HIGH ORDER METHODS

- Rationale
- Higher Order Finite Differences
- Higher Order Finite Elements
- Discontinuous Galerkin Methods
- Finite Point Methods

EFFORT vs ACCURACY (1)

Optimal Effort:

Assume even error distribution (optimal mesh) initially; then

$$Eff \geq c_1 h^{-d}$$

d: dimensionality of problem

Error:

$$\|u - u^h\| = c_2 h^{p+1} |u|_{p+1}$$

p: order of approximation for the elements

Desired: attain $\|u - u^h\| \to 0$ **without** $Eff \to \infty$

$$Eff \cdot \|u - u^h\| = c_3 h^{p+1-d} |u|_{p+1}$$

Worst Case Scenario (e.g. Turbulence)

Dimension	Eff * Err	Decrease with $h \to 0$
1-D	h^p	$p \geq 1$
2-D	h^{p-1}	$p \geq 2$
3-D	h^{p-2}	$p \geq 3$

$\Rightarrow$ strive for elements of higher order

EFFORT vs ACCURACY (2)

<u>However</u>: Redeeming factors:

- 1-D features in 2/3-D:
    - boundary layers
    - shocks
    - contact discontinuities

  $\Rightarrow$ linear elements sufficient
- Engineering Accuracy $O(1\%)$
- Unknown Physics

FINITE DIFFERENCES (1)

First Order Derivatives:

Define: $u' = u_{,x} = \partial u / \partial x$

$$u'_0 = \frac{1}{2h}\left(u_1 - u_{-1}\right) \quad , \quad -\frac{1}{6}h^2 u_{,xxx}$$

$$u'_0 = \frac{1}{12h}\left(-u_2 + 8u_1 - 8u_{-1} + u_{-2}\right) \quad , \quad +\frac{1}{30}h^4 u_{,xxxxx}$$

$$u'_1 + 4u'_0 + u'_{-1} = \frac{3}{h}\left(u_1 - u_{-1}\right) \quad , \quad +\frac{1}{30}h^4 u_{,xxxxx}$$

FINITE DIFFERENCES (2)

Compact First Order Derivatives:

$$\alpha u'_{-1} + u'_0 + \alpha u'_1 = \frac{a}{2}\left(u_1 - u_{-1}\right) + \frac{b}{4}\left(u_2 - u_{-2}\right)$$

Table: Compact Differening Schemes

Scheme	α	a	b	Order
E2	0	1	0	2
E4	0	4/3	-1/3	4
C4	1/4	3/2	0	4
C6	1/3	14/9	1/9	6

FINITE DIFFERENCES (3)

<u>Second</u> Order Derivatives:

Define: $u'' = u_{,xx} = \partial^2 u/\partial x^2$

$$u''_0 = \frac{1}{h^2}\left(u_1 - 2u_0 + u_{-1}\right) \quad , \quad -\frac{1}{12}h^2 u_{,xxxx}$$

$$u''_0 = \frac{1}{12h^2}\left(-u_2 + 16u_1 - 30u_0 + 16u_{-1} - u_{-2}\right) \;\; , \;\; +\frac{1}{90}h^4 u_{,xxxxxx}$$

$$u''_1 + 10u''_0 + u''_{-1} = \frac{12}{h^2}\left(u_1 - 2u_0 + u_{-1}\right) \quad , \quad +\frac{1}{20}h^4 u_{,xxxxxx}$$

Derived By:

a) Combining u''_0 for $h, 2h, 3h, ..$ etc.

b) Combining $u''_0, u''_{-1}, u''_1, ..$ etc.

FINITE DIFFERENCES (4)

<u>Attempt Flux Form</u>:

$$u_0' = \frac{1}{2h}\left(u_1 - u_{-1}\right) = \frac{1}{h}\left(\frac{u_1 + u_0}{2} - \frac{u_0 + u_{-1}}{2}\right)$$

$\Rightarrow$ 2nd Order Flux:

$$f_{ij} = \frac{u_i + u_j}{2}$$

$$u_0' = \frac{1}{12h}\left(-u_2 + 8u_1 - 8u_{-1} + u_{-2}\right) = \frac{1}{2h}\left(u_1 - u_{-1}\right)$$

$$+\frac{1}{12h}\left(-u_2 + 2u_1 - 2u_{-1} + u_{-2}\right)$$

$$-u_2+2u_1-2u_{-1}+u_{-2} = -u_2+u_0+u_1-u_{-1}+u_1-u_0-u_{-1}+u_{-2}$$

$$= \left(-(u_2 - u_0) + (u_1 - u_{-1})\right) - \left(-(u_1 - u_{-1}) + (u_0 - u_{-2})\right)$$

