Technische Universität Braunschweig

Introduction to PDEs and Numerical Methods
Lecture 7-8.
Finite difference methods – stability, concsistency, convergence

Dr. Noemi Friedman, 29.11, 06.12.2017.

Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)

- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)

- Numerical solutions of PDEs:

  - Finite difference method

  - Finite element method

Overview of this lecture

- Finite difference operators
- The heat equation
  - Analytical solution
  - Semidiscretization- Method of Lines (spatial discretization)
  - Time discretization
    - Euler foward
    - Euler backward
    - The theta-method

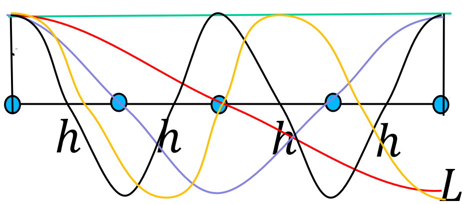
  - Consisteny, stability, convergence
  - Heat equation in higher dimension

Von Neumann Stability Analysis

Stability checking with Von Neumann Stability Analysis (Fourier stability analysis**)**

Let's suppose our solution has the form of:

$$e^{ik_m x} = \cos(k_m x) + i\,\sin(k_m x)$$



$m = 0,1,2,3,4$

$$u(t,x) = \sum_{m=0}^{\infty} A_m(t) e^{ik_m x}$$ (Fourier-expansion)

With the wave number: $k_m = \dfrac{m\pi}{L}$ $\quad m = 0..M \quad$ $M = \dfrac{L}{h}$

(Shannon's theorem)

Let's suppose that the solution in time changes exponentialy

$$A_m(t) = e^{\alpha_m t}$$ where α_m: constant

The solution takes the form after discretisation: $t = n\Delta t, \qquad x = jh$

$$\boxed{u(n,j) = \sum_{m=0}^{M} G(k_m)^n e^{ik_m jh}}$$

$$G(k_m)^n = A_m(t) = e^{\alpha_m n\Delta t} = \left(e^{\alpha_m \Delta t}\right)^n$$ **gain factor/ amplifyer**

Von Neumann Stability Analysis – stability of Euler forward

in simpler form

$$\boxed{u(n,j) = \sum_{k=0}^{M} G(k)^n e^{ikjh}}$$

$u_{n,j}(k) = G(k)^n e^{ikjh}$ for one frequency

Example: let's check the stability of the following scheme for the instationary heat equation:

$$u_{n+1,j} - u_{n,j} = \Delta t \frac{\beta^2}{h^2}\left(u_{n,j-1} - 2u_{n,j} + u_{n,j+1}\right)$$

(Euler forward, three point spatial discr.)

$$G(k)^{n+1} e^{ikjh} - G(k)^n e^{ikjh} = \Delta t \frac{\beta^2}{h^2}\left(G(k)^n e^{ik(j-1)h} - 2G(k)^n e^{ikjh} + G(k)^n e^{ik(j+1)h}\right)$$

$/G(k)^n$

$$G(k) e^{ikjh} - e^{ikjh} = \Delta t \frac{\beta^2}{h^2}\left(e^{ik(j-1)h} - 2e^{ikjh} + e^{ik(j+1)h}\right)$$

$/e^{ikjh}$

$$G(k) - 1 = \Delta t \frac{\beta^2}{h^2}\left(e^{-ikh} - 2 + e^{ikh}\right)$$

$$\boxed{e^{ikh} + e^{-ikh} = 2\cos(kh)}$$

$$G(k) = 1 + 2\Delta t \frac{\beta^2}{h^2}(\cos(kh) - 1)$$

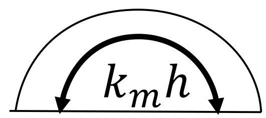
Von Neumann Stability Analysis – stability of Euler forward

The gain factor:

$$G(k) = 1 + 2\Delta t \frac{\beta^2}{h^2}(\cos(kh) - 1)$$

In a more precise form:

$$G(k_m) = 1 + 2\Delta t \frac{\beta^2}{h^2}\overbrace{(\cos(k_m h) - 1)}^{-2 < C_m > 0}$$



$$k_m = \frac{m\pi}{L}$$

$$m = 0..M$$

$$M = \frac{L}{h}$$

Stability requirement: $|G(k_m)| \leq 1$

$\max(G(k_m))$: $m = 0 \Longrightarrow G(k_0) = 1 + 2\Delta t \frac{\beta^2}{h^2}(1 - 1) = 1$

lowest frequency

$\min(G(k_m))$: $m = M \Longrightarrow G(k_M) = 1 + 2\Delta t \frac{\beta^2}{h^2}(-1 - 1) = 1 - 4\Delta t \frac{\beta^2}{h^2}$

highest frequency

Von Neumann Stability Analysis – stability of Euler forward

Stability requirement:

$$G(k_M) = 1 - 4\Delta t \frac{\beta^2}{h^2} \geq -1$$

$$4\Delta t \frac{\beta^2}{h^2} \leq 2$$

$$\boxed{\Delta t \leq \frac{h^2}{2\beta^2}}$$

Scheme for the heat equation is only stable if this condition is satisfied. (conditionally stable)

