FROM APPROXIMATION TO OPERATORS

Before: given u, approximate: $\|u - u^h\| \rightarrow min$

Now: given $L(u) = 0$ approximate: $\|L(u) - L(u^h)\| \rightarrow min$

$\Rightarrow \qquad \|L(u^h)\| \rightarrow min$

Minimize error:

$$\epsilon_L^h = L(u^h) = L(N^i \hat{u}_i)$$

using WRM

$$\int_\Omega W^i \epsilon_L^h d\Omega = 0 \quad , \quad i = 1, M$$

Choice of N^i, W^i defines the method:

- N^i polynomial, $W^i = \delta(x_i)$: FDM
- N^i polynomial, $W^i = 1$ if x$\subset \Omega_{el}$, 0 otherwise : FVM
- N^i polynomial, $W^i = N^i$: GFEM
- N^i polynomial, $W^i \neq N^i$: Petrov-GFEM
- N^i spectral, $W^i = \delta(x_i)$: SEM

LAPLACE OPERATOR

Given

$$\nabla^2 u = 0 \quad in \quad \Omega \quad , \quad u = 0 \quad on \quad \Gamma$$

WRM

$$\int_\Omega W^i \ \nabla^2 N^j \ d\Omega \ \hat{u}_j = 0$$

$\Rightarrow$

- N^j should have defined 2nd order derivatives
  $\Rightarrow C^1$-continuous across elements
- W^i can be the δ-function

Integration by parts:

$$-\int_\Omega \nabla W^i \cdot \nabla N^j \ d\Omega \ \hat{u}_j = 0$$

$\Rightarrow$

- Order of max(derivative) reduced $\Rightarrow$ can use wider space of trial/weight functions
- N^j should have defined 1st order derivatives
  $\Rightarrow C^0$-continuous across elements
- W^i can not be the δ-function

MINIMIZATION PROBLEM

$$I_{rz} = \int_\Omega [\nabla \epsilon^h]^2 \; d\Omega = \int_\Omega [\nabla (u^h - u)]^2 \; d\Omega \to min$$

$\Rightarrow$

$$\delta I_{rz} = \delta \hat{u}_i \int_\Omega \nabla N^i \cdot (\nabla N^j \hat{u}_j - \nabla u) \; d\Omega = 0$$

but

$$-\int_\Omega \nabla N^i \cdot \nabla u \; d\Omega = \int_\Omega N^i \nabla^2 u \; d\Omega = 0$$

$\Rightarrow$

$$\delta I_{rz} = \delta \hat{u}_i \int_\Omega \nabla N^i \cdot \nabla N^j \; d\Omega \; \hat{u}_j = 0$$

$\Rightarrow$ equivalent to Galerkin WRM

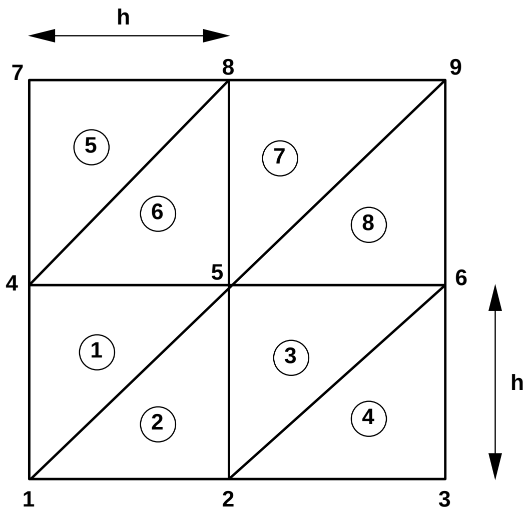
$\Rightarrow$ choice of W^i from same set as N^i optimal

EXAMPLE

Regular triangular mesh - assemble element contributions to produce the equation for a typical interior point for the Poisson-operator

