Meshfree Collocation Solution of PDEs via Interpolating Moving Least Squares

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Motivation

Meshfree methods are methods which work just on sets of points without the knot-to-knot connectivity, i.e. without a mesh. They are capable of avoiding the difficulties encountered in conventional methods, such as, large mesh deformations, remeshing in moving boundary value problems, and meshing complex geomentries.

Meshfree collocation methods are simple in its concept, easy to implement and fast numerical methods.

Since the accurate solution of PDEs requires interpolation at given data points, one must use singular weight functions, which introduces trouble in using the interpolating method.

Interpolating Moving Least Squares

- Minimize weighted squared error functional

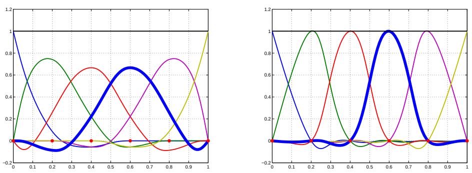
$$E_x(\boldsymbol{a}) = \sum_{i=1}^{N} w_i(\boldsymbol{x}) \left[\, p(\boldsymbol{x_i}) - u_i \,\right]^2 \longrightarrow \min,$$

- Weighting functions

$$\lim_{|\boldsymbol{x}-\boldsymbol{x}_i|\to 0} w(\boldsymbol{x}-\boldsymbol{x}_i) = \infty \qquad \Rightarrow \qquad w(\boldsymbol{\zeta}) := \frac{1}{\zeta^\alpha}$$

- Kernel functions

$$\varphi(\boldsymbol{x}) := \boldsymbol{W}(\boldsymbol{x})\boldsymbol{B}\left(\boldsymbol{B}^{\mathsf{T}}\boldsymbol{W}(\boldsymbol{x})\boldsymbol{B}\right)^{-1}\boldsymbol{b}(\boldsymbol{x})$$



Approximation (MLS) and interpolation (IMLS) cases

- Kronecker's delta property and partition of unity

$$\varphi_j(\boldsymbol{x}_i) = \delta_{ij} \qquad\qquad \sum_{i=1}^{N} \varphi_j(\boldsymbol{x}) = 1$$

Singularity problem

- Regularization $\varepsilon > 0$
- Split the matirx taking out the singularity
- Use Sherman-Morrison formula: connection between the inverse of the original and of the disturbed matrices
- Taylor series for $\varphi(x+\varepsilon)$ and letting $\varepsilon \to 0$
  - ◦ formulas for $D^\alpha\varphi(\boldsymbol{x})$

Boundary Value Problems

- General formulation of BVP

$$\boldsymbol{\mathcal{L}u} = \boldsymbol{f_\Omega} \quad \text{in} \quad \Omega \subset \mathbb{R}^d$$

$$\boldsymbol{\mathcal{B}u} = \boldsymbol{f_\Gamma} \quad \text{on} \quad \Gamma = \partial\Omega$$

- Meshfree collocation solution in strong form

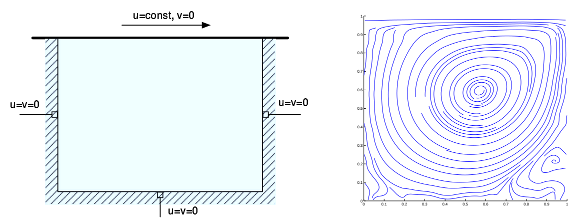
$$D^\alpha u^h(\boldsymbol{x}) = \sum_{j=1}^{N} D^\alpha \varphi_j(\boldsymbol{x}) \cdot u_j$$

- Finite difference operators on irregular set of nodes
  - ◦ explicit local formulas for $D^\alpha\varphi_j(\boldsymbol{x})$
- Well-posedness conditions for the implementation of von Neumann boudary conditions
- Positivity conditions for meshfree collocation methods

Incompressible Navier-Stokes

- Chorin's pressure-correction method
  - ◦ Intermediate velocity
  - ◦ Laplace equation for pressure and corrected velocity values
- Explicit time discretization
  - ◦ CFL-condition
- Babuška-Brezzi condition - prevents possible pressure oscillations
  - ◦ pressure and velocity evaluated on different point sets

Lid-driven cavity model problem



As an example we simulate the lid-driven cavity flow. The lid of a square container filled with a fluid moves with a constant velocity and thereby sets the fluid in motion.

On the upper boundary the velocity u in $x-$direction is set to be constant, simulating the moving lid. On all other boundaries non-slip boundary conditions are imposed ($u = v = 0$). The Figure shows the steady-state solution streaklines for $Re = 1000$.