Introduction to PDEs and Numerical Methods

Lecture 1:

Introduction

Dr. Noemi Friedman, 18.10.2017.

Basic information on the course

- Course Title:

  Introduction to PDEs and Numerical Methods

- Lecturers:

  Noémi Friedman
  n.friedman@tu-bs.de
  Mühlenpfordtstr. 23, 8th floor
  Room: 819

  Jaroslav Vondřejc
  j.vondrejc@tu-bs.de
  Mühlenpfordtstr. 23, 8th floor
  Room: 822

- Assistant (small tutorials):

Basic information on the course

- Credits and work load:

  5 credits: 6-7 hours/week

- Pre-requisits:

  Differential operators,
  elementary knowledge of PDEs,
  basics of linear algebra,
  basic coding skills

- Requisits:

  Weekly assignments in group of two or three (min 50%)
  Written exam:

- Software used:

  MATLAB, FEniCS (Python interface)

Recommended literature

- Script:
  See webpage: https://www.tu-braunschweig.de/wire/lehre/ws17/pde1

- Highly recommended book:
  Mark S. Gockenbach: Partial Differential Equations, Analytical and Numerical Methods

- Other usefull literature:
  - Michael T. Heath: Scientific Computing, an Introductory Survey
  - Alfio Quarteroni, Fausto Saleri: Scientific Computing with MATLAB

- On-line brush-up the basics:
  Essence of linear algebra (youtube, 3blue1brown, essence of linear algebra), link: https://www.youtube.com/watch?v=kjBOesZCoqc&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
  Essence of calculus (youtube, 3blue1brown, essence calculus), link: https://www.youtube.com/watch?v=kjBOesZCoqc&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
  Basics about PDEs (youtube, commutant, PDE1)
  : https://www.youtube.com/watch?v=LYsIBqjQTdI&list=PLF6061160B55B020

Information about the assignments

- Homework assignments in groups (max. group of three)
- Submission of homework
    - Written homework

      Submit on the beginning of the tutorial

      (include cover sheet with subject name (PDE1), matriculation number of students, assignment number, date)
    - For program codes:

      e-mail: wire.pde@gmail.com

      subject: assignment# NAMES

      (#: number of the assignment, NAMES: names of students)

      (e.g.: assignment1 J. Smith, K. Park)

- Consultation:
    - Jaroslav Vondřejc
    - Noemi Friedman (after the lecture, office hours by arrangement, please, take appointment first by e-mail: n.friedman@tu-bs.de)

Definition: ODEs PDEs

- **Partial Differential Equation:**
Equation specifying a relation between the partial derivative(s) of an unknown multivariable function and maybe the function itself:

