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Introduction to PDEs and Numerical Methods
Lecture 12.
Gelerkin method and the FEM

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Overview of this lecture

1. Recap: the weighted residual methods
   1. Abstract formulation
   2. Methods:
      1. Petrov-Galerkin method
         1. Pointwise collocation
         2. Subdomain collocation $\varphi_i = \chi_{\Omega_i}$
      2. Bubnov-Galerkin method
         1. (Least squares)
         2. FEM
2. (Equivalence of the strong form, the weak form and the minimisation problem)
   (see reading assignment (Chapter 5.4. in the Gockenbach book)
3. Orthogonality of the Galerkin method
4. The Céa's theorem
5. Discretisation, the Finite Element Method
6. 1D equation with the FEM
7. Inhomogenous boundary conditions

1. Recap
Weighted residual method

Instead of solving $Lu(x) = f$ (strong form)

Residual: $r = Lu - f = 0$

The residual should be zero in a weighted avarage sense:

$$\langle r, v \rangle = \langle Lu - f, v \rangle = 0 \quad \forall v \in V$$

Abstract form:

$$\langle Lu, v \rangle = \langle f, v \rangle \quad \forall v \in V$$

$$a(u, v) = l\langle \, v \rangle \quad \forall v \in V$$

1. Recap: Discretisation

Further simplifications (discretize to finite dimensional space)

- Approximate the solution with some basis/shape functions:

$$u_h(x) = \sum_i u_i \Phi_i(x)$$

- Instead of solving it for all $v(x) \in V$, select finite subspace for the weighting functions V_h:

$$v_h(x) = \sum_i u_i \varphi_i(x)$$

$$a(u_h, v_h) = l\langle v_h \rangle \quad \forall v_h \in V_h$$

$$a(u_h, \varphi_i) = l\langle \varphi_i \rangle \quad \forall i$$

$$a\left(\sum_j u_j \Phi_j(x), \varphi_i\right) = l\langle \varphi_i \rangle \quad \forall i$$

$$\sum_j u_j(x) \underbrace{a\left(\Phi_j(x), \varphi_i\right)} = \underbrace{l\langle \varphi_i \rangle} \quad \forall i$$

$$K_{ij} := a\left(\Phi_j(x), \varphi_i\right) \qquad f_i := l\langle \varphi_i \rangle \quad \Rightarrow \quad \mathbf{Ku} = \mathbf{f}$$

1. Recap: Discretisation

Descretized weak form

$$\sum_j u_j(x)\, \underbrace{a\big(\Phi_j(x), \varphi_i\big)}_{K_{ij} := a\left(\Phi_j(x), \varphi_i\right)} = \underbrace{l\langle\varphi_i\rangle}_{f_i := l\langle\varphi_i\rangle} \quad \forall i$$

$\Rightarrow$ $\mathbf{Ku} = \mathbf{f}$

How to choose the subspace? How to choose the weighting functions$\varphi_i(x)$?

- True solution can be well approximated by an element of the subspace
- Efficient computation

Definition of the **Dirac delta $\boldsymbol{\delta(x)}$**

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Definition of the **characteristic function $\boldsymbol{\chi_\Omega}$**

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Petrov-Galerkin method $(\Phi_i \neq \varphi_i)$

Pointwise collocation $\varphi_i = \delta(x - x_i)$

Subdomain collocation $\varphi_i = \chi_{\Omega_i}$

Bubnov-Galerkin method $(\Phi_i = \varphi_i)$

FEM: Galerkin method with subspace of *piecewise polynomial functions*

1. Weighted residual method (Bubnov Galerkin)
The mimimum mean square estimator

Least squares:

Minimising the square integral S of the residual r

$$r = Lu_h - f = L\left(\sum_i u_i\,\phi_i(x)\right) - f = \sum_i u_i\,L\big(\phi_i(x)\big) - f \qquad \mathrm{S} = \int_\Omega r^2\,d\Omega$$

At the minimumpoint the gradient of S should cancel:

$$\frac{\partial S}{\partial u_i} = \frac{\partial}{\partial u_i}\int_\Omega r^2\,d\Omega = \int_\Omega \frac{\partial}{\partial u_i}(r^2)\,d\Omega = 2\int_\Omega r\,\frac{\partial}{\partial u_i}\,d\Omega = 2\int_\Omega rL\big(\phi_i(x)\big)d\Omega$$

$$\langle r, L(\phi_i)\rangle = \langle Lu_h - f, L(\phi_i)\rangle = 0$$

3. Orthogonality of the Galerkin method

Let's say we know that there is a unique solution in $V = H_1^0$.
But V is an infinite dimensional space

$\Downarrow$

Let's narrow down the space, where we are trying to find the solution, to a finite dimensional space

example:

instead of finding $u(x) \in \ H_1^0$
we try to find the coefficients α_j of a „proxi model" (ansatz function):

$$u_h\,(x) = \sum_{j=1}^{n} \alpha_j \phi_j\,(x)$$

where

$\phi_j(\mathrm{x})$: known (linearly independent) basis or ansatz functions

$u_h\,(x)$: the approximation of the solution $u(x)$, which is in an n-dimensional space:

$$u_h \in V_h = \mathrm{span}\{\phi_1, \phi_2, \dots \phi_n\}$$

3. Orthogonality of the Galerkin method

Let's fix this subspace to a specific V_h (that is, we fix the ansatz functions in our examples)

⇓

How do we get the best approximation of the solution in this space from the equation:

$$\boxed{Lu(\mathbf{x}) = f(\mathbf{x})}$$

Our goal is to minimize the difference in between the solution and the approximation:

$$\text{error} = \|u(\mathbf{x}) - u_h(\mathbf{x})\| < \|u(\mathbf{x}) - \mathrm{z}(\mathbf{x})\| \; \forall \mathrm{z}(\mathbf{x}) \in V_h$$

⇒ **Best approximation** to $u(\mathbf{x})$ from V_h

The best approximation u_h to u from V_h is the one where the error is orthogonal to the space of V_h, that is to all possible $\mathrm{z}(\mathbf{x}) \in V_h$.
Instead of writing it for all z (as z is an n-dimensional space) we can write $\forall \phi_i \; i = 1..n$

$$\langle (u(\mathbf{x}) - u_h(\mathbf{x})), \phi_i(\mathbf{x}) \rangle = 0 \quad i = 1..n$$

