LAW OF THE WALL (1)

$$\frac{U}{u_*} = 2.5 \ \ln\left(\frac{yu_*}{\nu}\right) + B \quad , \quad \tau_w = \rho u_*^2 \qquad (*)$$

U : Tangential Velocity at y

ν: Molecular Kinematic Viscosity

B : Roughness-Constant (Smooth Wall $B = 5.5$)

u_*: Shear Speed

<u>Idea</u>: Given U, y, ρ, B : Compute $u_* \Rightarrow \tau_w$

Approximation of (*) by the 1/7th-Law

$$\frac{yu_*^s}{\nu} = 0.15 \left(\frac{yU}{\nu}\right)^{\frac{7}{8}} \quad ,$$

$$\frac{U}{u_*} = 2.5 \ \ln 0.15 + B + 2.19 \ \ln\left(\frac{yU}{\nu}\right)$$

LAW OF THE WALL (2)

Laminar: $\frac{yU}{\nu} < 130.3$

$$\frac{U}{u_*} = \left(\frac{yU}{\nu} \right)^{\frac{1}{2}}$$

Turbulent: $\frac{yU}{\nu} > 130.3$

$$\frac{U}{u_*} = 2.5 \ \ln 0.15 + B + 2.19 \ \ln \left(\frac{yU}{\nu} \right)$$

SMAGORINSKY

Idea: Model Subgrid Effects Via

$$\mu_t = c_s l^2 |\omega|$$

where

$$l^2 = \frac{1}{(\mathbf{k} \times \nabla N^i)^2} \quad , \quad \mathbf{k} = \frac{\mathbf{v} \times \omega}{|\mathbf{v} \times \omega|}$$

and (important)

- Shear Stresses at Walls from Law of the Wall

Pros:

- Simple
- Fast
- Good for Transient Problems, Overall Effects

Cons:

- Inconsistent for $h \to 0$?

BALDWIN-LOMAX

Inner part: $y < y_c$

$$\mu_{t_i} = \rho l^2 |\omega| \quad , \quad l = \kappa y \left(1 - e^{-\frac{y}{A}}\right)$$

Outer part: $y > y_c$

$$\mu_{t_o} = \alpha C_{cp} \rho Y_{max} F_{max} \left[1 + 5.5(y/\delta)^6\right]^{-1}$$

$$F(y) = y|\omega| \left(1 - e^{-\frac{y}{A}}\right) \quad , \quad \delta = \frac{Y_{max}}{0.65}$$

$$C_{cp} = 1.2 \ , \ \alpha = 0.0168 \ , \ \kappa = 0.4 \ , \ A = \frac{26.0}{\sqrt{Re \ |\omega|_w}}$$

y_c : Determined by $\mu_{t_i} = \mu_{t_o}$

$e^{\frac{y}{A}}$: van Driest Damping Factor

Pros:

- Simple
- Algebraic $\Rightarrow$ Fast
- Good for Attached, Simple Flows

Cons:

- Special Data Structures (1-D Normals)
- Separated Flows Iffy
- No Wake-Effects

BALDWIN-BARTH (1)

Define:

$$R_t = \frac{\nu_t}{\nu} Re$$

Then

$$\nu\tilde{R}_{t,t} + \mathbf{v} \cdot \nabla \nu\tilde{R}_t = (c_{\epsilon_2} f_2 - c_{\epsilon_1})\sqrt{P\ \nu\tilde{R}_t}$$
$$+\nabla \cdot \left(\nu + \frac{\nu_t}{\sigma_R}\right) \nabla \nu\tilde{R}_t - \left(\frac{1}{\sigma_R} + \frac{1}{\sigma_\epsilon}\right) \nabla \nu_t \cdot \nabla \nu\tilde{R}_t$$

where

$\nu_t = c_\mu D_1 D_2 \nu\tilde{R}_t$

$D_1 = 1 - exp\left(-y^+/A^+\right)$

$D_2 = 1 - exp\left(-y^+/A_2^+\right)$

$P = \nu_t \left(v^i_{,j} + v^j_{,i}\right) v^i_{,j} - \frac{2}{3}\nu_t \left(v^k_{,k}\right)^2$

and

BALDWIN-BARTH (2)

$$f_2(y^+) = \frac{c_{\epsilon_1}}{c_{\epsilon_2}} + (1 - \frac{c_{\epsilon_1}}{c_{\epsilon_2}})(\frac{1}{\kappa y^+} + D_1 D_2)\left(\sqrt{D_1 D_2}\right.$$