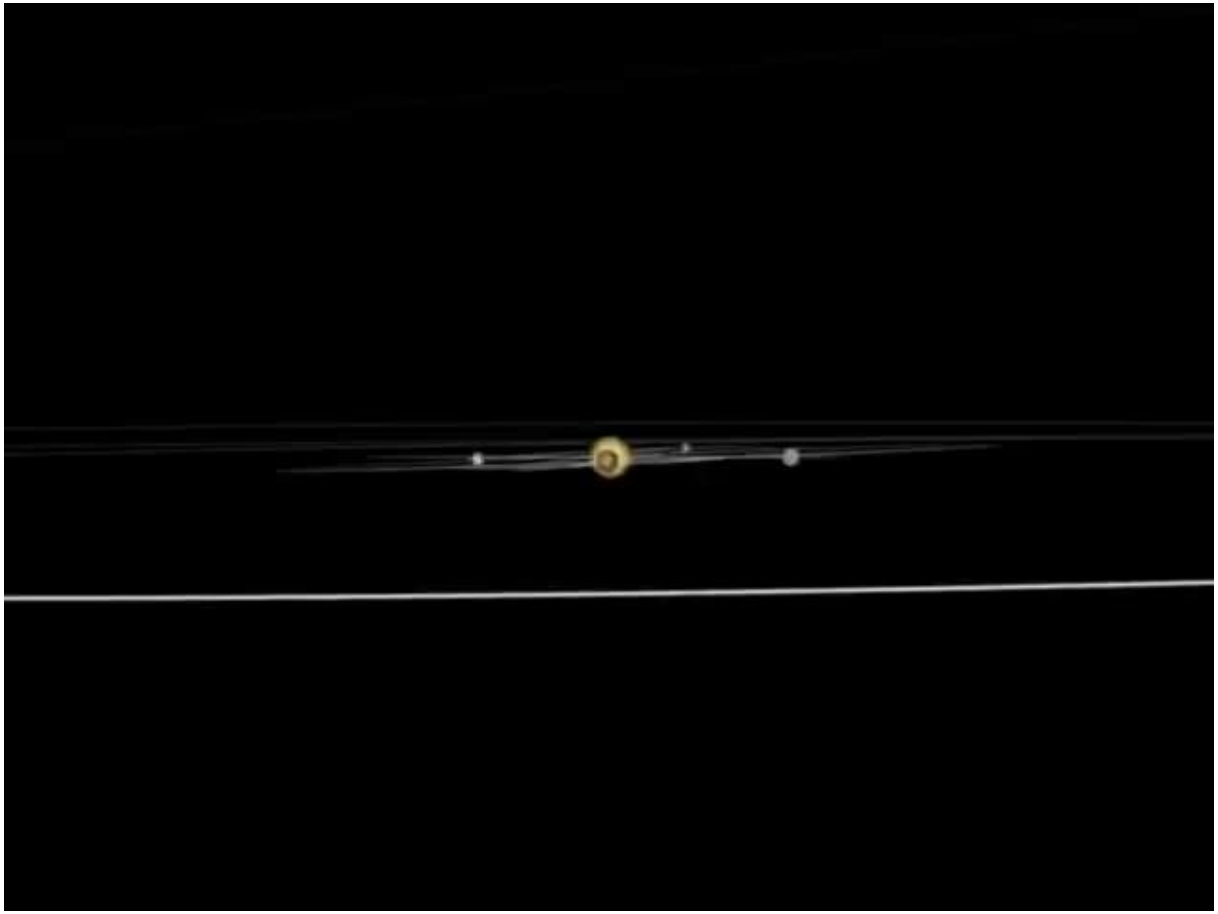
$$F\left(u(x,y,z,t), \frac{\partial u(x,y,z,t)}{\partial x}, \frac{\partial u(x,y,z,t)}{\partial y}, \ldots \frac{\partial^2 u(x,y,z,t)}{\partial xy}, \ldots\right) = f(x,y,z,t)$$

- **Ordinary Differential Equation:**
Equation specifying a relation between the derivative(s) of an unknown **univariable function** and maybe the function itself:

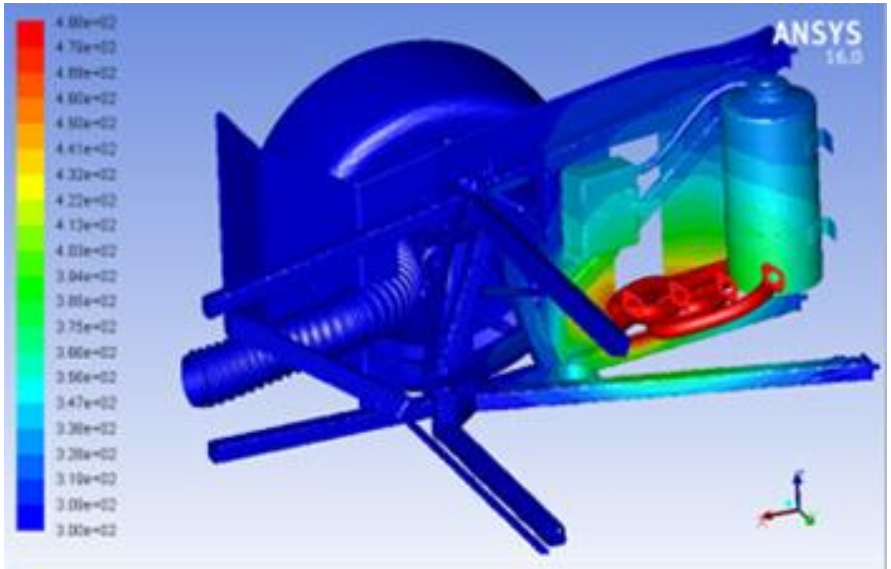
$$F\left(u(t), \frac{du(t)}{\partial t}, \qquad \frac{d^2u(t)}{dt^2}, \ldots\right) = f(t)$$

Boundary Value Problem (BVP), Initial Boundary Value Problem (IBVP): PDE with initial/boundary conditions