3. Orthogonality of the Galerkin method

Plugging in the proxi model to the orthogonality condition we have:

$$\left\langle \left(u - \sum_{j=1}^{n} u_j \phi_j \right), \phi_i \right\rangle = 0 \quad i = 1..n$$

Rearranging the equation we get:

$$\langle u, \phi_i \rangle - \sum_{j=1}^{n} u_j \langle \phi_j, \phi_i \rangle = 0 \quad i = 1..n$$

$$\sum_{j=1}^{n} u_j \underbrace{\langle \phi_j, \phi_i \rangle}_{K_{ij}} = \underbrace{\langle u, \phi_i \rangle}_{f_j} \quad i = 1..n \quad \Rightarrow \quad \sum_{j=1}^{n} u_j K_{ij} = f_j \quad \Rightarrow \quad \mathbf{K}\boldsymbol{u} = \mathbf{f}$$

3. Orthogonality of the Galerkin method

$$\boxed{\mathbf{K}\boldsymbol{\alpha} = \mathbf{f}}$$

where

$K_{ij} = \langle \phi_j(x), \phi_i(x) \rangle$: can be calculated from the basis functions and the inner product of the given space (if the basis is orthonormal, the matrix is the identity matrix

$f_j = \langle u, \phi_i \rangle$: ?

u_j: The coefficients that we are looking for

But how do we get $f_j = \langle u, \phi_i \rangle$?

We know $\boxed{\langle Lu, \phi_i \rangle_{L2} = \langle f, \phi_i \rangle_{L2}}$

If the left hand side of the weak equation is a V-elliptic, bounded bilinear functional, that is also symmetric, then it can be written as:

$$\langle Lu, \phi_i \rangle_{L2} = a(u, \phi_i) = \langle u, \phi_i \rangle_E \Rightarrow f_j = \langle u, \phi_i \rangle_E = \langle f, \omega \rangle_{L2}$$

3. Orthogonality of the Galerkin method

That means that with Galerkin method we orthogonalize the projection, that is, we minimize the error in the energy space.

And the matrix equation, that we can calculate the coefficients from, will have the form:

$$\boxed{\mathbf{K}\boldsymbol{u} = \mathbf{f}}$$

with

$$K_{ij} = \langle \phi_j, \phi_i \rangle_E = a(\phi_j, \phi_i)$$

$$f_j = \langle u, \phi_i \rangle_E = \langle f, \phi_i \rangle_{L2}$$

Example: $-EA\dfrac{d^2u}{dx^2} = p(x)$ $\quad\quad x = [0, l], \;\; u(0) = 0, \;\; u(l) = 0$

$$Lu = -EA\frac{\partial^2 u(x)}{\partial^2 x} \quad\quad f = p$$

$$K_{ij} = \langle \phi_j(x), \phi_i(x) \rangle_E = \int_0^l EA \frac{d\phi_j(x)}{dx}\frac{d\phi_i(x)}{dx} dx \quad\quad f_j = \langle f, \phi_i \rangle_{L2} = \int_0^l p(x)\phi_i(x) dx$$

3. Orthogonality of the Galerkin method (the same but differently)

The solution of the BVP satisfies the initial weak formulation:

$$a(u, v) = l(v)\ \forall v \in V$$

and as $V_h \subset V$, it also satisfies

$$a(u, v) = l(v)\ \forall v \in V_h$$

Similarly, the approximation of the solution u_h satisfies:

$$a(u_h, v) = l(v)\ \forall v \in V_h$$

Subtracting the last two equations :

$$a(u, v) - a(u_h, v) = 0 \qquad \forall v \in V_h$$

$$\boxed{a(u - u_h, v) = 0 \qquad \forall v \in V_h}$$

$\Rightarrow$ In case $a(u, v)$ is a symmetric bilinear term

$\Uparrow$

Galerkin method gives the best approximation of the true solution in the given subspace V_h in the energy norm

4. What if the the bilinear form is not symmetric? Céa's theorem

Conclusion from before:

$a(\cdot,\cdot)$: symmetric $\Rightarrow$ Gallerkin method gives the best approximation in the energy norm

But what about the error in other norms?

According to Céa's theorem, even without $a(\cdot,\cdot)$ being symmetric, **the error of the approximation of Galerkin will be allways bounded**:

$$\boxed{\|u - u_h\| \leq \frac{M}{\delta} \|u - v\| \quad \forall v \in V_h}$$

Where M and δ are constants from the conditions of boundedness and V-ellipticity of the bilinear term $a(\cdot,\cdot)$:

$$a(u, v) \leq M\|u\|\|v\|$$
$$a(u, u) \geq \delta\|u\|^2$$

and $\|u - v\|$ is the norm of the difference in between the true solution and any $v \in V_h$ This term depends on the n-dimensional space V_h chosen, and the space where the true solution lies in.

5. Discretisation

How to choose (Φ_i)?

- True solution can be well approximated by an element of the subspace
- Efficient and robust computation of $\mathbf{K}\boldsymbol{u} = \mathbf{f}$
  - $\mathbf{K}$ is sparse/diagonal
  - $\mathbf{K}$ is not ill-conditioned
  - Determination of and derivations with Φ_i is easy
  - Integration/derivative of Φ_i is easily computable

Examples:

- Polynomials
- First N eigenfunctions of the PDE: $L\Phi_\mathrm{i}(\mathbf{x}) = \lambda_i \Phi_i(x)$
- Trigonometric functions
- Piecewise polynomials $\Rightarrow$ **Finite Element Method**

5. Discretisation
Galerkin method – choosing basis/weighting function

How about using monomials for $\Phi_i(\mathbf{x})$:

$$\Phi_0(\mathrm{x}) = 1, \Phi_1(\mathrm{x}) = \mathrm{x}, \Phi_2(\mathrm{x}) = \mathrm{x}^2, \Phi_3(\mathrm{x}) = x^3 \ldots$$

$$\frac{\partial \Phi_0(x)}{\partial x} = 0, \frac{\partial \Phi_1(x)}{\partial x} = 1, \frac{\partial \Phi_2(x)}{\partial x} = 2x, \frac{\partial \Phi_3(x)}{\partial x} = 3x^2 \ldots$$

$$K_{ij} = \int_0^L \frac{\partial \Phi_i(x)}{\partial x} \frac{\partial \Phi_j(x)}{\partial x} dx$$