$\Rightarrow$ High Order Flux:

$$f_{ij} = \frac{u_i + u_j}{2} + \frac{1}{6}\mathbf{l}_{ij} \cdot \left(\nabla u_i - \nabla u_j\right)$$

FINITE DIFFERENCES (5)

Attempt Flux Form:

$$u_0'' = \frac{1}{h^2}\left(u_1 - 2u_0 + u_{-1}\right) = \frac{1}{h^2}\big((u_1 - u_0) - (u_0 - u_)\big)$$

$\Rightarrow$ 2nd Order Flux:

$$f_{ij} = u_j - u_i$$

$$\begin{aligned} u_0'' &= \frac{1}{12h^2}\left(-u_2 + 16u_1 - 30u_0 + 16u_{-1} - u_{-2}\right) \\ &= \frac{1}{h^2}\left(u_1 - 2u_0 + u_{-1}\right) \\ &+ \frac{1}{12h^2}\left(-u_2 + 4u_1 - 6u_0 + 4u_{-1} - u_{-2}\right) \end{aligned}$$

$$\begin{aligned} \Delta u_{10} - \frac{1}{2}\left(\nabla u_0 + \nabla u_1\right) &= u_1 - u_0 - \frac{1}{2}\left(\frac{u_1 + u_{-1}}{2} + \frac{u_2 + u_0}{2}\right) \\ &= \frac{1}{4}\left(-u_2 + 3u_1 - 3u_0 + u_{-1}\right) \end{aligned}$$

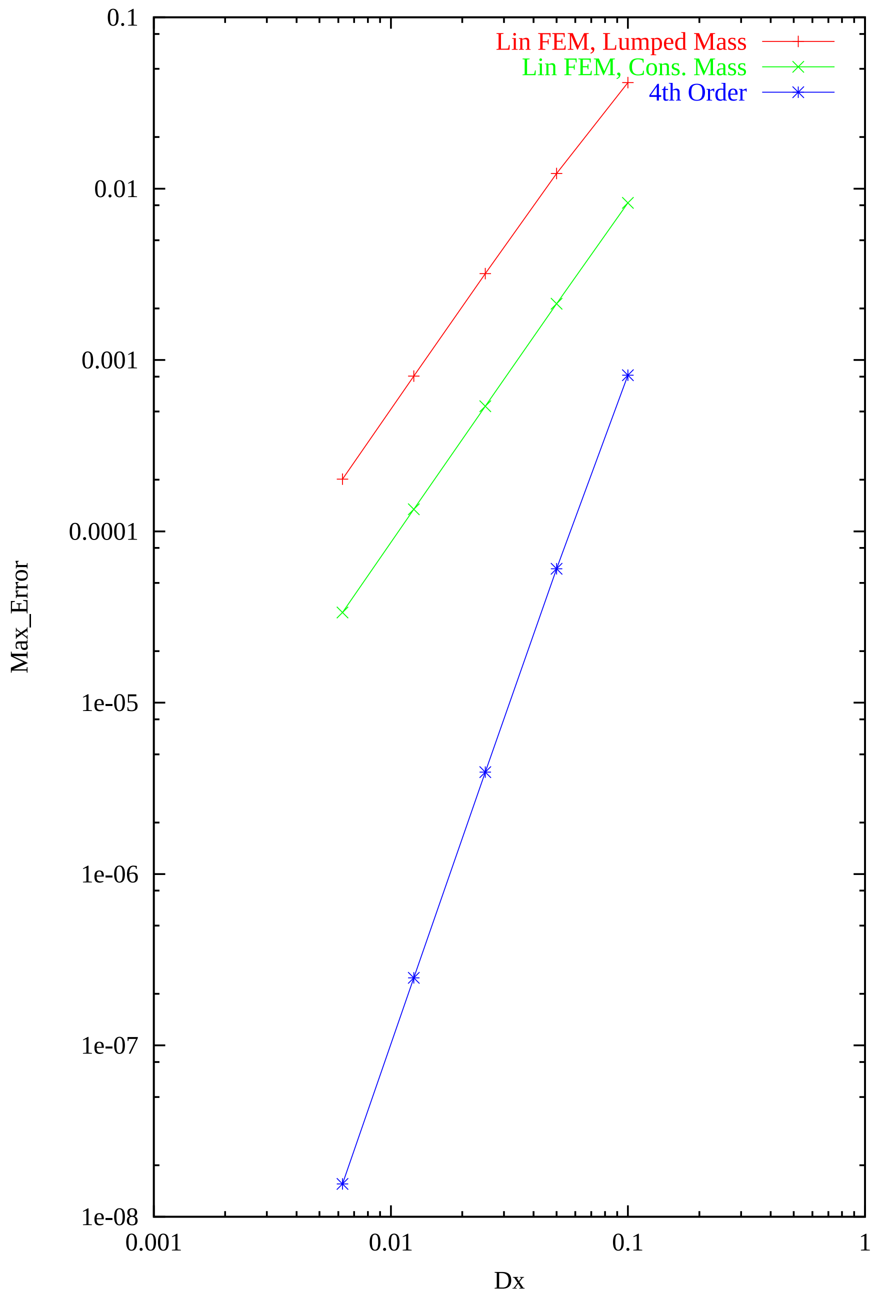
$\Rightarrow$ High Order Flux:

$$f_{ij} = u_j - u_i + \frac{1}{3}\left[u_j - u_i - \frac{1}{2}\mathbf{l}_{ij} \cdot \left(\nabla u_i + \nabla u_j\right)\right]$$

FINITE DIFFERENCES (6)

Example: 1-D, $h = const$

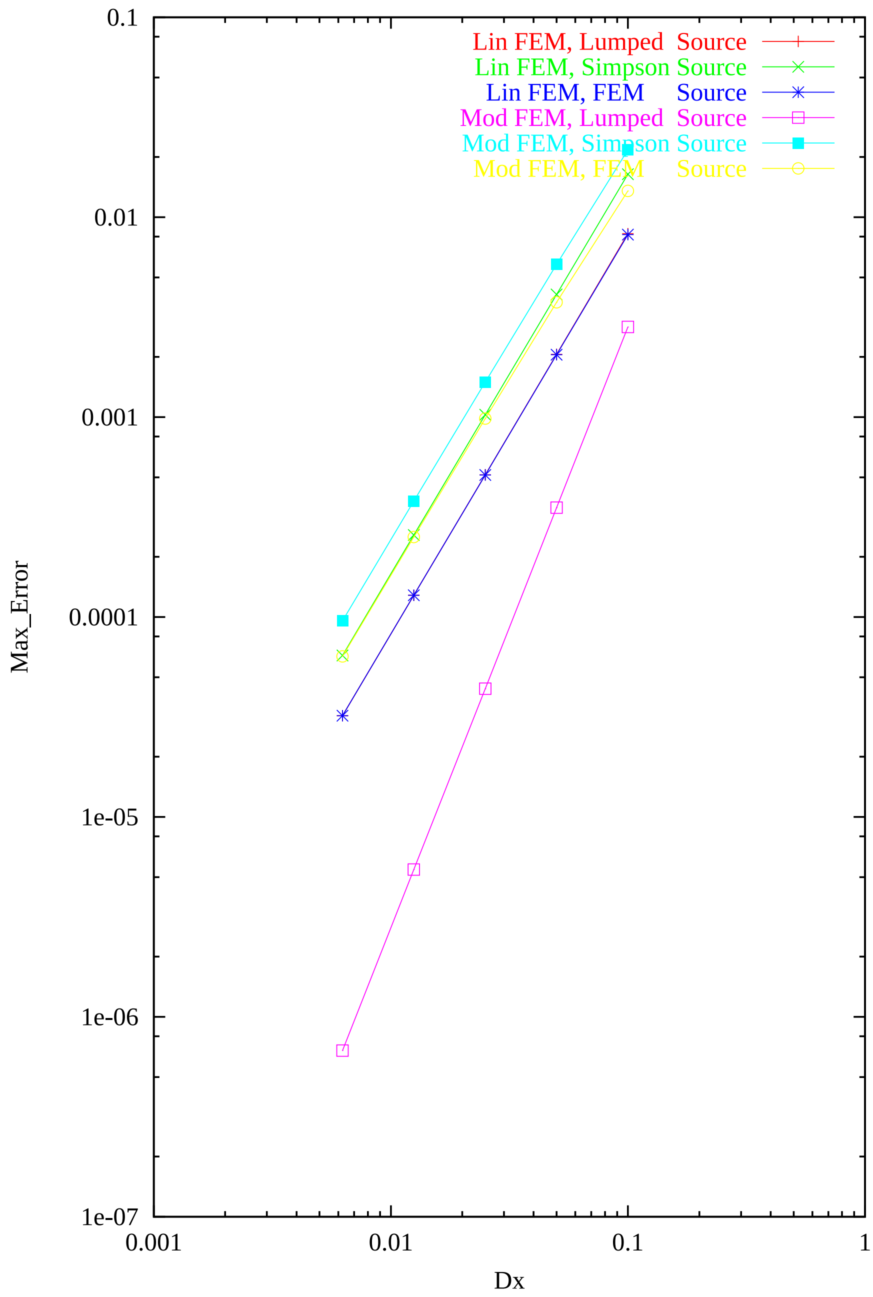
$$u_{,x} \quad , \quad u = sin(\omega x) \quad , \quad \omega = \pi \quad , \quad u(0) = u(1) = 0$$