Motivation – simulation of planets



Motivation – heat convection

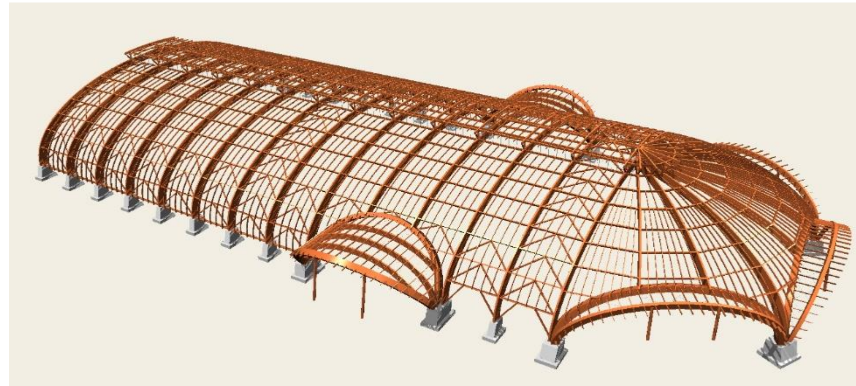


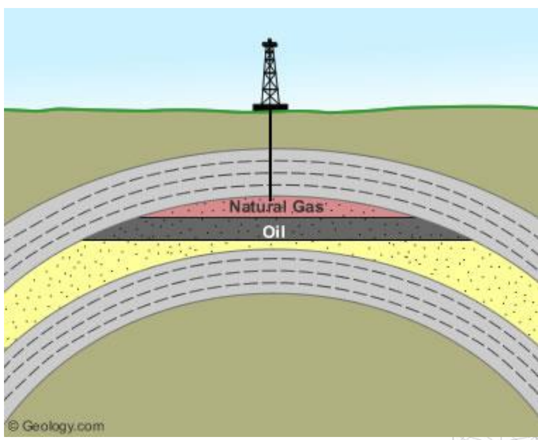
computed surface temperatures due to convective and radiative heat transfer from the exhaust manifold to surrounding objects

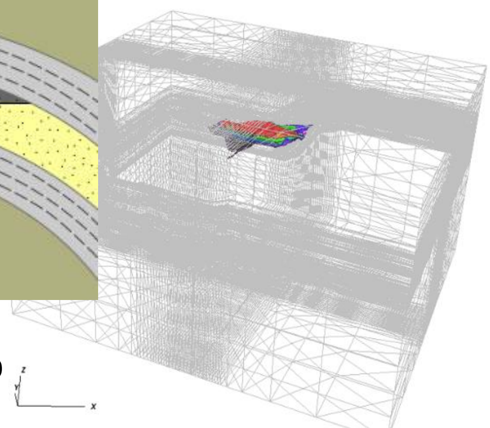
Source:
ANSYS http://www.ansys.com/staticassets/ANSYS/staticassets/product/16-highlights/underhood-simulation-surface-temps-heat-transfer-manifold-2.jpg