Example (the Poisson equation) :

$$K_{0i} = \int_0^L \frac{\partial \Phi_0(x)}{\partial x} \frac{\partial \Phi_i(x)}{\partial x} dx = \int_0^L 0 \cdot \frac{\partial \Phi_i(x)}{\partial x} dx = 0$$

$$K_{11} = \int_0^L \frac{\partial \Phi_1(x)}{\partial x} \frac{\partial \Phi_1(x)}{\partial x} dx = \int_0^L 1 \cdot 1 dx = L$$

$$K_{21} = \int_0^L \frac{\partial \Phi_2(x)}{\partial x} \frac{\partial \Phi_1(x)}{\partial x} dx = \int_0^L 1 \cdot 2x dx = L^2$$

$$K_{22} = \int_0^L 2x 2x dx = 4/3L^3 \quad K_{23} = \int_0^L 2x 3x^2 dx = 6/4L^4$$

$$K_{13} = \int_0^L 1 \cdot 3x^2 dx = L^3 \quad K_{33} = \int_0^L 3x^2 3x^2 dx = 9/5L^5$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & L^2 & L^3 \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{6}{4}L^4 \\ 0 & L^3 & \frac{6}{4}L^4 & \frac{9}{5}L^5 \end{bmatrix}$$

$$K_{ij} = \int_0^L ijx^{i-1}x^{j-1} dx = \int_0^L ijx^{(i+j-2)} dx = \frac{ij}{i+j-1} L^{i+j-1}$$

Elements in the stiffness matrix may have orders of magnitude difference

By increasing the number of the basis functions (degree),the higher the condition number of $\mathbf{K}$ may get.

Monomials are not a good choice if high degree is needed.

5. Discretisation
Galerkin method – choosing basis/weighting function

By increasing the number of the basis functions (degree),the higher the condition number of $\mathbf{K}$ may get. $\mathbf{K}$ is also not sparse.

Monomials are not a good choice if high degree is needed.

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & & \cdots \\ 0 & L & L^2 & L^3 & & \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{6}{4}L^4 & & \\ 0 & L^3 & \frac{6}{4}L^4 & \frac{9}{5}L^5 & & \\ & & & & \ddots & \\ \vdots & & & & & \frac{n^2}{2n-1}L^{2n-1} \end{bmatrix}$$

How to use polynomials then?

1. Orthogonalise monomials

   Orthogonalise the monomials $\Phi_i(\mathbf{x})$ with Gramm-Schmidt with respect to the inner product $\langle u, v\rangle_E = a(u,v) = \int u'v'\,dx$

2. Use piecewise low degree polynomials (Finite Element Method)

5. Discretisation of the Galerkin method – choosing basis/weighting function – orthogonal polynomials

Orthogonalise the monomials $\Phi_i(\mathbf{x})$ with Gramm-Schmidt

$$\Phi_0(\mathrm{x}) = 1, \Phi_1(\mathrm{x}) = \mathrm{x}, \Phi_2(\mathrm{x}) = \mathrm{x}^2, \Phi_3(\mathrm{x}) = x^3 \ldots$$

- First new basis funtion: $\boxed{\Psi_0 := \Phi_0 = 1}$
- Second new basis function Ψ_1:
  - 1. Orthogonal projection of Φ_1 onto Ψ_0: $\langle \Phi_1, \Psi_0 \rangle_E = \int_0^L \Phi_1{}' \Psi_0{}' \, dx = \int_0^L 1 \cdot 0 \, dx = 0$
  - 2. Subtract orthogonal part from Φ_1: $\Psi_1 := \Phi_1 - \langle \Phi_1, \Psi_0 \rangle_E \Psi_0 = \mathrm{x} - 0 = \mathrm{x}$
  - 3. Normalise Ψ_1: $\Psi_1 := \frac{\Psi_1}{\langle \Psi_1, \Psi_1 \rangle_E} = \frac{x}{\int_0^L 1 \cdot 1 dx} = \frac{x}{L}$

  The new second basis function is $\boxed{\Psi_1 = \frac{x}{L}}$

- Third new basis function Ψ_2:
  - 1. Orthogonal projection of Φ_2 onto Ψ_0 and Ψ_1:
    - $\langle \Phi_2, \Psi_0 \rangle_E = \int_0^L \Phi_2{}' \Psi_0{}' \, dx \int_0^L 2x \cdot 0 \, dx = 0$
    - $\langle \Phi_2, \Psi_1 \rangle_E = \int_0^L \Phi_2{}' \Psi_1{}' \, dx \int_0^L 2x \cdot 1/L \, dx = L$
  - 2. Subtract orthogonal parts from Φ_2: $\Psi_2 := \Phi_2 - \langle \Phi_2, \Psi_0 \rangle_E \Psi_0 - \langle \Phi_2, \Psi_1 \rangle_E \Psi_1 = \mathrm{x}^2 - 0 - \mathrm{L}\frac{x}{L} = \mathrm{x}^2 - \mathrm{x}$
  - 3. Normalise Ψ_2: $\Psi_2 := \frac{\Psi_2}{\langle \Psi_2, \Psi_2 \rangle_E} = \frac{\mathrm{x}^2 - \mathrm{x}}{\int_0^L (2x-1) \cdot (2x-1) dx} = \frac{\mathrm{x}^2 - \mathrm{x}}{\frac{4}{3}L^3 - 2L^2 + L}$

  The new basis function is $\boxed{\Psi_2 = \frac{\mathrm{x}^2 - \mathrm{x}}{\frac{4}{3}L^3 - 2L^2 + L}}$

- Etc…

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5. Discretisation
Galerkin method with nodal basis, piecewise linear shape functions

Lagrange/nodal basis:

$$\Phi_i(\mathbf{x}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Approximation will be exact in the mesh nodes

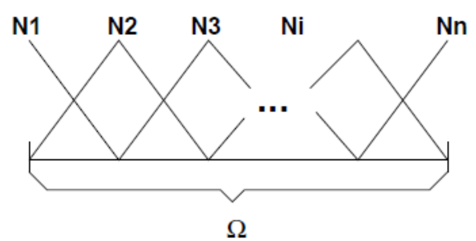
$$u(\mathbf{x}) \approx \sum_{i=1}^{N} u_i \, \Phi_i(\mathbf{x})$$