$$-\nabla^2 u = f$$

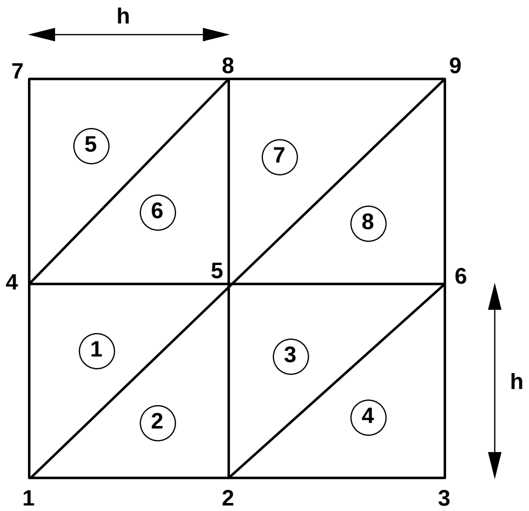
<u>Mesh:</u>



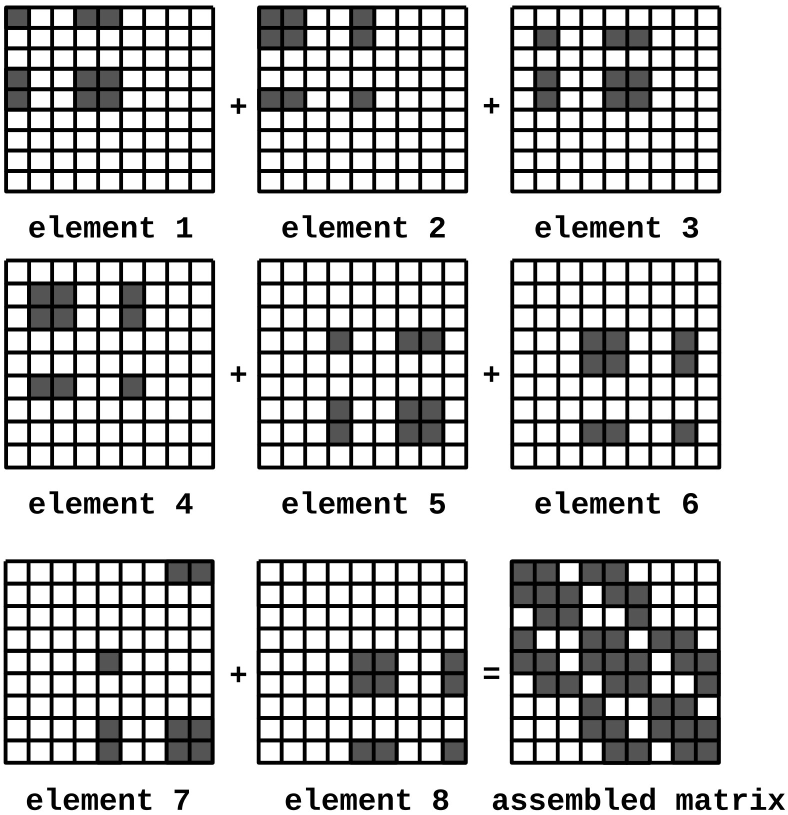
Example for Poisson Operator

Element	Nodes (A,B,C)		
1	1	5	4
2	2	5	1
3	2	6	5
4	2	3	6
5	4	8	7
6	4	5	8
7	5	9	8
8	5	6	9

Mesh:



Matrix Entries:



<u>Element 1</u>

Shape-Function Derivatives:

$$\begin{bmatrix} N^A \\ N^B \\ N^C \end{bmatrix}_{,x} = \frac{1}{h^2} \begin{bmatrix} 0 \\ h \\ -h \end{bmatrix} \quad , \quad \begin{bmatrix} N^A \\ N^B \\ N^C \end{bmatrix}_{,y} = \frac{1}{h^2} \begin{bmatrix} -h \\ 0 \\ h \end{bmatrix}$$

LHS contribution:

$$\mathbf{K}_1 \cdot \mathbf{u}_1 = \frac{1}{2h^2} \begin{bmatrix} h^2 & 0 & -h^2 \\ 0 & h^2 & -h^2 \\ -h^2 & -h^2 & 2h^2 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_5 \\ \hat{u}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_5 \\ \hat{u}_4 \end{bmatrix}$$

RHS contribution

$$\mathbf{M}^1 \cdot \mathbf{f}_1 = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \hat{f}_1 \\ \hat{f}_5 \\ \hat{f}_4 \end{bmatrix} = \frac{h^2}{24} \begin{bmatrix} 2\hat{f}_1 + \hat{f}_5 + \hat{f}_4 \\ \hat{f}_1 + 2\hat{f}_5 + \hat{f}_4 \\ \hat{f}_1 + \hat{f}_5 + 2\hat{f}_4 \end{bmatrix}$$

Element 2

Shape-Function Derivatives:

$$\begin{bmatrix} N^A \\ N^B \\ N^C \end{bmatrix}_{,x} = \frac{1}{h^2} \begin{bmatrix} -h \\ h \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} N^A \\ N^B \\ N^C \end{bmatrix}_{,y} = \frac{1}{h^2} \begin{bmatrix} 0 \\ -h \\ h \end{bmatrix}$$