If the gain factor is positive, the solution will not oscilate in time:

$$G(k_m) \geq 0$$

$$G(k_M) = 1 - 4\Delta t \frac{\beta^2}{h^2} \geq 0$$

$$\boxed{\frac{h^2}{4\beta^2} \geq \Delta t}$$

Solution will give oscillatory solution if this condition is not satisfied.

Von Neumann Stability Analysis – stability of Euler backward

Example 2: let's check the stability of the following scheme for the instationary heat equation:

$$\frac{u_{j,n+1}-u_{j,n}}{\Delta t} = \theta \frac{\beta^2}{h^2}\left(u_{j-1,n+1} - 2u_{j,n+1} + u_{j+1,n+1}\right) + \qquad \text{(Theta method)}$$

$$+(1-\theta)\frac{\beta^2}{h^2}\left(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}\right) + O(h^2) + O(\Delta t^p)$$

$$G(k)^{n+1}e^{ikjh} - G(k)^n e^{ikjh} = \Delta t\theta \frac{\beta^2}{h^2}\left(G(k)^{n+1}e^{ik(j-1)h} - 2G(k)^{n+1}e^{ikjh} + G(k)^{n+1}e^{ik(j+1)h}\right)$$

$$+\Delta t(1-\theta)\frac{\beta^2}{h^2}\left(G(k)^n e^{ik(j-1)h} - 2G(k)^n e^{ikjh} + G(k)^n e^{ik(j+1)h}\right)$$

$$/G(k)^n$$

$$G(k)e^{ikjh} - e^{ikjh} =$$

$$= \Delta t\theta \frac{\beta^2}{h^2} G(k)\left(e^{ik(j-1)h} - 2e^{ikjh} + e^{ik(j+1)h}\right) + \Delta t(1-\theta)\frac{\beta^2}{h^2}\left(e^{ik(j-1)h} - 2e^{ikjh} + e^{ik(j+1)h}\right)$$

$$/e^{ikjh}$$

$$G(k) - 1 = \Delta t\theta \frac{\beta^2}{h^2} G(k)\left(e^{-ikh} - 2 + e^{ikh}\right) + \Delta t(1-\theta)\frac{\beta^2}{h^2}\left(e^{-ikh} - 2 + e^{ikh}\right)$$

Von Neumann Stability Analysis – stability of Euler backward

Example 2: let's check the stability of the following scheme for the instationary heat equation:

$$G(k) - 1 = \Delta t \theta \frac{\beta^2}{h^2} G(k)\left(e^{-ikh} - 2 + e^{ikh}\right) + (1-\theta)\frac{\Delta t \beta^2}{h^2}\left(e^{-ikh} - 2 + e^{ikh}\right)$$

$$G(k) - \underbrace{\theta \Delta t \frac{\beta^2}{h^2}}_{:= r} G(k)\left(e^{-ikh} - 2 + e^{ikh}\right) = (1-\theta)\underbrace{\frac{\Delta t \beta^2}{h^2}}_{:= r}\left(e^{-ikh} - 2 + e^{ikh}\right) + 1$$

$$\boxed{e^{ikh} + e^{-ikh} = 2\cos(kh)}$$

$$G(k) - \theta r G(k)(2\cos(kh) - 2) = (1-\theta) r (2\cos(kh) - 2) + 1$$

$$G(k)(1 + 2\theta r - 2\theta r \cos(kh)) = 2(1-\theta) r (\cos(kh) - 1) + 1$$

$$G(k) = \frac{2(1-\theta) r (\cos(kh) - 1) + 1}{(1 + 2\theta r - 2\theta r \cos(kh))}$$

Von Neumann Stability Analysis – stability of Euler backward

The gain factor: $$G(k_m) = \frac{2(1-\theta)r(\cos(k_m h) \; - 1) + 1}{(1 + 2\theta r - 2\theta r\cos(k_m h))}$$

Lowest frequency: $k_0 = 0$

$$G(0) = \frac{2(1-\theta)r(1 \; - 1) + 1}{(1 + 2\theta r - 2\theta r)} = 1$$

Highest frequency: $k_M = M\frac{\pi}{L} = \frac{L\pi}{Lh}$ $$G(k_M) = \frac{2(1-\theta)r(-1 \; - 1) + 1}{(1 + 2\theta r + 2\theta r)} = \frac{1 - 4(1-\theta)r}{1 + 4\theta r}$$