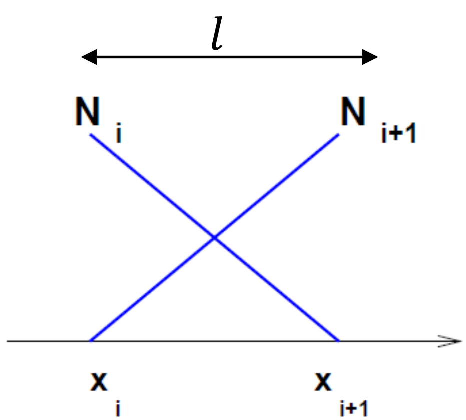
nodal values of solution

$$u(\mathbf{x}_j) = \sum_{i=1}^{N} u_i \, \Phi_i(\mathbf{x}_j)$$

$$\mathbf{Ku} = \mathbf{f}$$

Piecewise linear basis functions (hat functions):





$$\Phi_i(x) = \mathrm{N_i(x)} = \begin{cases} \dfrac{x - x_{i-1}}{l} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} + x}{l} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases}$$

6. How to solve PDE with FEM with nodal basis, piecewise linear shape functions

Finite Element method with piecewise linear functions in 1D, hom DBC

1) Weak formulation of the PDE, definition of the ‚energy' inner product (the bilinear functional, a) and and the linear functional (l)

$$a(u,v) = l(v)$$

2) Define approximating subspace by definition of a mesh (nodes 0,1,..N, with coordinates, elements) and setup the hat functions on them

$$\Phi_i(x) = \mathrm{N_i(x)} = \begin{cases} \dfrac{x - x_{i-1}}{l} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} + x}{l} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases} \qquad i = 1..N-1$$

3) Compute the elements of the stiffness matrix (Grammian) – evaluation of integrals

$$K_{ij} = a\big(N_i(x), N_j(x)\big) = \big\langle N_i(x), N_j(x)\big\rangle_E \quad i,j = 1..N-1$$

4) Compute the elements of the vector of the right hand side – evaluation of integrals

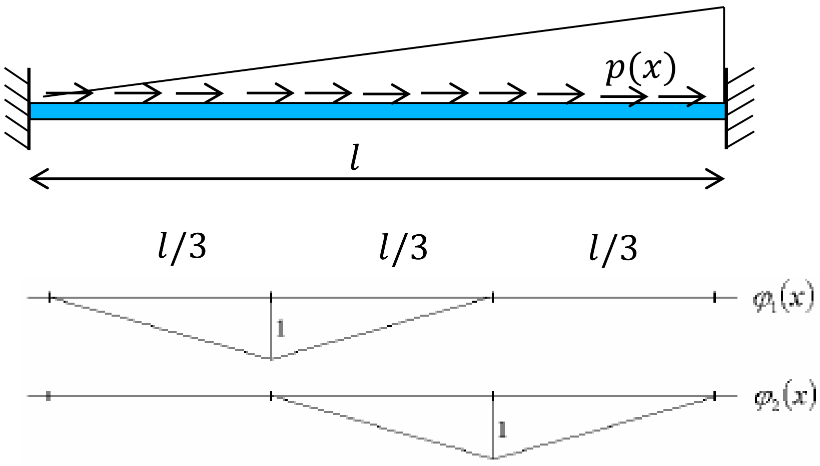
$$f_i = l(N_i),\ i = 1..N-1$$

5) Solve the system of equations: $\mathbf{Ku} = \mathbf{f}$

for $\mathbf{u}$, which gives the solution at the nodes.

The solution in between the nodes can be calculated from: $u(\mathrm{x}) \approx \sum_{i=1}^{N} u_i\, N_i(\mathrm{x})$

6.
1D Example with linear nodal basis



$$p(x) = ax$$

Strong form: $-EA\dfrac{d^2u}{dx^2} = p(x)$

$$u(0) = u(l) = 0$$

Weak form:

$$\int_0^l EA\frac{du}{dx}\frac{d\varphi}{dx}dx = \int_0^l p(x)\varphi(x)dx$$

Discretisation of the weak form:

$$u(\mathbf{x}) \approx \sum_{i=1}^{2} u_i\,\varphi_i(\mathbf{x})$$

with the basis functions:

$$\varphi_1 = \begin{cases} \dfrac{3x}{l} & \text{for} \quad x \le l/3 \\ 2-\dfrac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 0 & \text{for} \quad 2l/3 \le x \end{cases} \qquad \frac{\partial\varphi_1}{\partial x} = \begin{cases} \dfrac{3}{l} \\ -\dfrac{3}{l} \\ 0 \end{cases}$$

$$\sum_{i=1}^{2} u_i \underbrace{EA\int_l \frac{\partial\varphi_i(x)}{\partial x}\frac{\partial\varphi_j(x)}{\partial x}dx}_{K_{ij}} = \underbrace{\int_l f(x)\varphi_j(x)dx}_{f_j}$$

$$\varphi_2 = \begin{cases} 0 & \text{for} \quad x \le l/3 \\ -1+\dfrac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 3-\dfrac{3x}{l} & \text{for} \quad 2l/3 \le x \end{cases} \qquad \frac{\partial\varphi_2}{\partial x} = \begin{cases} 0 \\ \dfrac{3}{l} \\ -\dfrac{3}{l} \end{cases}$$