FINITE DIFFERENCES (7)

Example: 1-D, $h = const$

$$u_{,xx} = s \quad , \quad s = -\omega^2 sin(\omega x) \quad , \quad \omega = \pi \quad , \quad u(0) = u(1) = 0$$



FINITE DIFFERENCES (8)

Need Low-Pass Filtering Schemes (Kreiss Theorem)

$$\alpha_f \hat{u}_{-1} + \hat{u}_0 + \alpha_f \hat{u}_1 = \sum_{n=0}^{N} \frac{a_n}{2} \left(u_n - u_{-n} \right)$$

α_f, a_n Derived from Taylor/Fourier Analysis

FINITE DIFFERENCES (9)

Higher Order Flux-Form Formulae:

2nd Order:

$$F_{i+1/2} = \frac{1}{2}\left(f_i + f_{i+1}\right)$$

4th Order:

$$F_{i+1/2} = \frac{7}{12}\left(f_i + f_{i+1}\right) - \frac{1}{12}\left(f_{i-1} + f_{i+2}\right)$$

6th Order:

$$F_{i+1/2} = \frac{37}{60}\left(f_i + f_{i+1}\right) - \frac{4}{60}\left(f_{i-1} + f_{i+2}\right) + \frac{1}{60}\left(f_{i-2} + f_{i+3}\right)$$

8th Order:

$$F_{i+1/2} = \frac{533}{840}\left(f_i + f_{i+1}\right) - \frac{139}{840}\left(f_{i-1} + f_{i+2}\right)$$

$$+\frac{29}{840}\left(f_{i-2} + f_{i+3}\right) - \frac{1}{840}\left(f_{i-3} + f_{i+4}\right)$$

FINITE DIFFERENCES (10)

- Work: Linear Increase With Order (2/4/6/..)

  $w_{dof}^{FD} = pd$

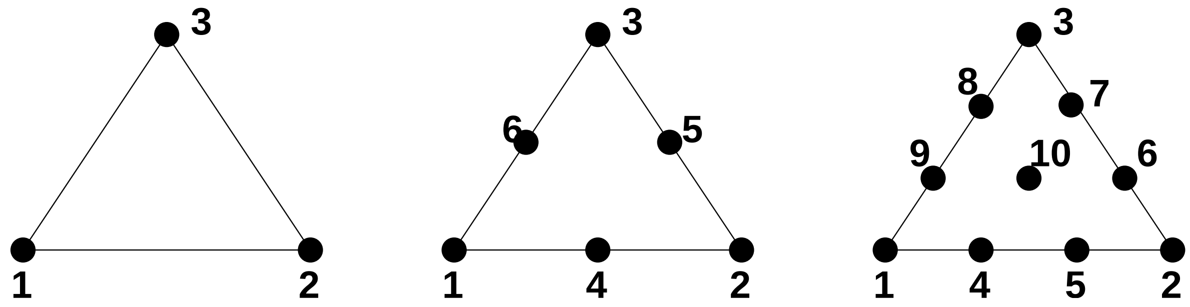
- Rationale:
  - For LES, Need Many Points