$$\left. + \frac{y^+}{\sqrt{D_1 D_2}}\left(\frac{1}{A^+} exp\left(-\frac{y^+}{A^+}\right) D_2 + \frac{1}{A_2^+} exp\left(-\frac{y^+}{A_2^+}\right) D_1\right)\right)$$

$$\frac{1}{\sigma_\epsilon} = (c_{\epsilon_2} - c_{\epsilon_1})\sqrt{c_\mu}/\kappa^2 \quad , \quad \sigma_R = \sigma_\epsilon$$

$$\kappa = 0.41 \quad , \quad c_{\epsilon_1} = 1.2 \quad , \quad c_{\epsilon_2} = 2.0$$

$$c_\mu = 0.09 \quad , \quad A^+ = 26.0 \quad , \quad A_2^+ = 10.0$$

Boundary Conditions

- Solid Walls: $\nu \tilde{R}_t = 0$
- Inflow: $\nu \tilde{R}_t = \nu \tilde{R}_{t\infty}$
- Outflow: extrapolate $\nu \tilde{R}_t$

BALDWIN-BARTH (3)

Pros:

- 1-Eqn. $\Rightarrow$ Transport Character Preserved
- 1-Eqn. $\Rightarrow$ Relatively Fast
- $y^+ = 30.0$

Cons:

- Special Data Structures (Distance-Function)
- Stiff Source Term
- Mixed Term Requires Careful Numerical Treatment

SPALART-ALLMARAS (1)

Define: $\tilde{\nu}$: Modified Kinematic Eddy Viscosity

$$\mu_t = \rho \tilde{\nu} f_{\nu 1} Re$$

Then

$$\rho \tilde{\nu}_{,t} + \rho \mathbf{v} \cdot \nabla \tilde{\nu} = c_{b1} \tilde{S} \rho \tilde{\nu}$$

$$+\frac{1}{\sigma}\left[\nabla \cdot (\mu + \rho \tilde{\nu}) \nabla \tilde{\nu} + c_{b2} \rho \left(\nabla \tilde{\nu}\right)^2\right] - c_{\omega 1} \rho f_{\omega} \left(\frac{\tilde{\nu}}{d}\right)^2$$

where

$\tilde{S} = |\omega| + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{\nu 2}$

$f_{\nu 2} = 1 - \frac{\Xi}{1 + \Xi f_{\nu 1}}$

SPALART-ALLMARAS (2)

$$\kappa = 0.41 \quad , \quad c_{b1} = 0.1355 \quad , \quad c_{b2} = 0.6220$$

$$c_{\nu 1} = 7.1 \quad , \quad c_{\nu 2} = 5.0$$

$$c_{t3} = 1.1 \quad , \quad c_{t4} = 2.0$$

$$c_{\omega 1} = 3.279875 \quad , \quad c_{\omega 2} = 0.3 \quad , \quad c_{\omega 3} = 2.0$$

Boundary Conditions

- Solid Walls: $\tilde{\nu} = 0$
- Inflow: $\tilde{\nu} = \tilde{\nu}_\infty$
- Outflow: extrapolate $\tilde{\nu}$

SPALART-ALLMARAS (3)

Pros:

- 1-Eqn. $\Rightarrow$ Transport Character Preserved
- 1-Eqn. $\Rightarrow$ Relatively Fast
- $y^+ = 30.0$

Cons:

- Special Data Structures (Distance-Function)
- Stiff Source Term
- Mixed Term Requires Careful Numerical Treatment

$k - \epsilon$: FIELD EQUATIONS

$$k_{,t} + \mathbf{v} \cdot \nabla k = \nabla \frac{\nu_t}{\sigma_k} \nabla k + P - \epsilon$$

$$\epsilon_{,t} + \mathbf{v} \cdot \nabla \epsilon = \nabla \frac{\nu_t}{\sigma_\epsilon} \nabla \epsilon + c_1 \frac{\epsilon}{k} P - c_2 \frac{\epsilon^2}{k}$$

$$\nu_t = c_\mu \frac{k^2}{\epsilon} \quad , \quad P = \nu_t \left(v^i_{,j} + v^j_{,i} \right) v^i_{,j}$$