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6.
1D Example with linear nodal basis

$$\sum_{i=1}^{N} u_i \underbrace{EA \int_l \frac{\partial \varphi_i(x)}{\partial x}\frac{\partial \varphi_j(x)}{\partial x} dx}_{K_{ij}} = \underbrace{\int_l f(x)\varphi_j(x)dx}_{f_j}$$

$$\varphi_1 = \begin{cases} \frac{3x}{l} & \text{for} \quad x \le l/3 \\ 2 - \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 0 & \text{for} \quad 2l/3 \le x \end{cases} \qquad \frac{\partial \varphi_1}{\partial x} = \begin{cases} \frac{3}{l} \\ -\frac{3}{l} \\ 0 \end{cases}$$

$$K_{ij} = EA \int_l \frac{\partial \varphi_i(x)}{\partial x}\frac{\partial \varphi_j(x)}{\partial x} dx$$

$$\varphi_2 = \begin{cases} 0 & \text{for} \quad x \le l/3 \\ -1 + \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 3 - \frac{3x}{l} & \text{for} \quad 2l/3 \le x \end{cases} \qquad \frac{\partial \varphi_2}{\partial x} = \begin{cases} 0 \\ \frac{3}{l} \\ -\frac{3}{l} \end{cases}$$

$$\frac{K_{11}}{EA} = \int_0^{l/3} \frac{\partial \varphi_1(x)}{\partial x}\frac{\partial \varphi_1(x)}{\partial x} dx + \int_{l/3}^{2l/3} \frac{\partial \varphi_1(x)}{\partial x}\frac{\partial \varphi_1(x)}{\partial x} dx + \int_{2l/3}^{l} \frac{\partial \varphi_1(x)}{\partial x}\frac{\partial \varphi_1(x)}{\partial x} dx$$

$$\frac{K_{11}}{EA} = \int_0^{l/3} \frac{3}{l}\frac{3}{l} dx + \int_{l/3}^{2l/3} \frac{-3}{l}\frac{-3}{l} dx + \int_{2l/3}^{l} 0 dx$$

$$K_{11} = EA\left(\frac{3}{l} + \frac{3}{l} +\right) = \frac{6}{l}\mathrm{EA}$$

6.
1D Example with linear nodal basis

$$K_{12} = \int_l EA \frac{\partial \varphi_1(x)}{\partial x}\frac{\partial \varphi_2(x)}{\partial x} dx$$

$$\varphi_1 = \begin{cases} \frac{3x}{l} & \text{for} \quad x \le l/3 \\ 2 - \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 0 & \text{for} \quad 2l/3 \le x \end{cases} \qquad \frac{\partial \varphi_1}{\partial x} = \begin{cases} \frac{3}{l} \\ -\frac{3}{l} \\ 0 \end{cases}$$

$$\frac{K_{12}}{EA} = \int_0^{l/3} \frac{3}{l} 0 dx + \int_{l/3}^{2l/3} \frac{-3}{l}\frac{+3}{l} dx + \int_{2l/3}^{l} 0 \frac{-3}{l} dx$$

$$\frac{K_{11}}{EA} = 0 - \frac{3}{l} + 0 = \frac{-3}{l}$$

$$\varphi_2 = \begin{cases} 0 & \text{for} \quad x \le l/3 \\ -1 + \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 3 - \frac{3x}{l} & \text{for} \quad 2l/3 \le x \end{cases} \qquad \frac{\partial \varphi_2}{\partial x} = \begin{cases} 0 \\ \frac{3}{l} \\ -\frac{3}{l} \end{cases}$$

$$K_{21} = EA \int_l \frac{\partial \varphi_2(x)}{\partial x}\frac{\partial \varphi_1(x)}{\partial x} dx = K_{12} = \frac{-3}{l}\mathrm{EA}$$

$$\frac{K_{22}}{EA} = \int_l \frac{\partial \varphi_2(x)}{\partial x}\frac{\partial \varphi_2(x)}{\partial x} dx = \int_0^{l/3} 0 \cdot 0 dx + \int_{l/3}^{2l/3} \frac{3}{l}\frac{3}{l} dx + \int_{2l/3}^{l} \frac{-3}{l}\frac{-3}{l} dx$$

$$\frac{K_{22}}{EA} = 0 + \frac{3}{l} + \frac{3}{l} = \frac{6}{l}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \frac{3EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

6.
1D Example with linear nodal basis

$$\boxed{f_j = \int_l p(x)\varphi_j(x)dx}$$

$$\varphi_1 = \begin{cases} \frac{3x}{l} & \text{for} \quad x \le l/3 \\ 2 - \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 0 & \text{for} \quad 2l/3 \le x \end{cases}$$

$$\varphi_2 = \begin{cases} 0 & \text{for} \quad x \le l/3 \\ -1 + \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 3 - \frac{3x}{l} & \text{for} \quad 2l/3 \le x \end{cases}$$

$$f_1 = \int_l p(x)\varphi_1(x)dx$$

$$f_1 = \int_0^{l/3} ax\frac{3x}{l}dx + \int_{l/3}^{2l/3} ax(2 - \frac{3x}{l})dx + \int_{2l/3}^{l} ax \cdot 0dx$$

$$f_1 = \frac{al^2}{27} + \left(\frac{3al^2}{9} - \frac{7al^2}{27}\right) + 0 = \frac{al^2}{9}$$

<u>Home assignment:</u>

Calculate $f_2 = \int_l p(x)\varphi_2(x)dx$

Solve the system of equations $\frac{3EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

for u_1 and u_2. Draw the function u(x).

6.
1D Example with linear nodal basis

Home assignment answer:

$$\varphi_2 = \begin{cases} 0 & \text{for} \quad x \le l/3 \\ -1 + \frac{3x}{l} & \text{for} \quad l/3 \le x \le 2l/3 \\ 3 - \frac{3x}{l} & \text{for} \quad 2l/3 \le x \end{cases}$$

$$\boxed{f_j = \int_l p(x)\varphi_j(x)dx}$$

$$f_2 = \int_l p(x)\varphi_2(x)dx$$

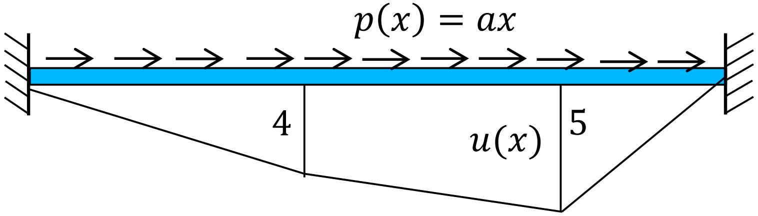
$$f_2 = \int_0^{l/3} ax \cdot 0\, dx + \int_{l/3}^{2l/3} ax\left(-1 + \frac{3x}{l}\right)dx + \int_{2l/3}^{l} ax\left(3 - \frac{3x}{l}\right)dx$$

$$f_2 = 0 + \left(-\frac{3al^2}{18} + \frac{7al^2}{27}\right) + \left(\frac{15al^2}{18} - \frac{19al^2}{27}\right) = \frac{2al^2}{9}$$

