GALERKIN (1)

$$\int_\Omega N^i(\mathbf{u}^h_{,t} + \nabla \cdot \mathbf{F}) d\Omega = 0$$

$$\int_\Omega N^i \left[N^j (\hat{\mathbf{u}}_j)_{,t} + \nabla \cdot \mathbf{F}(N^j \hat{\mathbf{u}}_j) \right] d\Omega = 0$$

In order to save CPU, use:

$$\mathbf{F}(N^j \hat{\mathbf{u}}_j) = N^j \mathbf{F}(\hat{\mathbf{u}}_j)$$

$\Rightarrow$

$$\int_\Omega N^i N^j d\Omega \ (\hat{\mathbf{u}}_j)_{,t} + \int_\Omega N^i \nabla \cdot N^j d\Omega \ \mathbf{F}(\hat{\mathbf{u}}_j) = 0$$

or

$$\mathbf{M}_c \cdot \hat{\mathbf{u}}_{,t} = \mathbf{r} \quad , \quad \mathbf{r} = \mathbf{r}(\mathbf{u})$$

<u>Remarks:</u>

- Integration by parts possible; if N^i linear, same as FVM
- Again use:

$$\int_\Omega .. = \sum_{el} \int_{\Omega_{el}} ..$$

GALERKIN (2)

To see what happens: 1-D, h=const., linear advection:

$$u_{,t} + a u_{,x} = 0$$

For each element

$$\mathbf{M}_c = \int N^i N^j dx = \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad ,$$

$$\mathbf{D}_1 = a \int N^i N^j_{,x} = a \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{D}_2 = - \int N^i_{,x} N^j_{,x} dx = -\frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assemble the element contributions

$$\frac{h}{6} \begin{bmatrix} 2 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 4 & 1 \\ & & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_{,t} =$$

$$\frac{-a}{2} \begin{bmatrix} -1 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

GALERKIN (3)

With mass-lumping, at node i :

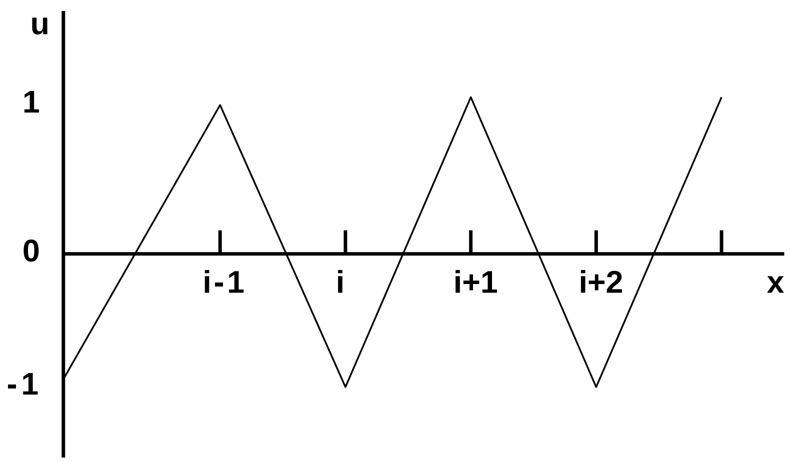
$$(u_i)_{,t} = -\frac{a}{2h}\,(u_{i+1} - u_{i-1})$$

$\Rightarrow$ same as central differencing

<u>Stability:</u> RHS

$$r_i = -\frac{1}{2h}\,(u_{i+1} - u_{i-1})$$

Only every 2nd node coupled $\Rightarrow$ zero-energy modes
$\Rightarrow$ Chequerboard



$\Rightarrow$ need stabilizing terms

GALERKIN (4)

1. 2nd order (TVD) : $h^2 \frac{\partial^2}{\partial x^2}$

$$F_{i+1} - 2F_i + F_{i-1} = 4$$

2. 4th order : $h^4 \frac{\partial^4}{\partial x^4}$

$$F_{i+2} - 4F_{i+1} + 6F_i - 4F_{i-1} + F_{i-2} = 16$$

- Simulate ∇^4 as $\nabla^2\nabla^2$ (two passes of ∇^2)