Stability requirement: $|G(k_M)| \leq 1$

$$\frac{1 - 4(1-\theta)r}{1 + 4\theta r} \geq -1 \Rightarrow (1 - 2\theta)r \leq 1/2$$

$\theta \geq 1/2$

Unconditionally stable

$\theta < 1/2$

Restriction on the timestep:

$$r < \frac{1}{2(1-2\theta)}$$

Von Neumann Stability Analysis – stability of Euler backward check positivity

Requirement to get **non-oscillatory sollution**: $G(k_M) \geq 0$

$$\frac{1-4(1-\theta)r}{1+4\theta r} \geq 0 \Rightarrow (1-2\theta)r \leq 1/2$$

Restriction on the timestep:

$$\boxed{r < \frac{1}{4(1-2\theta)}}$$

Requirement to get **positive sollution**:

$$\frac{u_{j,n+1}-u_{j,n}}{\Delta t} = \theta \frac{\beta^2}{h^2}\left(u_{j-1,n+1} - 2u_{j,n+1} + u_{j+1,n+1}\right) +$$
$$+(1-\theta)\frac{\beta^2}{h^2}\left(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}\right) + O(h^2) + O(\Delta t^p)$$

$$u_{j,n+1} = \theta \Delta t \frac{\beta^2}{h^2}\left(u_{j-1,n+1} - 2u_{j,n+1} + u_{j+1,n+1}\right) +$$
$$+(1-\theta)\Delta t \frac{\beta^2}{h^2}\left(u_{j-1,n} \boxed{-2u_{j,n}} + u_{j+1,n}\right) + u_{j,n} + O(h^2) + O(\Delta t^p)$$

$$\boxed{r < \frac{1}{2(1-2\theta)}}$$

Stability, consistency, convergence – clear up definitions

Important definitions:

Well-posedness (in the sense of Hadamard)
- solution exists
- the solution is unique
- continious dependence on the initial data

- e.g.: heat equation, Laplace-equation

Ill-posed problems

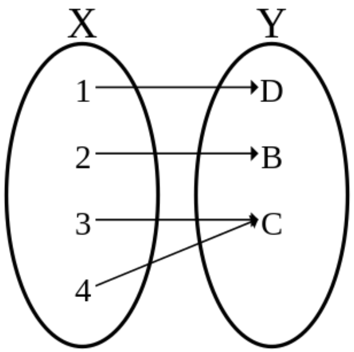
That are not well-posed in the sense of Hadamard
e.g.: inverse problems, like the inverse of the heat equation

Stability, consistency, convergence - introduction

Well-posedness differently: $\mathcal{L}x = y$ $\quad \mathcal{L}$: X→Y

Surjective

$\mathcal{L}$: X→Y is surjective, if every element y in Y has a corresponding element x in X such that f(x) = y. The function f may map more than one element of X to the same element of Y. (For all yЄY I can find a solution in X)



$$\forall y \in Y, \exists x \in X \quad \mathcal{L}x = y$$

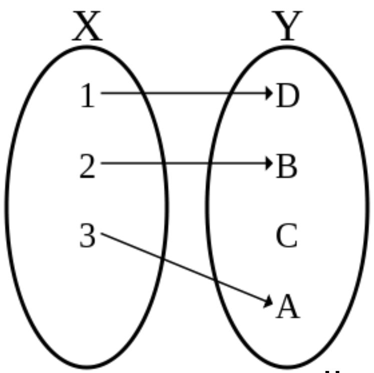
+

~~The function $g : \mathbf{R} \to \mathbf{R}$ defined by~~ $g(x) = x^2$

The function $g : \mathbf{R} \to \mathbf{R}^+$ defined by $g(x) = x^2$

Injective (one-to-one mapping)

every element of Y is the image of at most one element of X



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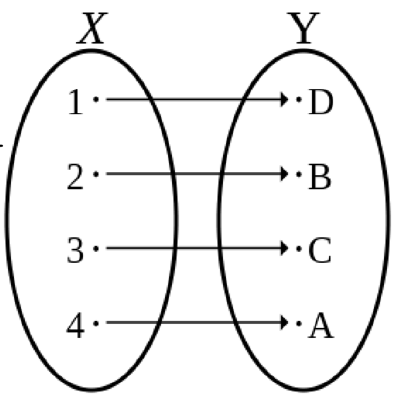
$$\mathcal{L}x_1 = \mathcal{L}x_2 \Rightarrow x_1 = x_2$$