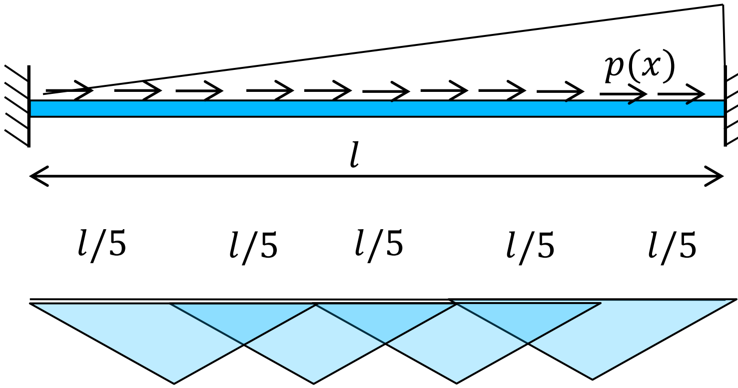
$$\boxed{f_2 = \frac{2al^2}{9}}$$

Solve the system of equations $\frac{3EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{al^2}{9}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{al^3}{81EA}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$



6. Compiling elementwisely
1D Example with linear nodal basis



$p(x) = ax$

Strong form: $-EA\dfrac{d^2u}{dx^2} = p(x)$

$$u(0) = u(l) = 0$$

Weak form:

$$\int_0^l EA\frac{du}{dx}\frac{d\varphi}{dx}dx = \int_0^l p(x)\varphi(x)dx$$

Discretisation of the weak form:

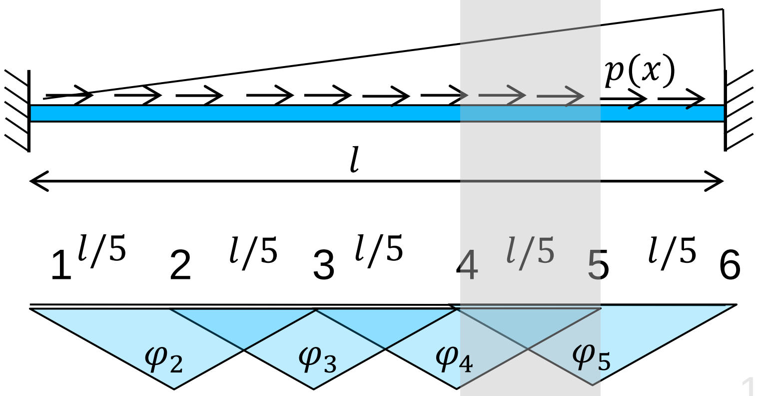
$$u(\mathbf{x}) \approx \sum_{i=1}^{4} u_i\,\varphi_i(\mathbf{x})$$

$$\sum_{i=1}^{4} u_j\, EA\underbrace{\int_l \frac{\partial\varphi_i(x)}{\partial x}\frac{\partial\varphi_j(x)}{\partial x}dx}_{K_{ij}} = \underbrace{\int_l p(x)\varphi_j(x)dx}_{f_j}$$

Not efficient to calculate all the elements of the stiffness matrix one by one!

Calculate element stiffness matrices and assemble

6. Compiling elementwisely
1D Example with linear nodal basis



instead:
Compute stiffness matrix elementwisely and then assemble

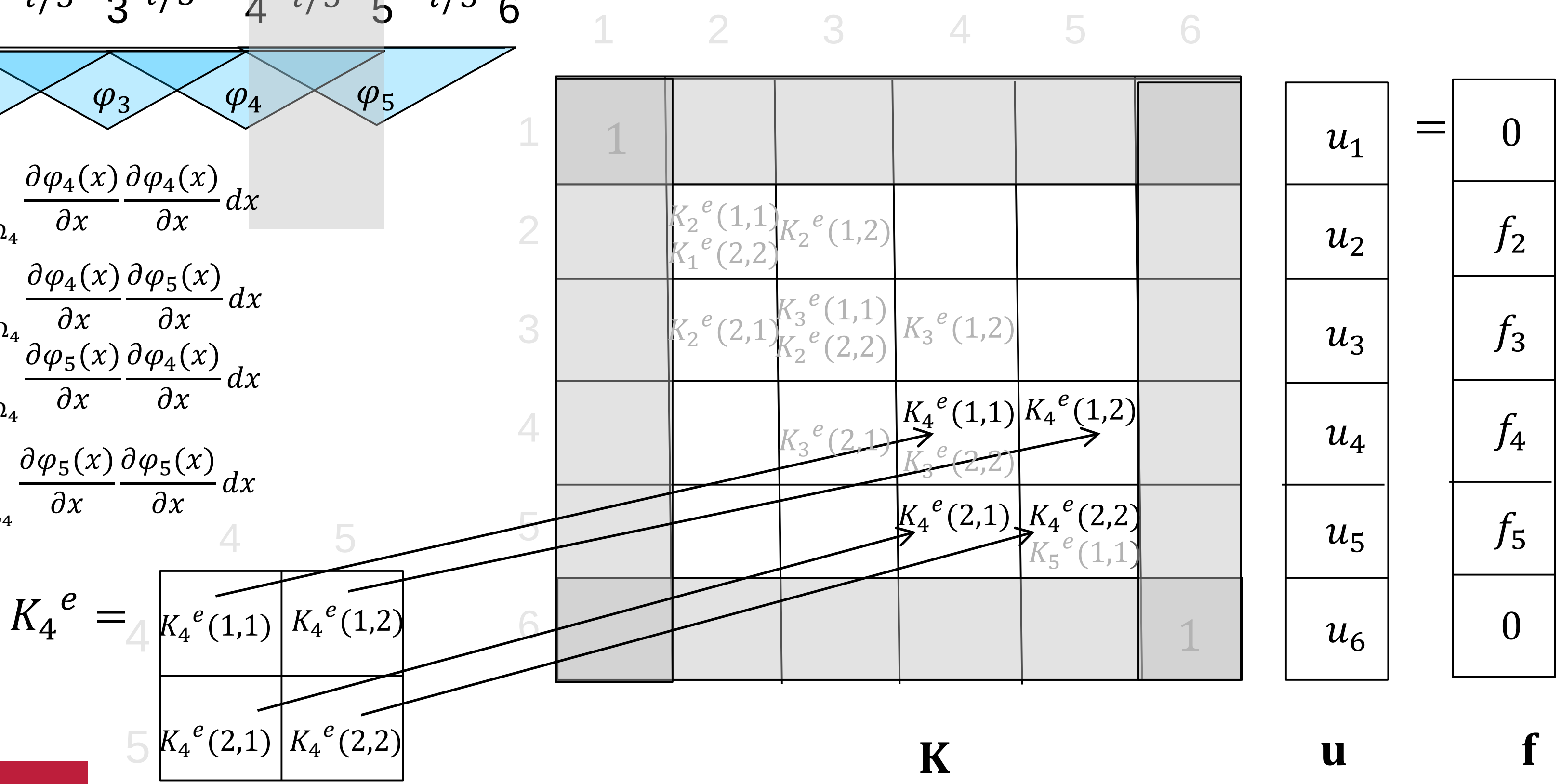
$$K_4{}^e(1,1) = EA \int_{\Omega_4} \frac{\partial \varphi_4(x)}{\partial x} \frac{\partial \varphi_4(x)}{\partial x} dx$$

