CONSERVATION LAWS (1)

Quantities Conserved:

- Mass
- Momentum (Newton' Law)
- Energy (First Law of Thermodynamics)

General Form:

$$Change = Production(Destruction)$$

In Eulerian Frame:

$$Change|_{fixed\mathbf{x}} + Transport = Diffusion + Production$$

Or:

$$\Phi_{,t} + \nabla \cdot \mathbf{v}\Phi = \nabla \cdot q + S$$

COMPRESSIBLE NAVIER-STOKES EQUATIONS (1)

Define:

ρ : density
v_i: fluid velocity in direction x_i
p : pressure
e : specific total energy
T : temperature
q_i: heat flux in direction x_i
σ_{ij}: (deviatoric) stress tensor

Then: (L.M.H. Navier (1827), G.G. Stokes (1845))

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = \nabla \cdot \mathbf{F}^v + \mathbf{S}$$

$$\mathbf{u} = \left\{ \begin{array}{r} \rho \\ \rho v_i \\ \rho e \end{array} \right\} \; , \; \mathbf{F}^a_j = \left\{ \begin{array}{r} \rho v_j \\ \rho v_i v_j + p\delta_{ij} \\ v_j(\rho e + p) \end{array} \right\} \; , \; \mathbf{F}^v_j = \left\{ \begin{array}{r} 0 \\ \sigma_{ij} \\ v_l \sigma_{lj} - q_j \end{array} \right\}$$

COMPRESSIBLE NAVIER-STOKES EQUATIONS (2)

State Equations

In General:

$$p = p(\rho, T)$$

Gases:

$$p = \frac{R}{m}\rho T = r\rho T$$

R : Absolute Gas Constant
($R = 8.312 \cdot \ m^2 kg/sec^2 \ ^oKmol$)
m : Mass per mol
(Mass of 22.4 Liters of Gas at 0^oC and $760 mmHg$)

Gas	O_2	N_2	H_2	Air
m $[g/mol]$	32	28.016	2.016	29
c $[m/sec]$	330	353	1316	347

COMPRESSIBLE NAVIER-STOKES EQUATIONS (3)

$$\epsilon = e - \frac{1}{2} v_j v_j = c_v T$$

ϵ : Internal Energy
c_v: Specific Heat at Constant Volume

$$h = \epsilon + \frac{p}{\rho} = c_p T$$

h: Enthalpy
c_p: Specific Heat at Constant Pressure

$\Rightarrow$ For **Ideal** Gases:

$$p = (\gamma - 1)\rho[e - \frac{1}{2} v_j v_j]$$

γ: Ratio of Specific Heats ($= c_p/c_v$)
Valid up to 800K (Molecular Vibration)

COMPRESSIBLE NAVIER-STOKES EQUATIONS (4)

Viscous Stress Tensor

$$\sigma_{ij} = \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \lambda \frac{\partial v_k}{\partial x_k} \delta_{ij}$$

- So-Called Newtonian Fluid
- In General: $\sigma_{ij} = \sigma_{ij}(\nabla \mathbf{v})$

Stokes Hypothesis:

$$\lambda = -\frac{2\mu}{3}$$

Sutherland's Law:

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{3}{2}} \cdot \frac{T_\infty + 110}{T + 110}$$

- Valid for gases

COMPRESSIBLE NAVIER-STOKES EQUATIONS (5)

Fourier's Law:

$$\mathbf{q} = -k\nabla T$$

k: conductivity

For Gases:

$$\frac{k}{k_\infty} = \frac{\mu}{\mu_\infty}$$

COMPRESSIBLE NAVIER-STOKES EQUATIONS (6)

Entropy: s

$$T ds = de + pd\left(\frac{1}{\rho}\right) = dh - \frac{dp}{\rho}$$

$\Rightarrow$

$$T ds = c_p\left(dT - \frac{\gamma - 1}{\gamma}\frac{T}{p}dp\right)$$

$\Rightarrow$

$$s - s_0 = c_p\left(ln\left(\frac{T}{T_0}\right) - \frac{\gamma - 1}{\gamma}ln\left(\frac{p}{p_0}\right)\right)$$

$\Rightarrow$ For Isentropic Flows ($ds = 0$):

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$\Rightarrow$ With $p = r\rho T$, $h = c_p T$:

$$\frac{h}{h_0} = \frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

COMPRESSIBLE NAVIER-STOKES EQUATIONS (7)

Speed of Sound:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \frac{p}{\rho} = (\gamma - 1) c_p T = (\gamma - 1) h$$

Iso-Enthalpic Flows Revisited:

$$h = c_p T + \frac{\mathbf{v}^2}{2} = c_p T_0$$

T_0: Stagnation Temperature

$$T_0 = T \cdot \left(1 + \frac{\mathbf{v}^2}{2 c_p T}\right) = T \cdot \left(1 + \frac{(\gamma - 1)\mathbf{v}^2}{2\gamma \frac{p}{\rho}}\right)$$

$$= T \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)$$

M: $= |\mathbf{v}|/c$: Mach-Number

COMPRESSIBLE NAVIER-STOKES EQUATIONS (8)

Isentropic Flows:

$$\frac{\rho}{\rho_0} = \left(\frac{h}{h_0}\right)^{\frac{1}{\gamma-1}} = \left(1 - \frac{\mathbf{v}^2}{2h_0}\right)^{\frac{1}{\gamma-1}}$$

and:

$$c^2 = \gamma\frac{p}{\rho} = (\gamma-1)h = (\gamma-1)(h_0 - \frac{\mathbf{v}^2}{2})$$