Bijective



Continious dependence on the ininital data

The inverse/solution operator is uniformly bounded

$$\|\mathcal{L}^{-1}\| < C \Longrightarrow \|\mathbf{A}^{-1}\mathbf{y}\| < C\|\mathbf{y}\|$$

Norm of an operator(example): $\mathbf{Ax} = \mathbf{b} \quad \|\mathbf{A}\| = \sup\limits_{\substack{\mathbf{x}\in X \\ \mathbf{x}\neq 0}} \dfrac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|}$

Stability, consistency, convergence - introduction

$$\mathcal{L}u = \dot{u} + u_{xx} = f \qquad \mathcal{L}_h u_h = f_h \qquad \text{(discretized in space)}$$

Numerical stability
Even if an operator is well-posed in the sense of Hadamard, it may suffer from **numerical instability** when solved with finite precision, or with errors in the data.

$$\|\mathcal{L}^{-1}\| \leq C \quad \text{but} \quad \cancel{\left\|\mathcal{L}_h^{-1}\right\| \leq C}$$

A method is **numerically instable** if the round-off or truncation **errors** can be **amplified**, causing the error to grow exponentially

Ill-conditioned
A well-posed operator may be **ill-conditioned**, that is a small error in the initial data can result in much larger errors in the answers.
(indicated by a large condition number)

Stability, consistency, convergence - introduction

Consistency

A certain finite difference method is consistent if:

$$\lim_{\Delta t,h\to 0}\left\|\mathcal{L}(u)-\mathcal{L}_{\Delta t,h}(u)\right\|=0$$ (method approximates the differential equation)

where $\mathcal{L}(u)$: original operator

$\mathcal{L}_{\Delta t,h}(u)$: approximated operator (discretized)

<u>For example:</u> $\mathcal{L}(u)=u'$

from the Taylor expansion $u'=\dfrac{u(x+h)-u(x)}{h}+O(h)$

$$\mathcal{L}_h(u)=\frac{u(x+h)-u(x)}{h}$$ (first order method)

$$\left\|u'-\frac{u(x+h)-u(x)}{h}\right\|\le Ch \Longrightarrow \lim_{h\to 0}\|\mathcal{L}(u)-\mathcal{L}_h(u)\|=0$$

Stability, consistency, convergence - introduction

Convergence

A finite difference method is convergent if:

$$\lim_{\Delta t, h \to 0} \|u - u_h\| = 0$$

Solution of the FD method (numerical approximation) gets closer to the exact solution of the PDE as the discretisation is made finer.

where u: analytical solution

u_h: approximated solution

Difficult to show, **but**

<u>Lax Richtmyer theorem</u>
A **consistent** finite difference method for a **well-posed, linear** initial value problem is convergent if and only if it is **stable**.

→ Instead of analysing convergency check consistency and stability

Stability, consistency, convergence - introduction

Lax Richtmyer theorem

A **consistent** finite difference method for a **well-posed, linear** initial value problem is convergent if and only if it is **stable**.

$$\|u-u_h\| = \left\|\mathcal{L}^{-1}f - \mathcal{L}_h^{-1}f_h\right\|$$

convergency:

$$\lim_{h\to 0}\|u-u_h\| = 0$$

$$\|u-u_h\| = \left\|\mathcal{L}^{-1}f \underbrace{- \mathcal{L}_h^{-1}f + \mathcal{L}_h^{-1}f}_{=0} - \mathcal{L}_h^{-1}f_h\right\|$$

$$\leq \left\|\mathcal{L}^{-1}f - \mathcal{L}_h^{-1}f\right\| + \left\|\mathcal{L}_h^{-1}f - \mathcal{L}_h^{-1}f_h\right\|$$ (triangular inequality)

$$= \left\|\mathcal{L}_h^{-1}(\mathcal{L}_h u - \mathcal{L}u)\right\| + \left\|\mathcal{L}_h^{-1}(f - f_h)\right\|$$ (linearity of $\mathcal{L}_h^{-1}$)

$$\leq \mathrm{C}\underbrace{\|\mathcal{L}_h u - \mathcal{L}u\|}_{\to 0 \text{ as } h\to 0} + \mathrm{C}\underbrace{\|f - f_h\|}_{\to 0 \text{ as } h\to 0}$$

(stability)

(consistency)

$$\left\|\mathcal{L}^{-1}f - \mathcal{L}_h^{-1}f\right\| = \left\|\mathcal{L}^{-1}\mathcal{L}u - \mathcal{L}_h^{-1}\mathcal{L}u\right\| = \left\|u - \mathcal{L}_h^{-1}\mathcal{L}u\right\| = \left\|\underbrace{\mathcal{L}_h^{-1}\mathcal{L}_h}_{=1} u - \mathcal{L}_h^{-1}\mathcal{L}u\right\| = \left\|\mathcal{L}_h^{-1}(\mathcal{L}_h u - \mathcal{L}u)\right\|$$