$$K_4{}^e(1,2) = EA \int_{\Omega_4} \frac{\partial \varphi_4(x)}{\partial x} \frac{\partial \varphi_5(x)}{\partial x} dx$$

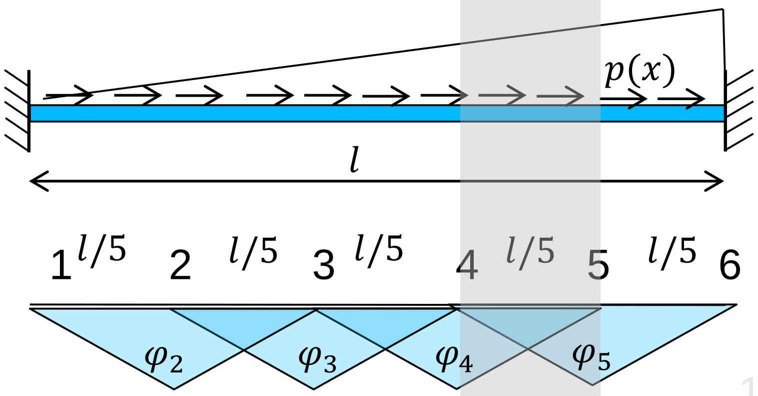
$$K_4{}^e(2,1) = EA \int_{\Omega_4} \frac{\partial \varphi_5(x)}{\partial x} \frac{\partial \varphi_4(x)}{\partial x} dx$$

$$K_4{}^e(2,2) = EA \int_{\Omega_4} \frac{\partial \varphi_5(x)}{\partial x} \frac{\partial \varphi_5(x)}{\partial x} dx$$

Global stiffness matrix



6. Compiling elementwisely
1D Example with linear nodal basis

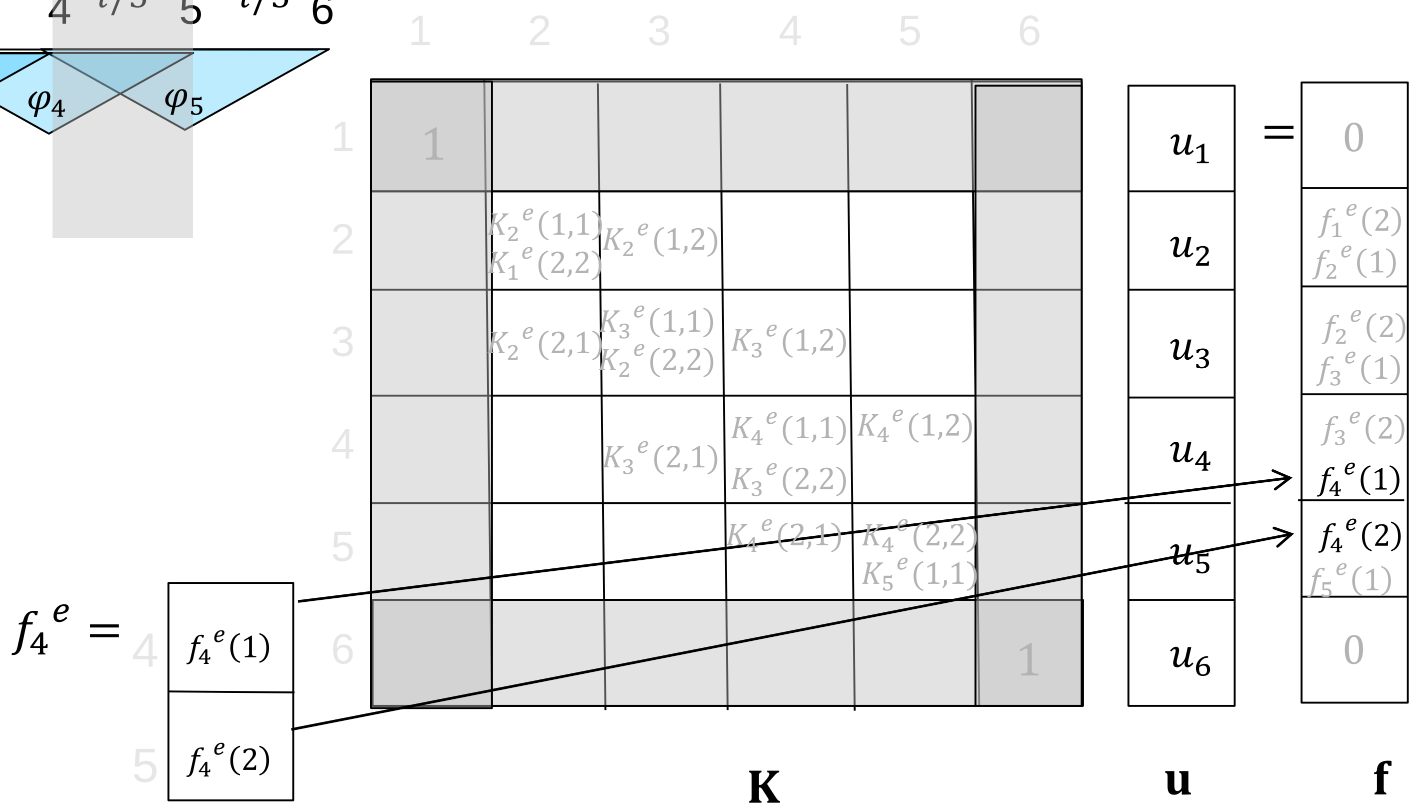


instead:
Compute stiffness matrix elementwisely and then assemble

$$f_4{}^e(1) = \int_{\Omega_4} p(x)\varphi_4(x)dx$$

$$f_4{}^e(2) = \int_{\Omega_4} p(x)\varphi_5(x)dx$$

$$f_4{}^e = \frac{l^e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}p(x_4)\\ p(x_5)\end{bmatrix}$$