LHS contribution:

$$\mathbf{K}_2 \cdot \mathbf{u}_2 = \frac{1}{2h^2} \begin{bmatrix} h^2 & -h^2 & 0 \\ -h^2 & 2h^2 & -h^2 \\ 0 & -h^2 & h^2 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_5 \end{bmatrix}$$

RHS contribution

$$\mathbf{M}^1 \cdot \mathbf{f}_1 = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_5 \end{bmatrix} = \frac{h^2}{24} \begin{bmatrix} 2\hat{f}_1 + \hat{f}_2 + \hat{f}_5 \\ \hat{f}_1 + 2\hat{f}_2 + \hat{f}_5 \\ \hat{f}_1 + \hat{f}_2 + 2\hat{f}_5 \end{bmatrix}$$

Fully assembled form of equation 5

$$4\hat{u}_5 - \hat{u}_2 - \hat{u}_4 - \hat{u}_6 - \hat{u}_8 = \frac{h^2}{12}(6\hat{f}_5 + \hat{f}_1 + \hat{f}_2 + \hat{f}_6 + \hat{f}_4 + \hat{f}_8 + \hat{f}_9)$$

Compare this with finite difference expansion for

$$\left[-\nabla^2 u - f\right]_{node\ 5} = 0$$

$$4\hat{u}_5 - \hat{u}_2 - \hat{u}_4 - \hat{u}_6 - \hat{u}_8 = h^2 \hat{f}_5$$

SIMPLE SMOOTHER

<u>Desired:</u> Fast approximation to: $\nabla h^2 \nabla u$

For linear triangles:

$$\frac{12}{A} \cdot (\mathbf{M}_l - \mathbf{M}_c)_{el} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- Same can be shown for linear tetrahedra

RECOVERY OF FIRST ORDER DERIVATIVES

Given:

$$u \approx N^i \hat{u}_i$$

Approximate:

$$\frac{\partial u}{\partial s} \approx \tilde{N}^i \hat{u}'_i$$

$\Rightarrow$

$$\frac{\partial u}{\partial s} \approx \frac{\partial N^i}{\partial s} \hat{u}_i$$

$\Rightarrow$

$$\tilde{N}^i \hat{u}'_i \approx \frac{\partial N^k}{\partial s} \hat{u}_k$$

WRM:

$$\int_\Omega W^i \tilde{N}^j d\Omega \; \hat{u}'_j = \int_\Omega W^i \frac{\partial N^j}{\partial s} d\Omega \; \hat{u}_j$$

Galerkin:

$$\mathbf{M}_c \mathbf{u}' = \int_\Omega N^i N^j d\Omega \; \hat{u}'_j = \int_\Omega N^i \frac{\partial N^j}{\partial s} d\Omega \; \hat{u}_j$$

RECOVERY OF SECOND ORDER DERIVATIVES (1)

Possibilities:

a) Repeated Evaluation of First Derivatives

b) Direct Evaluation of Second Derivatives (Faster)

$$\frac{\partial^2 u}{\partial s^2} \approx \tilde{N}^i \ \hat{u}_i^{''}$$

WRM:

$$\int_\Omega W^i \tilde{N}^j d\Omega \ \hat{u}_j^{''} = \int_\Omega W^i \frac{\partial^2 N^j}{\partial s^2} d\Omega \ \hat{u}_j$$

Integration by Parts:

$$\int_\Omega W^i \tilde{N}^j d\Omega \ \hat{u}_j^{''} = -\int_\Omega \frac{\partial W^i}{\partial s} \frac{\partial N^j}{\partial s} d\Omega \ \hat{u}_j + \int_\Gamma W^i n_s \frac{\partial N^j}{\partial s} d\Gamma \ \hat{u}_j$$

RECOVERY OF SECOND ORDER DERIVATIVES (2)

Galerkin:

$$\mathbf{M}_c \mathbf{u}^{''} = \int_\Omega N^i N^j d\Omega \ \hat{u}_j^{''} = -\int_\Omega \frac{\partial N^i}{\partial s} \frac{\partial N^j}{\partial s} d\Omega \ \hat{u}_j$$

$$+ \int_\Gamma N^i n_s \frac{\partial N^j}{\partial s} d\Gamma \ \hat{u}_j$$

Higher Order Derivatives: Recursively