Motivation – structural analysis

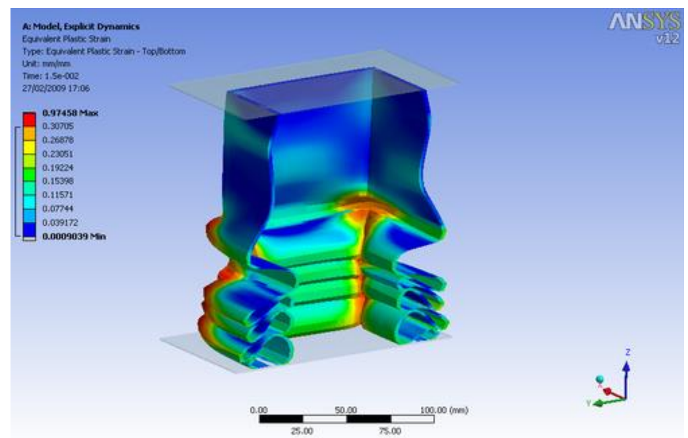
Source: Noemi Friedman







Source: Claudia Zaccarato



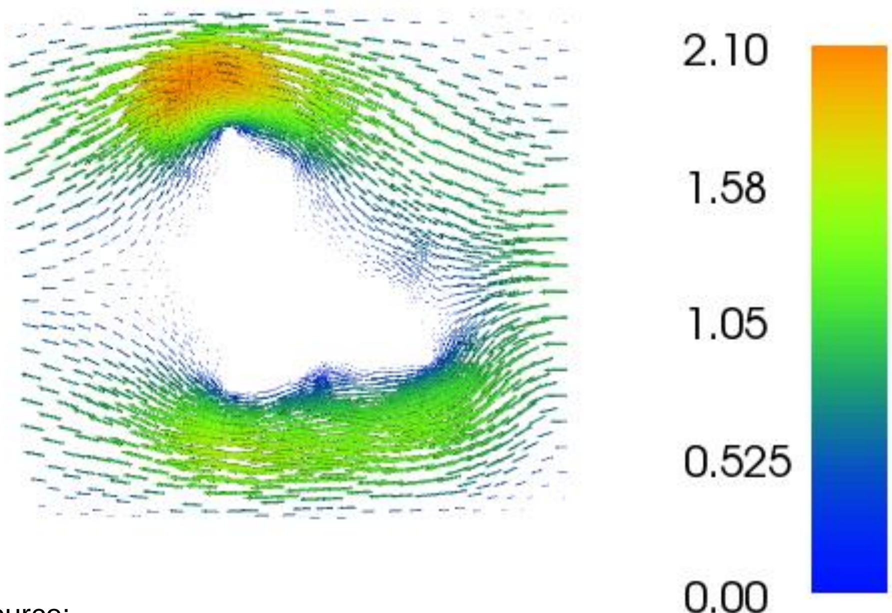
Source: ANSYS
http://wildeanalysis.co.uk/system/photos/838/preview/ansys_explicit_str.png?1273430962

Motivation – flow problems

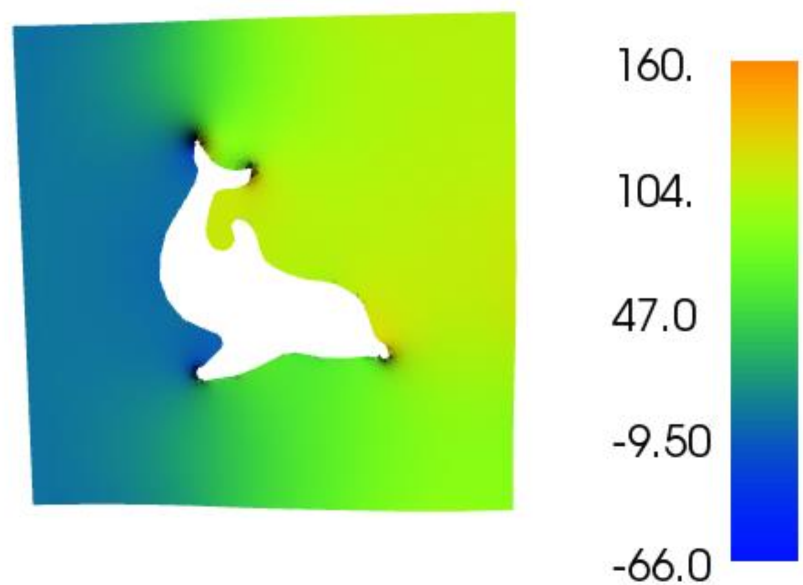
- The Stokes equation

$$-\nabla \cdot (\nabla u + p I) = f \qquad \text{in } \Omega$$
$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