6. Compiling elementwisely
Local/global coordinate system 1D

Idea:
coordinate transformation to have unit length elements ➡ element stiffnes matrix is the same for each element

$$\xi = \frac{x - x_i}{x_{i+1} - x_i} \Rightarrow \boxed{\xi = [0,1]} \qquad \frac{\partial \xi}{\partial x} = \frac{1}{l^e}$$

$$K_4{}^e(i,j) = EA \int_{\Omega_4} \frac{\partial \varphi_i(x)}{x} \frac{\partial \varphi_j(x)}{\partial x} dx$$

$$K_4{}^e(i,j) = EA \int_{\Omega_4} \frac{\partial \varphi_i(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\frac{1}{l^e}} \frac{\partial \varphi_j(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\frac{1}{l^e}} dx = \frac{EA}{(l_4{}^e)^2} \int_{\Omega_4} \frac{\partial \varphi_i(\xi)}{\partial \xi} \frac{\partial \varphi_j(\xi)}{\partial \xi} dx$$

$$K_4{}^e(i,j) = \frac{EA}{(l^e)^2} \int_0^1 \frac{\partial \varphi_i(\xi)}{\partial \xi} \frac{\partial \varphi_j(\xi)}{\partial \xi} \underbrace{\frac{dx}{d\xi}}_{l^e} d\xi = \frac{EA}{l^e} \int_0^1 \frac{\partial \varphi_i(\xi)}{\partial \xi} \frac{\partial \varphi_j(\xi)}{\partial \xi} d\xi$$

$$\Rightarrow \boxed{K_4{}^e = \frac{EA}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

7. Inhomogenous Dirichlet BC and Mixed BC
Inhomogenous Dirichlet BC

Consider the BVP with ihomogenous Dirichlet BC:

$$u'' = f \qquad \text{on } I = (0,1)$$
$$u = g \qquad \text{on } \partial I = \{0,1\} \quad u(0) = g(0) = 2 \quad u(1) = g(1) = 3,$$

Find $u \in V_g$ such that

$$a(u,v) = l(v) \quad \forall v \in V = H_0^1 \quad \text{with} \quad a(u,v) = \int_I u'v',$$
$$l(v) = \int_I fv = \int_I 2v$$

$$V_g = \{v \in L^2(I) | \nabla v \in L^2(I), v|_{\partial I} = g\},$$
$$V = H_0^1 = \{v \in L^2(I) | \nabla v \in L^2(I), v|_{\partial I} = 0\}$$

Let us introduce a function, that satisfies the inh. D.BC., e.g. $\omega = g \quad \text{on } \partial I = \{0,1\}$

$\omega \in H^1$

7. Inhomogenous Dirichlet BC and Mixed BC
Inhomogenous Dirichlet BC

The unknown function then can be written as:

$$u = \omega + u_0$$

Then the weak formulation will take the form:

$$a(\omega + u_0, v) = l(v) \ \forall v \in V$$

$$\boxed{a(\omega, v)} + a(u_0, v) = l(v) \ \forall v \in V$$

By rearranging (putting the known terms on the l.h.s.):

$$a(u_0, v) = l(v) \boxed{- a(\omega, v)} \quad \forall v \in V$$

Discretisation: $a(u_{0h}, v_h) = l\langle v_h \rangle \text{-} a(\omega, v_h) \quad \forall v_h \in V_h$ $\qquad V_h = \{\sum_{i=0}^{n} \alpha_i \varphi_i \text{ for } \alpha_i \in \mathbb{R}\}$

$$\sum_j u_j \, a\big(\varphi_j(x), \varphi_i\big) = l\langle \varphi_i \rangle \text{-} a(\omega, \varphi_i) \quad \forall i$$

7. Inhomogenous Dirichlet BC and Mixed BC
Mixed BC

Strong formulation: find $u \in C^2(I)$

$$-u''(x) + u(x) = f(x) \text{ on } I = (0,1)$$
$$u(0) = 0$$
$$u'(1) = 1$$

1) Multiply by test/weight function $v(x)$ and integrate

$$-\int_I u''(x)v(x)dx + \int_I u(x)v(x)dx = \int_I f(x)v(x)dx \qquad v(0) = 0$$

2.) Integration by parts

$$-\int_I u''(x)v(x)dx = \int_I u'(x)v'(x)dx - u'(x)v(x)|_0^1$$

3.) Apply boundary conditions ($u'(1) = 1$, $v(0) = 0$)

$$u'(x)v(x)|_0^1 = v(1)$$

7. Inhomogenous Dirichlet BC and Mixed BC
Mixed BC

$$-\underbrace{\int_I u''(x)v(x)dx}_{=\int_I u'(x)v'(x)dx - v(1)} + \int_I u(x)v(x)dx = \int_I f(x)v(x)dx$$

Weak formulation: For $V = \{H^1(I) \,|\, u(0) = 0\}$ find $u \in V$ such that

$$\int_I u'(x)v'(x)dx + \int_I u(x)v(x)dx = \int_I f(x)v(x)dx + v(1) \quad \forall v \in V$$

Abstract setting:

$$V = \{H^1(I) \,|\, u(0) = 0\}, \quad a(u,v) = \int_I u'(x)v'(x)dx + \int_I u(x)v(x)dx, \quad l(v) = \int_I f(x)v(x)dx + v(1)$$

$$a(u,v) = l(v) \quad \forall v \in V.$$