$c_\mu = 0.09$, $c_1 = 1.44$, $c_2 = 1.92$, $c_3 = 0.80$

$\sigma_k = 1.0$, $\sigma_\epsilon = 1,3$, $\sigma_T = 0.9$

$k - \epsilon$: BOUNDARY CONDITIONS AT WALLS

a) Momentum, Kinematics:

$$\mathbf{v} \cdot \mathbf{n} = 0$$

$\Rightarrow$ Same as Euler

b) Momentum, Kinetics:

$$\tau_w = \rho u_*^2$$

- u_* From Law of the Wall

c) k:

$$k_w = \frac{u_*^2}{\sqrt{c_\mu}}$$

d) ϵ:

$$\epsilon_w = \frac{u_*^3}{\kappa y}$$

$k - \epsilon$: BOUNDARY CONDITIONS AT IN/OUTFLOW

a) Inflow: Assume Level of Fluctuations:

$$k_\infty = (\alpha \mathbf{v}_\infty)^2$$

- Typical Value: $\alpha = 0.05$ (5%)

b) Inflow: Assume Ratio of $c_t = \mu_t / \mu_l$:

$$\epsilon_\infty = c_t \frac{\rho_\infty c_\mu k_\infty^2}{\mu_l}$$

c) Outflow:

- Extrapolate k
- Extrapolate ϵ

$k - \epsilon$ MODEL

Pros:

- Transport Character
- Wide Acceptance
- Acceptable Accuracy for Many Applications

Cons:

- Plane/Round Jet Anomaly
- Poor Description of Anisotropy
- Incomplete Curvature and Rotation Effects
- Poorly Adapted for Low-Re Near-Wall Turbulence
- Stiff Source Terms
- Problems if $k, \epsilon \leq 0$

LOW-Re MODELS (1)

Key Idea: Introduce Damping Functions

$$c_\mu, c_1, c_2 \;\rightarrow\; f_\mu c_\mu, f_1 c_1, f_2 c_2$$

Define:

$$\tau_w = \rho u_*^2 \quad , \quad y^+ = \frac{y u_*}{\nu} \quad , \quad R_t = \frac{k^2}{\nu_l \epsilon} \quad , \quad R_y = \frac{\sqrt{k} y}{\nu_l}$$

In the Sequel: $y = y^+$

Table 1 $k - \epsilon$ Damping Functions: f_μ

Model	f_μ
Ch	$1 - exp(-0.0115y)$
Sh	$1 - exp(-0.006y - 0.0733y^2 + 0.0062y^3 - 0.0005y^4)$
LB	$(1 - exp(-0.165R_k))^2 \, (1 + 20.5/R_t)$
NH	$(1 - exp(-0.0376y))^2$
NT	$(1 - exp(-0.0384y))^2 \left(1 + 4.1/R_t^{0.75}\right)$
JL	$exp(-2.5/(1 + R_t/50))$
LS	$exp(-3.4/(1 + R_t/50)^2)$
MK	$\left(1 + 3.45/\sqrt{R_t}\right)(1 - exp(-y/70))$
YS	$1 - exp(-0.004y - 0.0337y^2 + 0.0050y^3 - 0.0027y^4)$

LOW-Re MODELS (2)

Table 2 $k - \epsilon$ Damping Functions: f_1

Model	f_1
Ch	1
Sh	1
LB	$1 + (0.05/f_\mu)^3$
NH	1
NT	1
JL	1
LS	1
MK	1
YS	1

LOW-Re MODELS (3)

Table 3 $k - \epsilon$ Damping Functions: f_2

Model	f_2
Ch	$1 - 0.22exp(-R_t^2/36)$
Sh	$1 - 0.22exp(-R_t^2/36)$
LB	$1 - exp(-R_t^2)$
NH	$1 - 0.30exp(-R_t^2)$
NT	$\left(1 - 0.30exp(-(R_t/6.5)^2)\right)(1 - exp(-y/6))^2$
JL	$1 - 0.30exp(-R_t^2)$
LS	$1 - 0.30exp(-R_t^2)$
MK	$\left(1 - 0.22exp(-R_t^2/36)\right)(1 - exp(y/5))^2$
YS	$1 - 0.22exp(-R_t^2/36)$

LOW-Re MODELS (4)

Remarks:

- Most Set: $k_w = \epsilon_w = 0$
- Need Fine Mesh, Well Into Sublayer
- $\Rightarrow$ Stiffness
- Satisfactory Model Yet to be Found
- DNS Has Stimulated Model Development

FLOATING POINT ERRORS (1)

1. Positivity:

$$k, \epsilon \geq 0$$

2. Max Velocity Fraction Level:

$$k < (c_{fmax} \cdot \mathbf{v})^2$$

   - Typical Value: $c_{fmax} = 0.5$ (50% Of Velocity)

3. Min Laminar Fraction Level:

$$\nu_t > \gamma_{min}\nu_l \quad \Rightarrow \quad c_\mu k^2 > \gamma_{min}\nu_l\epsilon$$