(linearity of $\mathcal{L}_h^{-1}$)

Consistency

Check consistency

Derivatives are approximated with the help of the Taylor series

1. Derivation of a consistent finite difference operator

Example: derivation of $u'(x)$ used in the Richardson scheme

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \qquad (1)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \qquad (2)$$

Subtracting from eq. (1) eq. (2) results in:

$$u(x+h) - u(x-h) = 2u'(x)h + O(h^3)$$

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

$$\boxed{u'_k = \frac{u_{k+1} - u_{k-1}}{2h} + O(h^2)}$$

Consistency

2. Check consistency of an already defined scheme

Example: prove consistency of the DuFort-Frankel scheme

$$\frac{u_{n+1,j}-u_{n-1,j}}{2\Delta t}-\frac{\beta^2}{h^2}(u_{n,j-1}-(u_{n-1,j}+u_{n+1,j})+u_{n,j+1})=0$$

$$\begin{aligned}
u_{n+1,j} &= u(t+\Delta t,x)=u_{n,j}+\frac{\partial u}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2+O(\Delta t^3)\\
u_{n-1,j} &= u(t-\Delta t,x)=u_{n,j}-\frac{\partial u}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2+O(\Delta t^3)\\
u_{n,j+1} &= u(t,x+h)=u_{n,j}+\frac{\partial u}{\partial x}h+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}h^2+O(h^3)\\
u_{n,j-1} &= u(t,x-h)=u_{n,j}-\frac{\partial u}{\partial x}h+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}h^2+O(h^3)
\end{aligned}$$

$$\frac{\frac{\partial u}{\partial t}\Delta t+O(\Delta t^3)}{\Delta t}-\frac{\beta^2(2u_{n,j}+\frac{\partial^2 u}{\partial x^2}h^2+O(h^3))}{h^2}+\frac{\beta^2(2u_{n,j}+\frac{\partial^2 u}{\partial t^2}\Delta t^2+O(\Delta t^3))}{h^2}=0$$

Consistency

$$\frac{\frac{\partial u}{\partial t}\Delta t + O(\Delta t^3)}{\Delta t} - \frac{\beta^2(2u_{n,j} + \frac{\partial^2 u}{\partial x^2}h^2 + O(h^3))}{h^2} + \frac{\beta^2(2u_{n,j} + \frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3))}{h^2} = 0$$

$$\frac{\partial u}{\partial t} - \beta^2 \frac{\partial^2 u}{\partial x^2} + E = 0$$

$$E = \frac{\beta^2 \Delta t^2}{h^2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^2) + O(h)$$

The method is consistent if: $\lim_{\Delta t, h \to 0} |E| = 0$

The second and the last term will tend to zero as discretisation if refined, but the last term will only be zero if

$$\lim_{\Delta t, h \to 0} \left|\frac{\Delta t}{h}\right| = 0$$

For example if the stability condition of Euler forward is satisfied:

$\Delta t < \frac{h^2}{2\beta^2}$ $\Rightarrow$ $\Delta t = O(h^2)$ $\Rightarrow$ scheme is consistent

Stability

Stability checking from eigenvalue analysis:

- Method of lines

$$\mathbf{u}(t) = \sum_{j=1}^{N-1} \beta_j(t)\mathbf{v}_j \quad \beta_j(t) = \beta_j^0 e^{\lambda_j t}$$

$$\mathrm{eig}(\mathbf{A}) \le 0 \quad \Rightarrow$$ unconditionally satisfied

- Euler forward method

$$\mathbf{u}_{n+1} = \underbrace{(\mathrm{I} + \Delta t\mathbf{A})}_{\mathbf{B}}\mathbf{u}_n \qquad |\mathrm{eig}(\mathbf{B})| \le 1 \quad \Rightarrow \quad \Delta t < \frac{h^2}{2\beta^2}$$

- Euler backward method

$$\mathbf{u}_n = \underbrace{(\mathrm{I} - \Delta t\mathbf{A})}_{\mathbf{B}_1}\mathbf{u}_{n+1} \qquad \left|\mathrm{eig}(\mathbf{B}_1^{-1})\right| \le 1 \quad \Rightarrow$$ unconditionally satisfied