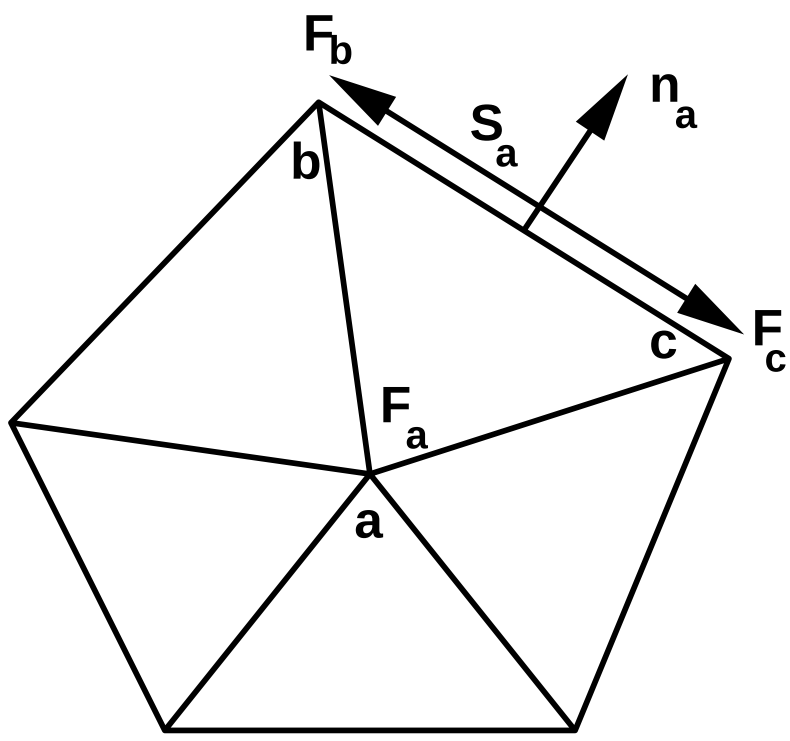
- Implicit matrix may also be assembled in this way

EQUIVALENCY WITH FVM (1)

$$\mathbf{r}^i = -\int_\Omega N^i \nabla \cdot N^j \ \mathbf{F}(\hat{\mathbf{u}}_j)\ d\Omega = \int_\Omega (\nabla N^i) N^j \cdot \ \mathbf{F}(\hat{\mathbf{u}}_j)\ d\Omega$$

For linear triangles

$$\mathbf{r}^i = \frac{A}{3}(\nabla N^i) \cdot \sum_{j_{el}} \mathbf{F}(\hat{\mathbf{u}}_j) = \frac{1}{3} \cdot \frac{1}{2}(s_i \mathbf{n}_i) \cdot \sum_{j_{el}} \mathbf{F}(\hat{\mathbf{u}}_j)$$



EQUIVALENCY WITH FVM (2)

For node a:

$$\mathbf{r}_a = \frac{s_a \mathbf{n}_a}{3} \cdot \left(\frac{\mathbf{F}_a}{2} + \frac{\mathbf{F}_b + \mathbf{F}_c}{2} \right)$$

but

$$\sum_{surr_{el}} s_a \mathbf{n}_a = 0$$

$\Rightarrow$ same as FVM

same can be shown for 3-D tetrahedra

LAX-WENDROFF (TAYLOR-GALERKIN)

Start with Taylor-series in time

$$\Delta \mathbf{u} = \Delta t\ \mathbf{u}_{,t} + \left. \frac{\Delta t^2}{2} \mathbf{u}_{,tt} \right|^{n+\Theta}$$

Then use repeatedly original equation

$$\mathbf{u}_{,t} = -\nabla \cdot \mathbf{F}$$

and

$$\Delta \mathbf{F} = \mathcal{A} \Delta \mathbf{u}$$

$\Rightarrow$

$$\Delta \mathbf{u} = -\Delta t\ \nabla \cdot \mathbf{F} + \left. \frac{\Delta t^2}{2} \nabla \cdot \mathcal{A} \cdot \nabla \cdot \mathbf{F} \right|^{n+\Theta}$$

but:

$$\left. \mathbf{F} \right|^{n+\Theta} = \mathbf{F}^n + \Theta \mathcal{A} \cdot \Delta \mathbf{u}$$

$\Rightarrow$

$$\left[1 - \Theta \frac{\Delta t^2}{2} \nabla \cdot \mathcal{A} \otimes \mathcal{A} \nabla \cdot \right] \Delta \mathbf{u} = -\Delta t\ \nabla \cdot \mathbf{F} + \frac{\Delta t^2}{2} \nabla \cdot \mathcal{A} \cdot \nabla \cdot \mathbf{F}$$

EXPEDITING THE RHS EVALUATION (1)