$u(\mathrm{x})$



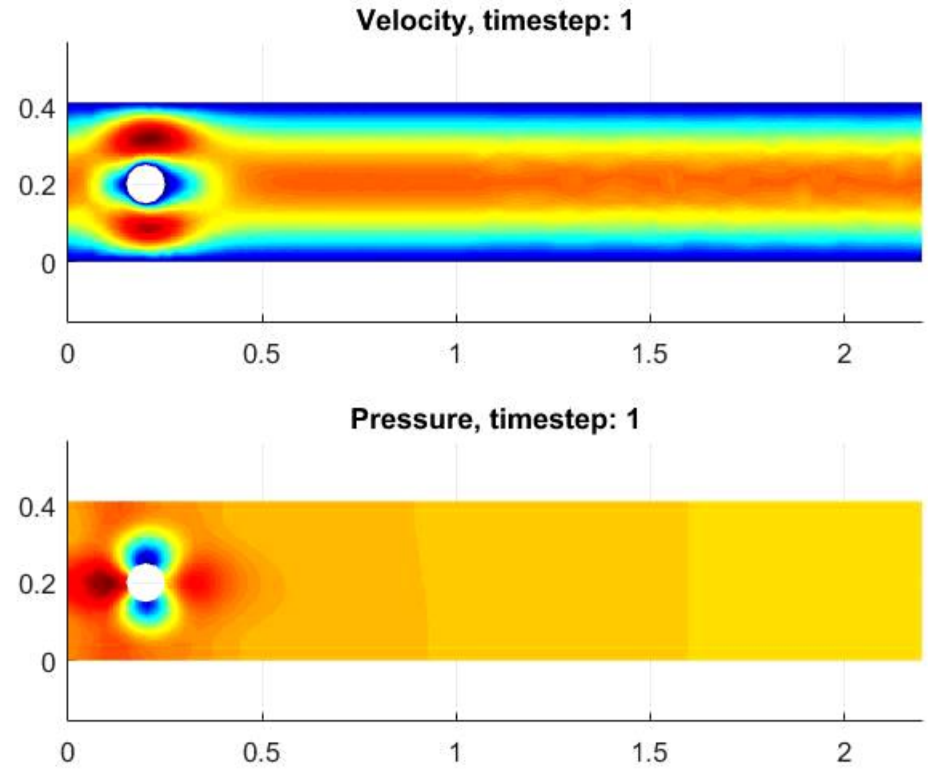
$p(\mathrm{x})$



Source:
FENICS documentation:
http://fenicsproject.org/documentation/dolfin/1.6.0/python/demo/documented/stokes-taylor-hood/python/documentation.html

Motivation – flow problems

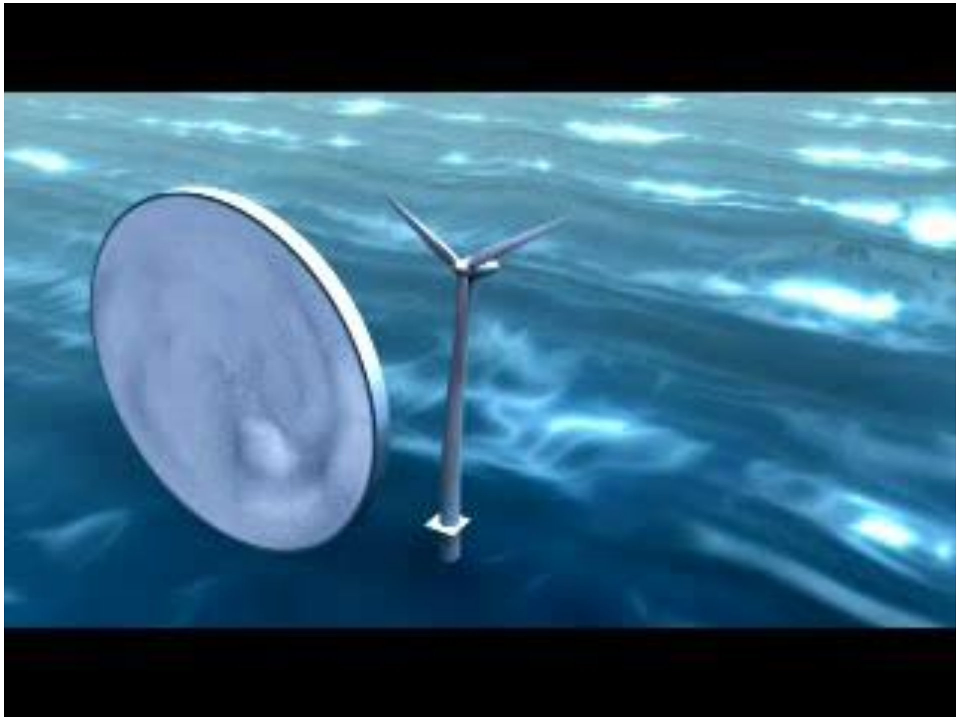
- The Navier-Stokes equation



Source:
WIRE

Motivation – flow-structure interaction

- The Navier Stokes equation



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Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)

- Analitical solution of elementary PDEs (Fourier series/transform, seperation of variables, Green’s function)

- Numerical solutions of PDEs:

  - Finite difference method

  - Finite Element Method (FEM)

  - Basics of numerical analysis
    - Convergence/consistence/stability
    - Solving linear system of equations

Overview of this lecture

- Basic definitions, motivation
- Differential operators: basic notations, divergence, Laplace, curl, grad