- Theta method

$$\underbrace{(\mathbf{I} - \theta\Delta t\mathbf{A})}_{\mathbf{B}_1}\mathbf{u}_{n+1} = \underbrace{(\mathbf{I} + (1-\theta)\Delta t\mathbf{A})}_{\mathbf{B}_2}\mathbf{u}_n$$

$$\left|\mathrm{eig}(\mathbf{B}_1^{-1}\mathbf{B}_2)\right| \le 1 \quad \Rightarrow$$

for $\theta \ge 1/2$**:** unconditionaly stable

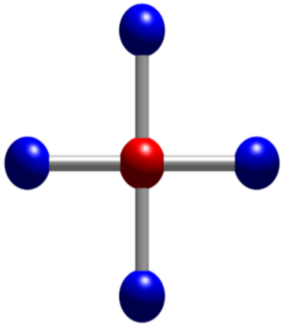
for $\theta < 1/2$**:** $\dfrac{\beta^2\Delta t}{h^2} < \dfrac{1}{2(1-2\theta)}$

Instationary heat equation in 3D

Let's consider the heat equation in 2 dimensions: $$\frac{\partial u}{\partial t} - \beta^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f$$

$$\begin{aligned}
\frac{\partial^2 u_{j,l}}{\partial x^2} &= \frac{1}{\Delta x^2}(u_{j-1,l} - 2u_{j,l} + u_{j+1,l}) \\
\frac{\partial^2 u_{j,l}}{\partial y^2} &= \frac{1}{\Delta y^2}(u_{j,l-1} - 2u_{j,l} + u_{j,l+1})
\end{aligned}$$

$$\begin{aligned}
x &= j \cdot \Delta x \\
y &= l \cdot \Delta y.
\end{aligned}$$



$$\frac{\partial u_{j,l}}{\partial t} - \frac{\beta^2}{\Delta x^2}(u_{j-1,l} - 2u_{j,l} + u_{j+1,l}) - \frac{\beta^2}{\Delta y^2}(u_{j,l-1} - 2u_{j,l} + u_{j,l+1}) = f$$

If dx=dy: $$\frac{\partial u_{j,l}}{\partial t} - \frac{\beta^2}{h^2}(-4u_{j,l} + u_{j-1,l} + u_{j+1,l} + u_{j,l-1} + u_{j,l+1}) = f.$$

After time discr. with theta method:

$$\begin{aligned}
\frac{u^{n+1}_{j,l} - u^n_{j,l}}{\Delta t} = \frac{\beta^2}{h^2}\big((1-\theta)&(-4u^n_{j,l} + u^n_{j-1,l} + u^n_{j+1,l} + u^n_{j,l-1} + u^n_{j,l+1}) + \\
\theta&(-4u^{n+1}_{j,l} + u^{n+1}_{j-1,l} + u^{n+1}_{j+1,l} + u^{n+1}_{j,l-1} + u^{n+1}_{j,l+1})\big) + f
\end{aligned}$$

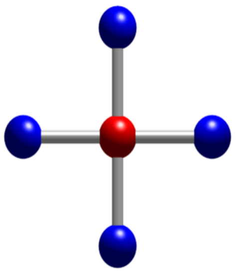
Instationary heat equation in 3D

$\mathbf{A}$: triangular matrix $\Rightarrow$ $\left.\begin{array}{l}\mathbf{B}_1 = (\mathrm{I} - \Delta t\mathbf{A}) \\ \mathbf{B}_{1\theta} = (\mathbf{I} - \theta\Delta t\mathbf{A})\end{array}\right\}$ tridiagonal matrices

Recall **2D instationary heat equation** : $\frac{\partial u}{\partial t} - \beta^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f$ (See Tutorial 3.)

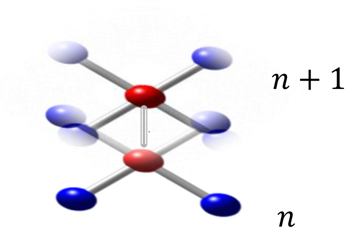
$$\frac{\partial u_{j,l}}{\partial t} - \frac{\beta^2}{\Delta x^2}(u_{j-1,l} - 2u_{j,l} + u_{j+1,l}) - \frac{\beta^2}{\Delta y^2}(u_{j,l-1} - 2u_{j,l} + u_{j,l+1}) = f$$

If dx=dy: $$\frac{\partial u_{j,l}}{\partial t} - \frac{\beta^2}{h^2}(-4u_{j,l} + u_{j-1,l} + u_{j+1,l} + u_{j,l-1} + u_{j,l+1}) = f.$$



After time discr. with theta method:

$$\frac{u_{j,l}^{n+1} - u_{j,l}^{n}}{\Delta t} = \frac{\beta^2}{h^2}((1-\theta)(-4u_{j,l}^{n} + u_{j-1,l}^{n} + u_{j+1,l}^{n} + u_{j,l-1}^{n} + u_{j,l+1}^{n}) + \theta(-4u_{j,l}^{n+1} + u_{j-1,l}^{n+1} + u_{j+1,l}^{n+1} + u_{j,l-1}^{n+1} + u_{j,l+1}^{n+1})) + f$$