Aim: avoid $\mathcal{A}$

- many entries
- real gas

$\Rightarrow$ Develop 2-step schemes

Half-Step: predict $\mathbf{u}^{n+0.5}$

$$\mathbf{u}^{n+0.5} = \mathbf{u} - \frac{\Delta t}{2} \nabla \cdot \mathbf{F}$$

Full Step: use $\mathbf{u}^{n+0.5}$

$$\Delta \mathbf{u} = -\Delta t \; \nabla \cdot \mathbf{F}(\mathbf{u}^{n+0.5}) = -\Delta t \; \nabla \cdot \mathbf{F}(\mathbf{u} - \frac{\Delta t}{2} \nabla \cdot \mathbf{F})$$

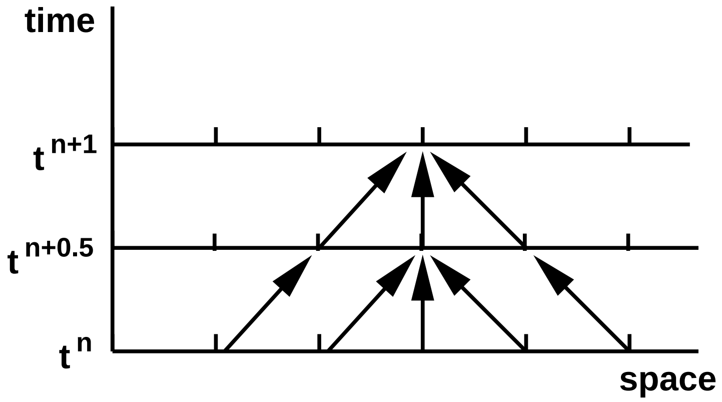
$$= -\Delta t \; \nabla \cdot \mathbf{F}(\mathbf{u}) + \frac{\Delta t^2}{2} \nabla \cdot \mathcal{A} \cdot \nabla \cdot \mathbf{F}(\mathbf{u})$$

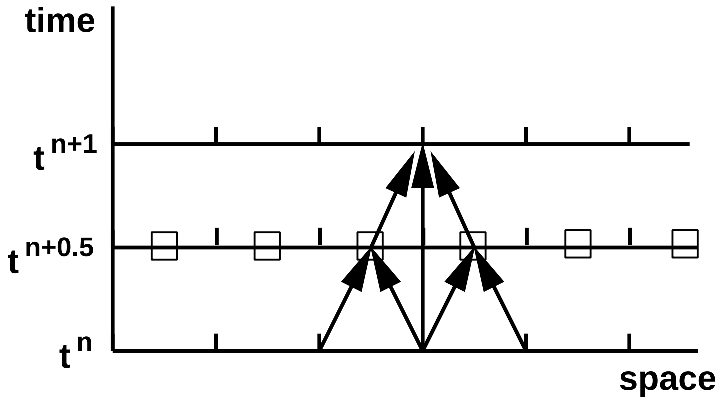
EXPEDITING THE RHS EVALUATION (2)

Several Possibilities

1.) N^i, N^i : 5-point stencil in 1-D (larger support)

2.) P^e, N^i : 3-point stencil in 1-D (same support)





LINEAR ELEMENTS (TRIANGLES, TETRAHEDRA)

Spatial discretization:

- at $t^{n+\frac{1}{2}} = t^n + \frac{1}{2}\Delta t$: $\mathbf{u}, \mathbf{F}$ piecewise constant

- at t^n , t^{n+1} : $\mathbf{u}, \mathbf{F}$ piecewise linear

a) <u>First step:</u> (GATHER, ADD)

$$\mathbf{u}_{el}^{n+\frac{1}{2}} = \frac{1}{Nnode} \cdot \sum_{i=1}^{Nnode} \mathbf{u}_i^n - \frac{\Delta t}{2} \cdot \sum_{i=1}^{Nnode} N_{,k}^i F_i^k$$