    $\Rightarrow$ Use Cartesian Grids

- Pros:
  - Easy
  - Flux Limiters Unserstood
    Mixing [Hybrid]
  - Linear Work Increase

- Cons:
  - Boundary Conditions Messy
  - Large Stencils for Low-Pass Filters

HIGHER ORDER FINITE ELEMENTS (1)



1) Linear Triangle

$$N^1 = \zeta_1 = 1 - \xi - \eta$$

$$N^2 = \zeta_2 = \xi$$

$$N^3 = \zeta_3 = \eta$$

HIGHER ORDER FINITE ELEMENTS (2)

2) Quadratic Triangle

$$N^1 = \zeta_1(2\zeta_1 - 1) = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$$

$$N^2 = \zeta_2(2\zeta_2 - 1) = \xi(2\xi - 1)$$

$$N^3 = \zeta_3(2\zeta_3 - 1) = \eta(2\eta - 1)$$

$$N^4 = 4\zeta_1\zeta_2 = 4\xi(1 - \xi - \eta)$$

$$N^5 = 4\zeta_2\zeta_3 = 4\xi\eta$$

$$N^6 = 4\zeta_1\zeta_3 = 4\eta(1 - \xi - \eta)$$

3) Cubic Triangle

$$N^1 = \tfrac{1}{2}\zeta_1(3\zeta_1 - 1)(3\zeta_1 - 2)$$

$$N^2 = \tfrac{1}{2}\zeta_2(3\zeta_2 - 1)(3\zeta_2 - 2)$$

$$N^3 = \tfrac{1}{2}\zeta_3(3\zeta_3 - 1)(3\zeta_3 - 2)$$

$$N^4 = \tfrac{9}{2}\zeta_1\zeta_2(3\zeta_1 - 1)$$

$$N^5 = \tfrac{9}{2}\zeta_1\zeta_2(3\zeta_2 - 1)$$

$$N^6 = \tfrac{9}{2}\zeta_2\zeta_3(3\zeta_2 - 1)$$

$$N^7 = \tfrac{9}{2}\zeta_2\zeta_3(3\zeta_3 - 1)$$

$$N^8 = \tfrac{9}{2}\zeta_3\zeta_1(3\zeta_3 - 1)$$

$$N^9 = \tfrac{9}{2}\zeta_3\zeta_1(3\zeta_1 - 1)$$

$$N^{10} = 27\zeta_1\zeta_2\zeta_3$$

HIGHER ORDER FINITE ELEMENTS (3)

<u>Work/Storage Increase</u>: Lagrange Polynomials

Degrees of Freedom: $n_{dof} = (1+p)^d \; w_{dof}^{FD} = pd$

Matrix Entries: $n_k = (1+p)^{2d}$

Work Per DOF (Linear): $w_{dof} = O(1+p)^d$

Work Per DOF (Nonlinear): $w_{dof}^{nl} = O(1+p)^{3d/2}$
(Gauss points: $n_g = O(1+p)^{d/2}$)

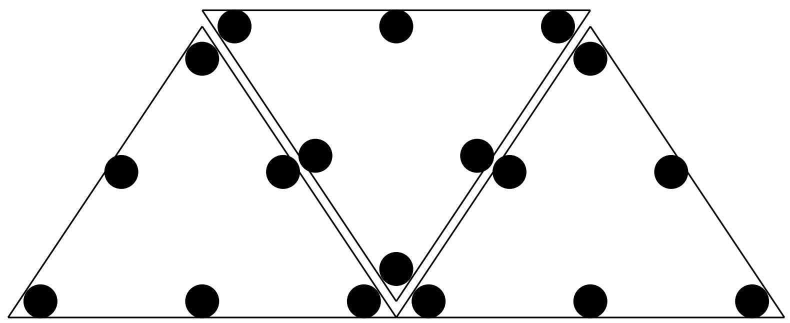
Effort per DOF as a Function of the Approximation

Order	w_{dof}^{3D}	C_r^{3D}	w_{dof}^{nl3D}	C_r^{nl3D}	w_{dof}^{FD3D}	C_r^{FD3D}
1	8	1.0	22.6	1.0	6	1.0
2	27	3.4	140.3	6.2	12	2.0
3	64	8.0	512.0	22.6	18	3.0
4	125	15.6	1397.5	61.8	24	4.0
5	216	27.0	3174.5	140.3	30	5.0
6	343	42.9	6352.4	280.7	36	6.0