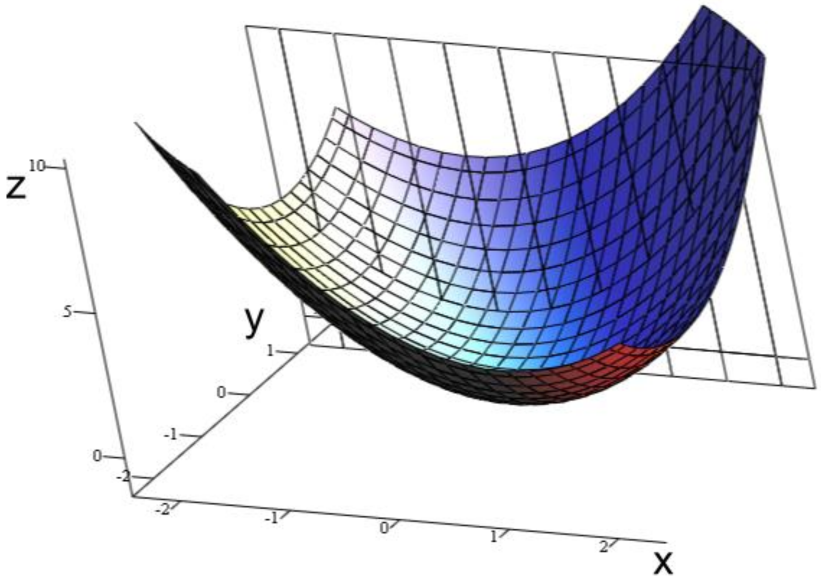
Differential operators – partial derivative

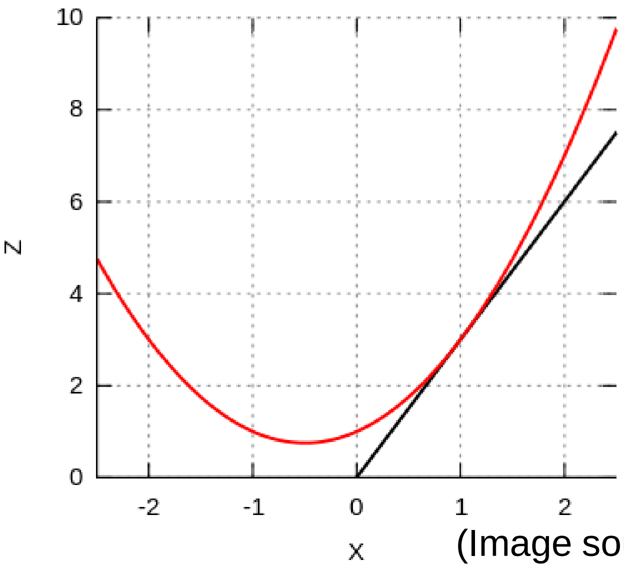
$\varphi(x, y, z, t)$

- Partial derivative: $\frac{\partial \varphi}{\partial x}, \partial_x \varphi, \ \varphi_x, \frac{\partial}{\partial x}\varphi, \ \varphi_{,x}, (\varphi')$ $\qquad \frac{\partial \varphi}{\partial t} = \dot{\varphi}$

Example:

$$z = \varphi(x, y) = x^2 + xy + y^2 \qquad \frac{\partial \varphi}{\partial x} = 2x + y \qquad \left.\frac{\partial \varphi}{\partial x}\right|_{x=1, y=1} = 3$$





(Image source: Wikipedia)

Differential operators – mixed derivative

$\varphi(x, y)$

- Mixed derivative: $\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$

Example:

$$f(x, y, z) = xy^2 \cos(z)$$

$$\frac{\partial f}{\partial x} = y^2 \cos(z) \qquad \frac{\partial^2 \varphi}{\partial x \partial y} = 2y\cos(z)$$

$$\frac{\partial f}{\partial y} = 2xy \cos(z) \qquad \frac{\partial^2 \varphi}{\partial y \partial x} = 2y\cos(z)$$

Differential operators – total derivative

$$\varphi(\boldsymbol{r}) = \varphi(x, y, z, t)$$

- Total derivative: $\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t}\frac{dt}{dt} + \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt}$
- Total differential (differential change of f): $d\varphi = \frac{\partial\varphi}{\partial t}dt + \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$

Example:

$\varphi(x, y) = x^2 + 2y$

$y(x) = x$

$\frac{\partial\varphi}{\partial x} = 2x$ partial derivative

$\frac{d\varphi}{dx} = \frac{\partial\varphi}{\partial x}\frac{dx}{dx} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = 2x + 2$ total derivative