NON-DIMENSIONAL FORM (1)

Aim: Establish Similarity for Flows

How: Define Reference Quantities

- Inflow Quantities
- Internal Characteristic Quantities

Typically:

L : Characteristic Length

$|\mathbf{v}_\infty|$: Inflow/Free-Stream Velocity

ρ_∞ : Inflow/Free-Stream Density

T_∞ : Inflow/Free-Stream Temperature

μ_∞ : Inflow/Free-Stream Viscosity

k_∞ : Inflow/Free-Stream Conductivity

NON-DIMENSIONAL FORM (2)

Define Non-Dimensional Quantities:

$$t^\star = \frac{t\ |\mathbf{v}_\infty|}{L} \quad , \quad x_i^\star = \frac{x_i}{L} \quad , \quad v_i^\star = \frac{v_i}{|\mathbf{v}_\infty|}$$

$$\rho^\star = \frac{\rho}{\rho_\infty} \quad , \quad T^\star = \frac{T}{T_\infty} \quad , \quad p^\star = \frac{p}{\rho_\infty|\mathbf{v}_\infty|^2} \quad , \quad e^\star = \frac{e}{|\mathbf{v}_\infty|^2}$$

$$\mu^\star = \frac{1}{Re_{\infty,L}} \cdot \frac{\mu}{\mu_\infty} \quad , \quad k^\star = \frac{1}{(\gamma-1)M_\infty^2 Pr Re_{\infty,L}} \cdot \frac{k}{k_\infty}$$

NON-DIMENSIONAL FORM (3)

Characteristic Numbers:

$Re_{\infty,L} = \frac{\rho_\infty |\mathbf{v}_\infty| L}{\mu_\infty}$: Reynolds-Number;

Ratio of: Intertial Forces:Viscous Forces

$M_\infty = \frac{|\mathbf{v}_\infty|}{c_\infty}$: Mach-Number;

Ratio of: Fluid Velocity:Speed of Sound

$Pr_\infty = \frac{c_p \mu_\infty}{k_\infty}$: Prandtl-Number;

Ratio of: Viscosity:Conductivity

$\gamma = \frac{c_p}{c_v}$:

Ratio of: Specific Heat at Constant Pressure:
Specific Heat at Constant Volume

NON-DIMENSIONAL FORM (4)

By Going to Dimensionless Form, Dropping the *:

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = \nabla \cdot \mathbf{F}^v + \mathbf{S}$$

$$\mathbf{u} = \left\{ \begin{array}{r} \rho \\ \rho v_i \\ \rho e \end{array} \right\} \; , \; \mathbf{F}^a_j = \left\{ \begin{array}{r} \rho v_j \\ \rho v_i v_j + p\delta_{ij} \\ v_j(\rho e + p) \end{array} \right\} \; , \; \mathbf{F}^v_j = \left\{ \begin{array}{r} 0 \\ \sigma_{ij} \\ v_l \sigma_{lj} - q_j \end{array} \right\}$$

$$p = (\gamma - 1)\rho[e - \frac{1}{2} v_j v_j]$$

$$\sigma_{ij} = \frac{1}{Re}\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij} \right)$$

$$\mathbf{q} = -\frac{1}{PrRe} k \nabla T$$

TYPICAL PARAMETERS AT 20^o C

Fluid	ρ $[kg/m^3]$	μ $[kg/m/sec]$	ν $[m^2/sec]$	Pr
Water	1000	$1.00 \cdot 10^{-3}$	$1.00 \cdot 10^{-6}$	7.030
Glycerine			$6.80 \cdot 10^{-4}$	
Mercury			$1.00 \cdot 10^{-7}$	0.023
Lubr. Oil			$8.92 \cdot 10^{-4}$	10-100
Air	1.21	$1.79 \cdot 10^{-5}$	$1.50 \cdot 10^{-5}$	0.710

TYPICAL REYNOLDS NUMBERS

Object	L [m]	V [m/sec]	Re
Protozoa	.001	0.01	$1.00 \cdot 10^1$
Piranha	0.15	0.30	$4.50 \cdot 10^4$
Manta-Ray	2.50	0.50	$1.25 \cdot 10^6$
Penguin	1.00	5.00	$5.00 \cdot 10^6$
White Shark	4.00	3.00	$1.20 \cdot 10^7$
Tuna	3.00	10.0	$3.00 \cdot 10^7$
Blue Whale	30.0	1.00	$3.00 \cdot 10^7$
Butterfly	0.05	4.00	$1.30 \cdot 10^4$
Albatros	0.30	10.0	$2.00 \cdot 10^5$
Car (Sedan)	4.00	25.0	$6.60 \cdot 10^6$
Truck	25.0	25.0	$4.20 \cdot 10^7$
B-747 WTO	6.00	40.0	$1.60 \cdot 10^7$
B-747 WCR	6.00	220.	$8.80 \cdot 10^7$
SR-71	20.0	2220	$2.90 \cdot 10^9$

LIMIT OF VALIDITY OF CONTINUUM APPROXIMATION

Define: Knudsen-Number

$$Kn = \frac{Mean\ Free\ Path}{Characteristic\ Length} \approx \frac{M_\infty}{Re_{\infty,L}}$$

For $Kn \leq 0.03$: Continuum Approximation Very Good

Exceptions:

- Large Mean Free Paths
  - Reentry Flows
- Small Characteristic Lengths
  - Micromachines