & & N_4(\xi,\eta) & \\ & N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) & & N_4(\xi,\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}
$$

Stiffness matrix:

$$
\mathbf{K}_{ij} = \int_{\Omega_{elm}} \begin{pmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} d\Omega_{elm} \quad i,j \in [1,\dots,4]
$$

$$
K_{ij} = \int_{\Omega} \nabla N_i(\mathbf{x}) \cdot \nabla N_j(\mathbf{x}) d\Omega
$$