Substantial derivative of flowfield quantities ($\mathrm{e.g.}\ p_s$: pressure observed by drifting sensor)

$$\frac{dp_s}{dt} = \frac{\partial p_s}{\partial t}\frac{dt}{dt} + \frac{\partial p_s}{\partial x}\frac{dx}{dt} + \frac{\partial p_s}{\partial y}\frac{dy}{dt} + \frac{\partial p_s}{\partial z}\frac{dz}{dt} = \frac{\partial p_s}{\partial t} + \frac{\partial p_s}{\partial x}u + \frac{\partial p_s}{\partial y}v + \frac{\partial p_s}{\partial z}w$$

$$= \frac{\partial p_s}{\partial t} + \mathrm{V} \cdot \nabla p_s$$

Differential operators – gradient

$\varphi(\boldsymbol{r}) = \varphi(x, y, z)$: $\mathbb{R}^3 \to \mathbb{R}$ (vector-scalar function)

- Nabla operator:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

- **direction:** greatest rate of increase of the function
- **magnitude:** the slope of the function in that direction

- Gradient:

$$\operatorname{grad} \varphi = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{e}_x + \frac{\partial\varphi}{\partial y}\vec{e}_y + \frac{\partial\varphi}{\partial z}\vec{e}_z = \begin{pmatrix} \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \\ \frac{\partial\varphi}{\partial z} \end{pmatrix}$$

Example:

$$f_1(x, y, z) = xy^2\cos(z)$$

$$\nabla f_1 = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f_1(x, y, z) = \begin{bmatrix} y^2\cos(z) \\ 2xy\cos(z) \\ -xy^2\sin(z) \end{bmatrix}$$

Differential operators – directional derivative

$$\varphi(\boldsymbol{r}) = \varphi(x,y,z)\colon \mathbb{R}^3 \to \mathbb{R}$$ (vector-scalar function)

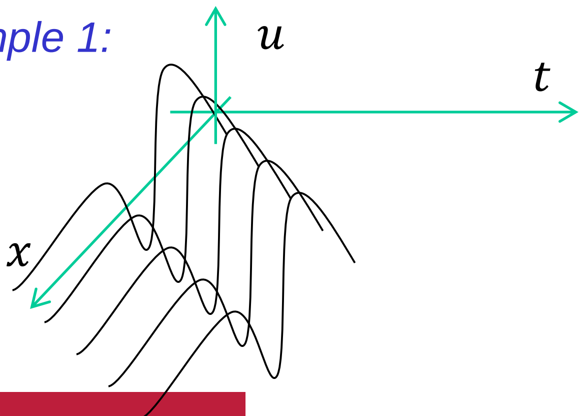
- Directional derivative:

$$D_{\boldsymbol{v}}\varphi(\boldsymbol{r}) = \lim_{h\to 0}\frac{\varphi(\boldsymbol{r}+h\boldsymbol{v})-\varphi(\boldsymbol{r})}{h} = \boldsymbol{v}\cdot\nabla\varphi(\boldsymbol{r}) = \boldsymbol{v}^T\nabla\varphi(\boldsymbol{r}) = \begin{bmatrix} v_x & v_y & v_z\end{bmatrix}\begin{bmatrix}\frac{\partial\varphi(x,y,z)}{\partial x}\\ \frac{\partial\varphi(x,y,z)}{\partial y}\\ \frac{\partial\varphi(x,y,z)}{\partial z}\end{bmatrix}$$

(normalised):

$$D_{\boldsymbol{v}}\varphi(\boldsymbol{r}) = \lim_{h\to 0}\frac{\varphi(\boldsymbol{r}+h\boldsymbol{v})-\varphi(\boldsymbol{r})}{h|\boldsymbol{v}|} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}\cdot\nabla\varphi(\boldsymbol{r})$$

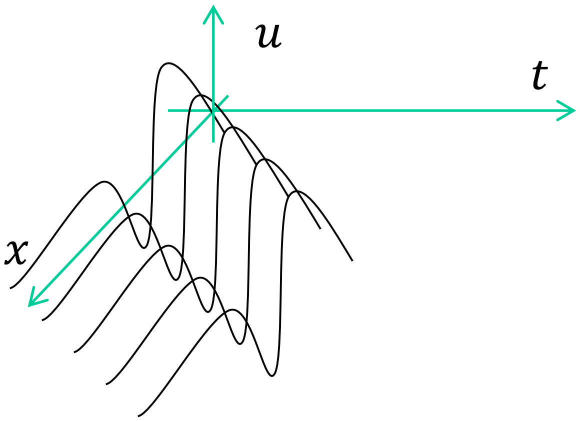
Example 1:



What is the differential equation to define a wave traveling with speed c?
In the direction $x-ct$ u is constant→ directional derivative is zero:

$$D_{\boldsymbol{v}}u(x,t) = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}\cdot\nabla u(x,t) = 0 \quad (\boldsymbol{v}\cdot\nabla u(x,t) = 0)$$

Differential operators – directional derivative



$$\boldsymbol{v} = \begin{bmatrix} c \\ 1 \end{bmatrix} \qquad \boldsymbol{v} \cdot \nabla\, u(x,t) = 0$$

$$\boldsymbol{v} \cdot \nabla\, u(x,t) = \boldsymbol{v}^T\, \nabla\, u(x,t) = [c \quad 1] \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial t} \end{bmatrix} = 0$$

Transport equation: $u_t + cu_x = 0$

Example 2:

Let's suppose $u = \sin(x - ct)$ is a solution of the transport equation.
What is its directional derivative in the direction: $\boldsymbol{v} = \begin{bmatrix} c \\ 1 \end{bmatrix}$

$$\boldsymbol{v} \cdot \nabla\, u(x,t) = \boldsymbol{v}^T\, \nabla\, u(x,t) = [c \quad 1] \begin{bmatrix} \cos(x - ct) \\ -c\cos(x - ct) \end{bmatrix} = 0$$

Differential operators - divergence

- Divergence:

of $\boldsymbol{g}(x,y,z)\colon \mathbb{R}^3 \to \mathbb{R}^3$
(of a vector field):

$$\boldsymbol{g}(x,y,z) = \begin{bmatrix} g_x(x,y,z) \\ g_y(x,y,z) \\ g_z(x,y,z) \end{bmatrix} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

$$\mathrm{div}\boldsymbol{g}(x,y,z) = \nabla \cdot \boldsymbol{g}(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

Example:

$$\mathrm{f}_2(x,y,z) = (xy^2, y^2z^3, xyz)^T$$

$$\nabla \cdot \mathrm{f}_2 = div\mathrm{f}_2 = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} xy^2 \\ y^2z^3 \\ xyz \end{bmatrix} = \frac{\partial(xy^2)}{\partial x} + \frac{\partial(y^2z^3)}{\partial y} + \frac{\partial(xyz)}{\partial z} = y^2 + 2yz^3 + xy$$

Differential operators - Laplace

- Laplace operator:

$$\Delta = \vec{\nabla}^2 = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

$$\Delta f(x,y,z) = \frac{\partial^2 f(x,y,z)}{\partial x^2} + \frac{\partial^2 f(x,y,z)}{\partial y^2} + \frac{\partial^2 f(x,y,z)}{\partial z^2}$$

Example:

$$f_3(x,y,z) = x^3 + y^2 z$$

$$\Delta f_3 = \nabla \cdot \nabla f_3(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f_3(x,y,z) = \frac{\partial^2}{\partial x^2} f_3 + \frac{\partial^2}{\partial y^2} f_3 + \frac{\partial^2}{\partial z^2} f_3 = 6x + 2z$$

Differential operators – rotation (curl)

- Rotation (curl):

of $\boldsymbol{g}(x,y,z)\colon \mathbb{R}^3 \to \mathbb{R}^3$
(of a vector field):

$$\boldsymbol{g}(x,y,z) = \begin{bmatrix} g_x(x,y,z) \\ g_y(x,y,z) \\ g_z(x,y,z) \end{bmatrix} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

$$rot\boldsymbol{g}(x,y,z) = \nabla \times \boldsymbol{g}(x,y,z) = det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{bmatrix} = \begin{bmatrix} \frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \\ \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \\ \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \end{bmatrix}$$

- **direction:** axis of rotation
- **magnitude:** magnitude of rotation

Example:

$$\mathrm{f}_2(x,y,z) = (xy^2, y^2z^3, xyz)^T$$

$$\nabla \times \mathrm{f}_2 == curl\mathrm{f}_2 = \begin{bmatrix} \frac{\partial(xyz)}{\partial y} - \frac{\partial(y^2z^3)}{\partial z} \\ -\frac{\partial(xyz)}{\partial x} + \frac{\partial(xy^2)}{\partial z} \\ \frac{\partial(y^2z^3)}{\partial x} - \frac{\partial(xy^2)}{\partial y} \end{bmatrix} = \begin{bmatrix} xz - 3y^2z^2) \\ -yz \\ -2xy \end{bmatrix}$$