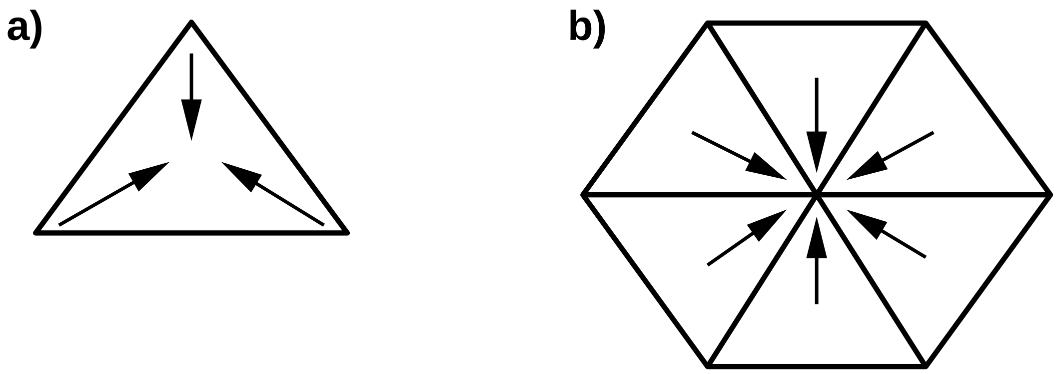
b) <u>Second step:</u> (SCATTER-ADD)

$$\int N^i N^j d\Omega \cdot \Delta \mathbf{u} = \frac{\Delta t}{Nnode} \cdot \sum_{el} VOL_{el} \cdot N_{,k}^i F_{el}^k \Big|^{n+\frac{1}{2}}$$

$\Rightarrow$

$$\mathbf{M}_c \cdot \Delta \mathbf{u}^n = \mathbf{r}^n \qquad (*)$$

DATA FLOW



SOLVING FOR THE CONSISTENT MASS MATRIX

Solve (*) as

$$\mathbf{M}_l \cdot (\Delta \mathbf{u}^n_{i+1} - \Delta \mathbf{u}^n_i) = \mathbf{r}^n - \mathbf{M}_c \cdot \Delta \mathbf{u}^n, \quad i = 0, .., niter$$

and $\Delta \mathbf{u}^n_0 = 0$

Usually $niter = 3$

ARTIFICIAL VISCOSITIES

Observation:

Need to do something special for shocks/discontinuities

$\Rightarrow$ need:

1.) Sensor: e.g. 1st/2nd order derivative

2.) Damping Operator: Laplacian

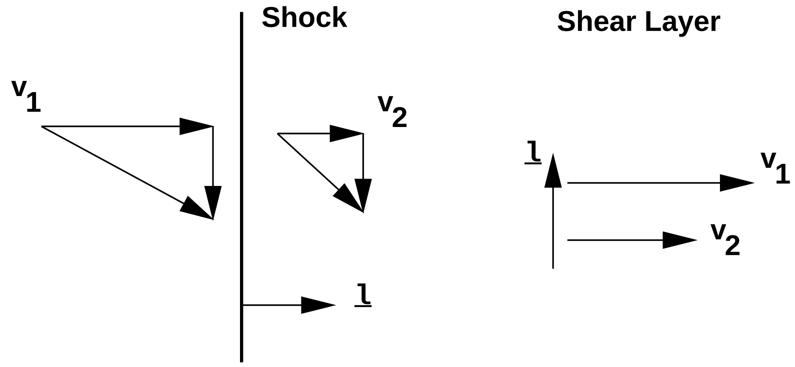
LAPIDUS

$$\mathbf{d} = \Delta t \cdot \frac{\partial}{\partial l}(|k^{ll}|\frac{\partial \mathbf{u}}{\partial l})$$

where

$$\mathbf{l} = \frac{\nabla|\mathbf{v}|}{|\nabla|\mathbf{v}||} \quad , \quad k^{ll} = c_1 \cdot h^2 \cdot \frac{\partial(\mathbf{v}\cdot\mathbf{l})}{\partial l}$$

- **invariant under coordinate rotation**
- **1-D $\Rightarrow$ cheap**
- **good effect at shocks**
- k^{ll} **vanishes at shear/ boundary layers and contact discontinuites**
- **identification of shock ($k^{ll} < 0$) or expansion ($k^{ll} > 0$) simple**



PRESSURE-BASED

$$\mathbf{d} = Cou\ \nabla \left(h^2 f(p)\right) \nabla \mathbf{u}$$

- invariant under coordinate rotation
- ∇^2 approximated by $(\mathbf{M}_l - \mathbf{M}_c) \Rightarrow$ fast
- should vanish at shear/ boundary layers and contact discontinuites
- independent of Δt, but dependent on Courant-nr. Cou
- take enthalpy for energy equation

a) Jameson:

$$f(p) = c_1 \frac{|\nabla h^2 \nabla p|}{\overline{p}}$$

- good for transonic flows

b) Peraire et al. :

$$f(p) = c_2 \frac{|\nabla h^2 \nabla p|}{|h \nabla p|}$$

- good for hypersonic flows