Instationary heat equation in 3D

$$f \coloneqq 0 \qquad \Delta t \frac{\beta^2}{h^2} := r$$

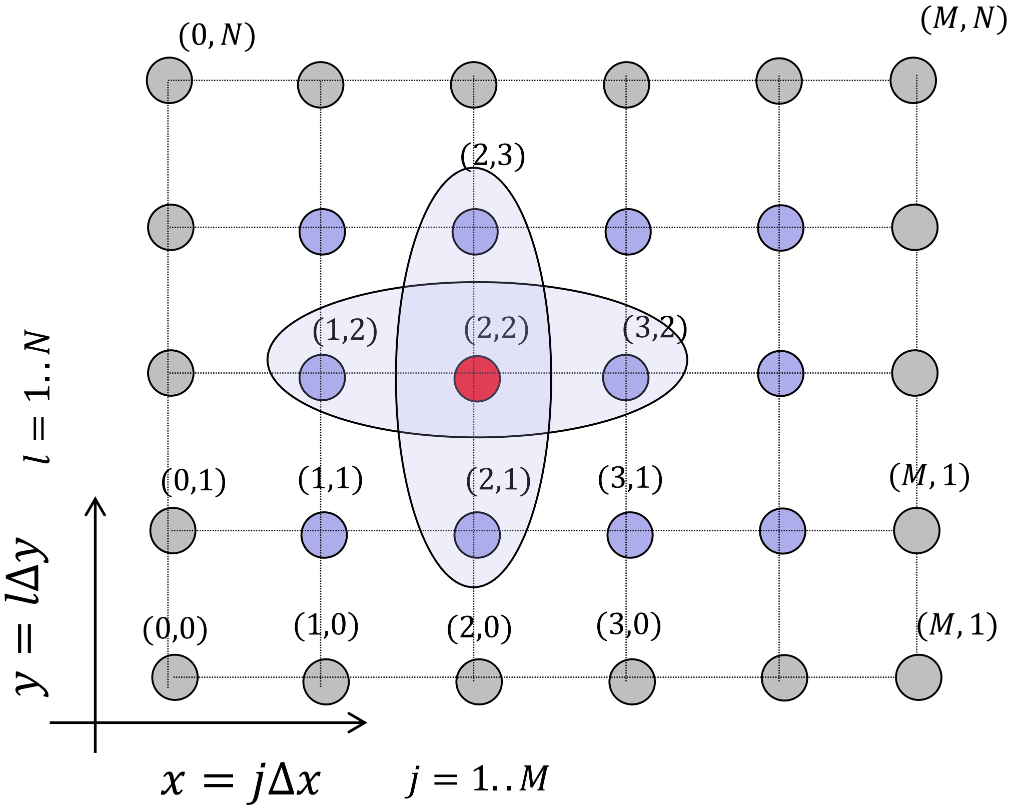
$$u^{n+1}{}_{j,l} - r\theta\left(-4u^{n+1}{}_{j,l} + u^{n+1}{}_{j-1,l} + u^{n+1}{}_{j+1,l} + u^{n+1}{}_{j,l-1} + u^{n+1}{}_{j,l+1}\right) =$$
$$= u^{n}{}_{j,l} + r(1-\theta)\left(-4u^{n}{}_{j,l} + u^{n}{}_{j-1,l} + u^{n}{}_{j+1,l} + u^{n}{}_{j,l-1} + u^{n}{}_{j,l+1}\right)$$

$\theta = 1$ Euler backward method

$$u^{n+1}{}_{j,l} - r\left(-4u^{n+1}{}_{j,l} + u^{n+1}{}_{j-1,l} + u^{n+1}{}_{j+1,l} + u^{n+1}{}_{j,l-1} + u^{n+1}{}_{j,l+1}\right) = u^{n}{}_{j,l}$$

Instationary heat equation in 3D

$$\mathbf{u}^n = \begin{bmatrix} {u^n}_{1,1} \\ {u^n}_{2,1} \\ {u^n}_{3,1} \\ \vdots \\ {u^n}_{M-1,1} \\ {u^n}_{1,2} \\ {u^n}_{2,2} \\ \vdots \\ {u^n}_{M-1,2} \\ \vdots \\ {u^n}_{1,N-1} \\ \vdots \\ {u^n}_{M-1,N-1} \end{bmatrix} \qquad \mathbf{u}^n = \begin{bmatrix} {u^n}_{1,1} \\ {u^n}_{2,1} \\ {u^n}_{3,1} \\ {u^n}_{4,1} \\ {u^n}_{1,2} \\ {u^n}_{2,2} \\ {u^n}_{3,2} \\ {u^n}_{4,2} \\ {u^n}_{1,3} \\ {u^n}_{2,3} \\ {u^n}_{3,3} \\ {u^n}_{4,3} \end{bmatrix}$$



with homogenous Dirichlet BC.