HIGHER ORDER FINITE ELEMENTS (4)

- Classic FEM Treatment
- Matrix Becomes Full/Dense
- Flux Limiters Unresolved
  What If Shock In Element ?
- Considerable Work Increase
- P-Multigrid/P-Preconditioning

DISCONTINUOUS GALERKIN (1)



Key Ideas:

- Treat Each Element Independently
- Shape Functions Not Continuous

$$\int_\Omega W^i \nabla \cdot \mathbf{F}(u) d\Omega = -\int_\Omega \nabla W^i \cdot \mathbf{F}(u) d\Omega + \int_\Gamma W^i \mathbf{n} \cdot \mathbf{F}(u) d\Gamma$$

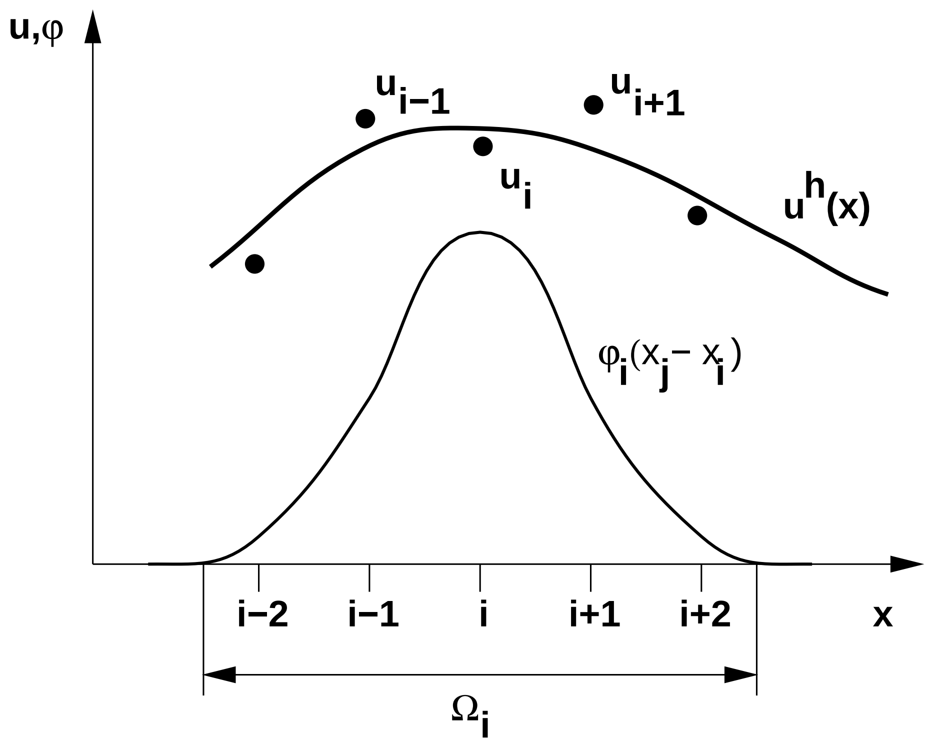
- Treat Domain Integrals as Before
- Use (Approximate) Riemann Solvers for Boundary Fluxes
- L/R States Close for High Order
  $\Rightarrow$ Simple Riemann Solver Works

DISCONTINUOUS GALERKIN (2)

- Viscous Terms ?
- Matrix Dense But Localized
- Dissipation via Riemann Solvers for Low-Order
- Dissipation if Shock in Element ?
- Considerable Work Increase
- P-Multigrid/P-Preconditioning

FINITE POINT METHODS (1)

- Generate Global Cloud of Points
- For Each Point: Obtain Local Cloud
- Choose Local Approximation
  (Monomials + Least Squares)
- Introduce Local Approximation into PDE
- Weighted Residuals ⇒ Algebraic Equations
  - Galerkin: Numerical Integration ⇒ Need Grid
  - Collocation: Truly Mesh Free



WLSQ Procedure

WEIGHTED LEAST SQUARES (WLSQ)

<u>Define</u>:

$u(\mathbf{x})$: Function
Ω_i: Local Interpolation Domain of $u(\mathbf{x})$
Ω_i: $\mathbf{x}_j \in \Omega_i; j = 1, n$

<u>Then</u>:

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) = p_k(\mathbf{x})\alpha_k = \mathbf{p}(\mathbf{x})^T \cdot \boldsymbol{\alpha} \quad , \quad k = 1, m$$

$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_m]^T$
$\mathbf{p}(\mathbf{x})$: 'Base Interpolating Functions'
(e.g. Monomials)

$$\mathbf{p}_2 = [1, x, y, z, x^2, xy, xz, y^2, yz, z^2]^T \quad (10)$$

$$\mathbf{p}_{2.5} = [1, x, y, z, x^2, xy, xz, y^2, yz, z^2,$$
$$x^2y, x^2z, xy^2, xyz, xz^2, y^2z, yz^2]^T \quad (17)$$

$$\mathbf{p}_3 = [1, x, y, z, x^2, xy, xz, y^2, yz, z^2,$$
$$x^3, x^2y, x^2z, xy^2, xyz, xz^2,$$
$$y^3, y^2z, yz^2, z^3]^T \quad (20)$$

WLSQ Cont.

Notation:

$u_j = u(\mathbf{x}_j)$, $u_j^h = u^h(\mathbf{x}_j)$
$\mathbf{p}_j = \mathbf{p}(\mathbf{x}_j)$
$\varphi_{ij} = \varphi(\mathbf{x}_j - \mathbf{x}_i)$

WLSQ $\Rightarrow$ Minimize:

$$J_i = \varphi_{ij}(u_j - u_j^h)^2 = \varphi_{ij}(u_j - \mathbf{p}_j^T \cdot \boldsymbol{\alpha})^2 \quad . \quad j = 1, ..., n$$

Remarks:

- Always Require: $n \geq m$
- $n = m \Rightarrow$ Interpolation

Minimization of $J_i(\alpha_j) \Rightarrow$

$$\mathbf{A} \cdot \boldsymbol{\alpha} = \mathbf{B} \cdot \mathbf{u}$$

i.e.

$$\boldsymbol{\alpha} = \mathbf{C} \cdot \mathbf{u} \quad , \quad \mathbf{C} = \mathbf{A}^{-1} \cdot \mathbf{B}$$

With

$$\mathbf{A} = \sum_{j=1}^{n} \varphi_{ij}(\mathbf{p}_j \otimes \mathbf{p}_j^T)$$

$$\mathbf{B} = [\varphi_{1i}\mathbf{p}_1, \varphi_{2i}\mathbf{p}_2, ..., \varphi_{ni}\mathbf{p}_n]$$

APPROXIMATION ORDER OF LOCAL CLOUD

Points Per Local Cloud:
- Tet Mesh: 15
- Cart Mesh: 27
- Empirical: 18-20

$\Rightarrow$ Could Approximate to Higher Order

- Test **A** of Higher Order Polynomials as Before
- Accept if Better

FLOW SOLVER (1)

Compressible Navier-Stokes:

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F} = 0$$

WLSQ:

$$\nabla \cdot \mathbf{F}\big|_i \approx \mathbf{r}^i = D_l^{ij} \mathbf{F}_j^l$$

Key Idea: Symmetrize

$$\mathbf{r}^i = D_l^{ij}\Big|_{j \neq i} (\mathbf{F}_j^l + \mathbf{F}_i^l) + (D_l^{ii} - D_l^{ij}\Big|_{j \neq i}) \mathbf{F}_i^l$$

or:

$$\mathbf{r}^i = D_l^{ij}\Big|_{j \neq i} (\mathbf{F}_j^l + \mathbf{F}_i^l) + \tilde{D}_l^{ii} \mathbf{F}_i^l$$

First Term:

- Equivalent to Galerkin WRM
- Equivalent to Central Differences
- $\Rightarrow$ Unstable $\Rightarrow$ Add Stabilizing Terms
    - Higher Order Damping (Requires Length)
    - Upwinding (No Length) [vanLeer, Roe, AUSM+, ..]

SUMMARY

- FPMs for Compressible and Incompressible Flow
- Key Ingredients:
    - Local Clouds via Delaunay
    - Quality Criteria
    - Upwind/Riemann Solvers
    - Limitors
    - Edge-Based Data Structure
    - Fully Parallel (Shared)
- Accuracy Comparable to FEM/FVM
- Efficiency: 2-3 Times Slower Than FEM [Low-Order]

- Efficiency TBD For Higher Order