   - Typical Value: $\gamma_{min} = 0.1$

4. Max Laminar Fraction Level:

$$\nu_t < \gamma_{max}\nu_l \quad \Rightarrow \quad \epsilon > \frac{c_\mu k^2}{\gamma_{max}\nu_l}$$

   - Typical Value: $\gamma_{max} = 1000$

FLOATING POINT ERRORS (2)

5. Drastic Change Limit:

$$\Delta\nu_t = c_\mu \left[2\frac{k}{\epsilon}\Delta k - \frac{k^2}{\epsilon^2}\Delta\epsilon \right] = c_\mu \frac{k^2}{\epsilon} \left[2\frac{\Delta k}{k} - \frac{\Delta\epsilon}{\epsilon} \right]$$

$\Rightarrow$

$$\frac{\Delta\nu_t}{\nu_t} = 2\frac{\Delta k}{k} - \frac{\Delta\epsilon}{\epsilon}$$

$\Rightarrow$

$$-\alpha < \frac{\Delta k}{k} \; , \; \frac{\Delta\epsilon}{\epsilon} < \alpha$$

- Typical Value: $\alpha = 0.3$

6. Upwinding of Advective Terms:

- Optimal Upwinding (SUPG, vanAlbada, MinMod, ...)
- Full Upwinding (Most Codes)

FLOATING POINT ERRORS (3)

7. Mofidy Equations:

   - Idea: Limits on ν_t Possible $\Rightarrow$ (Pelletier, Ilinca)

$$\epsilon = c_\mu k^2 \cdot \frac{\epsilon}{c_\mu k^2} = \frac{c_\mu k^2}{\nu_t}$$

$$\frac{\epsilon}{k} = c_\mu k \cdot \frac{\epsilon}{c_\mu k^2} = \frac{c_\mu k}{\nu_t}$$

$$\frac{\epsilon^2}{k} = c_\mu k\epsilon \cdot \frac{\epsilon}{c_\mu k^2} = \frac{c_\mu k\epsilon}{\nu_t}$$

FLOATING POINT ERRORS (4)

8. <u>Mofidy Equations</u>:
   - Work with new variables: $k = e^K, \epsilon = e^E$ (Pelletier, Ilinca)
   - Re-write equations, e.g. Incompressible Flows

$$\nu_t = c_\mu \frac{k^2}{\epsilon} = c_\mu e^{2K-E}$$

$$K_{,t} + \mathbf{v} \cdot \nabla K = \nabla \left(\nu_l + \frac{\nu_t}{\sigma_k} \right) \nabla K + \left(\nu_l + \frac{\nu_t}{\sigma_k} \right) \nabla K \cdot \nabla K$$

$$+ e^{-K} P - e^{E-K}$$

$$P = -\overline{v^i v^j} v^i_{,j} = \nu_t \left(v^i_{,j} + v^j_{,i} \right) v^i_{,j}$$

$$E_{,t} + \mathbf{v} \cdot \nabla E = \nabla \left(\nu_l + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla E + \left(\nu_l + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla E \cdot \nabla E$$

$$+ c_1 e^{-K} P - c_2 e^{E-K}$$

FLOATING POINT ERRORS (5)

9. <u>Implicit Treatment of Source-Terms</u>
   - <u>Several Options Possible</u>
     - Equation by Equation
     - Full Coupling

For Equation by Equation:

$$\left(1 + \Delta t \frac{2c_\mu k}{\nu_t}\right) \Delta k = \Delta k_{exp}$$

$$\left(1 + \Delta t \frac{c_2 c_\mu k}{\nu_t}\right) \Delta \epsilon = \Delta \epsilon_{exp}$$

FLOATING POINT ERRORS (6)

10. Implicit Source/Diffusion Terms: E.g. Eqn. By Eqn.

$$\left(1 - \Delta t \nabla \nu \nabla + \Delta t \frac{2 c_\mu k}{\nu_t}\right) \Delta k = \Delta k_{exp}$$

$$\left(1 - \Delta t \nabla \nu \nabla + \Delta t \frac{c_2 c_\mu k}{\nu_t}\right) \Delta \epsilon = \Delta \epsilon_{exp}$$

11. Implicit Source/Diffusion/Advection Terms:

$$\left(1 + \Delta t \mathbf{v} \cdot \nabla - \Delta t \nabla \nu \nabla + \Delta t \frac{2 c_\mu k}{\nu_t}\right) \Delta k = \Delta k_{exp}$$

$$\left(1 + \Delta t \mathbf{v} \cdot \nabla - \Delta t \nabla \nu \nabla + \Delta t \frac{c_2 c_\mu k}{\nu_t}\right) \Delta \epsilon = \Delta \epsilon_{exp}$$