$$ {u^{n+1}}_{j,l} - r\left(-4{u^{n+1}}_{j,l} + {u^{n+1}}_{j-1,l} + {u^{n+1}}_{j+1,l} + {u^{n+1}}_{j,l-1} + {u^{n+1}}_{j,l+1}\right) = {u^n}_{j,l} $$

Instationary heat equation in 3D

$$u^{n+1}{}_{j,l} - r\left(-4u^{n+1}{}_{j,l} + u^{n+1}{}_{j-1,l} + u^{n+1}{}_{j+1,l} + u^{n+1}{}_{j,l-1} + u^{n+1}{}_{j,l+1}\right) = u^{n}{}_{j,l}$$

with homogenous Dirichlet BC.

$$+r\begin{array}{cccc|cccc|cccc}
4 & -1 & & & -1 & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & \\
 & -1 & 4 & -1 & & & -1 & & & & & \\
 & & -1 & 4 & & & & -1 & & & & \\
\hline
-1 & & & & 4 & 1 & & & -1 & & & \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
 & & -1 & & & -1 & 4 & -1 & & & -1 & \\
 & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
 & & & & -1 & & & & 4 & -1 & & \\
 & & & & & -1 & & & -1 & 4 & -1 & \\
 & & & & & & -1 & & & -1 & 4 & -1 \\
 & & & & & & & -1 & & & -1 & 4
\end{array}
\begin{bmatrix}
u^{n+1}{}_{1,1} \\ u^{n+1}{}_{2,1} \\ u^{n+1}{}_{3,1} \\ u^{n+1}{}_{4,1} \\ u^{n+1}{}_{1,2} \\ u^{n+1}{}_{2,2} \\ u^{n+1}{}_{3,2} \\ u^{n+1}{}_{4,2} \\ u^{n+1}{}_{1,3} \\ u^{n+1}{}_{2,3} \\ u^{n+1}{}_{3,3} \\ u^{n+1}{}_{4,3}
\end{bmatrix}$$

Sparse matrix with bandwidth: 2M-1 (here 9) BUT the band itself is sparse, only five diagonals are nonzero

Instationary heat equation in 3D

$$u^{n+1}{}_{j,l} - r\left(-4u^{n+1}{}_{j,l} + u^{n+1}{}_{j-1,l} + u^{n+1}{}_{j+1,l} + u^{n+1}{}_{j,l-1} + u^{n+1}{}_{j,l+1}\right) = u^{n}{}_{j,l}$$

$$\mathbf{B_1}\mathbf{u}^{n+1} = \mathbf{u}^{n}$$

where

$$\mathbf{B_1} = \begin{bmatrix} B & C & & \\ C & B & C & \\ & \ddots & \ddots & \ddots \\ & & C & B \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1+4r & -r & & \\ -r & 1+4r & -r & \\ & \ddots & \ddots & \ddots \\ & & -r & 1+4r \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} -r & & & \\ & -r & & \\ & & \ddots & \\ & & & -r \end{bmatrix}$$

Reminder:
Stationary heat equation – what to solve?

Instationary heat equation with constant BC, and source term approaches a stationary state:

$$\frac{\partial}{\partial t}u(x,y,z,t) - \beta^2 \Delta u(x,y,z,t) = f(x,y,z) \quad \text{(parabolic)}$$

$$u(x,y,z,t) \to \tilde{u}(x,y,z) \quad \text{as} \quad t \to \infty$$

Equilibrium equation (stationary heat equation):

$$\frac{\partial}{\partial t}\tilde{u}(x,y,z) = 0 \quad \Rightarrow \quad -\beta^2 \Delta \tilde{u}(x,y,z) = f(x,y,z) \quad \text{(elliptic)}$$

Discretised form:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{A}\mathbf{u} = \mathbf{f} \qquad \longrightarrow \qquad \mathbf{A}\mathbf{u} = \mathbf{f}$$

Reminder:
Stationary heat equation – what to solve?

Conclusion

- instationary heat equation with implicit FD methods (Euler backward, Theta method)
- stationary heat equation

System of linear equations:

$$\mathbf{G}\mathbf{x} = \mathbf{b}$$

solve for $\mathbf{x}$

Where the G matrix is in general

- **sparse, banded**
- can get very **large** with refined spatial and temporal discretisation
- for 1D heat equation with three-point-stencils: **tridiagonal**
- for 1D heat equation with five-point-stencils: **pentadiagonal**
- for 2D heat equation: **banded with sparse band**