WHY INCOMPRESSIBLE ?

Pros:

- No c, $\Rightarrow$ detailed treatment of $\mathbf{v}$
- No e, $\Rightarrow$ faster

Cons:

- $c \rightarrow \infty$, $\Rightarrow$ need implicit/scaled scheme for p
- $\nabla \cdot \mathbf{v} = 0$, $\Rightarrow$ need to satisfy LBB

INCOMPRESSIBLE NAVIER-STOKES EQNS.

$$\mathbf{v}_{,t} + \mathbf{v}\nabla\mathbf{v} + \nabla p = \frac{1}{Re}\nabla^2\mathbf{v}$$

$$\nabla \cdot \mathbf{v} = 0$$

<u>Difficulty 1</u>: $\mathbf{v}\nabla\mathbf{v}$

<u>Difficulty 2</u>: $\nabla \cdot \mathbf{v} = 0$

ADVECTION OPERATOR (1)

Solution 1: Streamline Diffusion aka

Taylor/Galerkin
Lax Wendroff
Balancing Dissipation

$$-\mathbf{v}\nabla\mathbf{v} \quad \rightarrow \quad -\mathbf{v}\nabla\mathbf{v} + \alpha\nabla\mathbf{v} \otimes \mathbf{v}\nabla\mathbf{v}$$

with

$$\alpha = \alpha(Re_h)$$

ADVECTION OPERATOR (2)

Solution 2: Edge-Based Upwind Operator

$$r^i = D^{ij}\mathcal{F}_{ij} = D^{ij}(\mathbf{f}_i + \mathbf{f}_j)$$

$\mathbf{f}_i$: 'Fluxes Along Edges'

$$\mathbf{f}_i = S_k^{ij}\mathbf{F}_i^k \quad , \quad S_k^{ij} = \frac{d_k^{ij}}{D^{ij}} \quad , \quad D^{ij} = \sqrt{d_k^{ij} d_k^{ij}}$$

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j$$

$$\mathbf{f}_i = (S_k^{ij} v_i^k)\mathbf{v}_i \quad , \quad \mathbf{f}_j = (S_k^{ij} v_j^k)\mathbf{v}_j$$

Central Difference Flux → Consistent Numerical Flux

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j - |v^{ij}|(\mathbf{v}_i - \mathbf{v}_j)$$

$$v^{ij} = \frac{1}{2} S_k^{ij}\left(v_i^k + v_j^k\right)$$

First-Order → Second-Order Via MUSCL Limiting

DIV-CONSTRAINT (1)

Exact:

$$\nabla \cdot \mathbf{v}\nabla\mathbf{v} + \nabla \cdot \nabla p = 0$$

Numerical:

$$\mathbf{C}^t\mathbf{M}^{-1}\left[\mathbf{A}\mathbf{v} + \mathbf{C}\mathbf{p} + \mathbf{K}\mathbf{v}\right] = 0$$

Thus

$$\nabla \cdot \nabla p \approx \mathbf{C}^t\mathbf{M}^{-1}\mathbf{C}\mathbf{p}$$

May be **Unstable**

Resulting Operator: 1-D, Uniform Mesh

Table 1 $\mathbf{C}^t\mathbf{M}^{-1}\mathbf{C}$ for Different Velocity/ Pressure Comb.

Velocity	Pressure	Operator
p1	q0	(0,-1, 2,-1, 0)
p1	p1	(-1, 0, 2, 0,-1)
iso-p1	p1	(-1,-1, 4,-1,-1)

DERIVATION OF P1/P1 ELEMENT

Outline of Proof:

- take momentum PDE and weigh with N^b
- use fact that most integrals disappear
- express $\mathbf{v}^b$ as a function of p^i
- insert into continuity PDE

$\Rightarrow$ p1/p1+bubble is the same as p1/p1 with

$$\nabla \cdot \mathbf{v} = \nabla \beta \nabla p + \nabla \cdot \beta \mathbf{v} \cdot \nabla \mathbf{v}$$

with

$$\beta = \frac{1}{Nn^5 \left[\frac{1}{Re} \sum_j |\nabla N^j|^2 + \frac{\Delta t}{2} \sum_j \left(\mathbf{v} \cdot \nabla N^j \right)^2 \right]}$$

AN ALTERNATIVE DERIVATION

Acoustic Limit: $c >> |\mathbf{v}|$

$$c^{-2} p_{,t} + \nabla \cdot \mathbf{v} = 0$$

Taylor-expansion in time:

$$c^{-2} \Delta p = c^{-2} \Delta t \; p_{,t}|^{n} + c^{-2} \frac{\Delta t^2}{2} p_{,tt}|^{n+\theta}$$

$\Rightarrow$

$$\Delta t \; \nabla \cdot \mathbf{v} = \frac{\Delta t^2}{2} \left[\nabla^2 p + \nabla \cdot \mathbf{v} \cdot \nabla \mathbf{v} \right]^{n+\theta}$$

- Laplacian ‘Pressure-Diffusion’

MINI-P1/P1-ELEMENT TRADEOFFS

Mini-Element same as P1-Element with:

$$\nabla \cdot \mathbf{v} = \nabla \cdot \beta \left(\nabla p + \mathbf{v} \cdot \nabla \mathbf{v} \right)$$

$\nabla^2 p$: Stability

$\nabla \cdot \mathbf{v} \cdot \nabla \mathbf{v}$: Accuracy/Consistency

$\Rightarrow$ 2 Options:

a) more effort per element with original eqns
$\Rightarrow$ mixed elements

b) more effort per div-eqn with a simpler element
$\Rightarrow$ chosen here

DIV-CONSTRAINT (2)

Edge-Based Solvers:

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j \quad , \quad \mathbf{f}_i = S_k^{ij} v_i^k \quad , \quad \mathbf{f}_j = S_k^{ij} v_j^k$$

Consistent Numerical Flux:

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j - |\lambda^{ij}|(p_i - p_j)$$

$$\lambda^{ij} = \frac{\Delta t^{ij}}{l^{ij}}$$

Δt : characteristic Advective Timestep of Edge

l : Characteristic Advective Length of Edge

First-Order $\rightarrow$ Second-Order Via

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j - |\lambda^{ij}|(p_i - p_j + \frac{l^{ij}}{2}(\nabla p_i + \nabla p_j))$$

- $\Rightarrow$ Fourth-Order Damping For the Divergence Eqn.

DIV-CONSTRAINT (4)

Artificial Compressibility

$$\frac{1}{\beta^2} p_{,t} + \nabla \cdot \mathbf{v} = 0$$

- Treat as Hyperbolic System
- ‘Velocity of Sound’: β
- Roe-Matrix Will Combine $\mathbf{v}, p$ Modes

  $\Rightarrow$ More Elaborate Artificial Viscosity

TEMPORAL DISCRETIZATION: Δ-SCHEME

a) Advective/Diffusive Prediction: $\mathbf{v}^n \rightarrow \mathbf{v}^*$

$$\left[\frac{1}{\Delta t} - \frac{1}{Re}\nabla^2\right] \cdot (\mathbf{v}^* - \mathbf{v}^n) + \mathbf{v}^n \cdot \nabla \mathbf{v}^n + \nabla p^n = \frac{1}{Re}\nabla^2 \mathbf{v}^n$$

b) Pressure Correction: $p^n \rightarrow p^{n+1}$

$$\nabla \cdot \mathbf{v}^{n+1} = 0$$

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^*}{\Delta t} + \nabla(p^{n+1} - p^n) = 0$$

$\Rightarrow$

$$\nabla^2(p^{n+1} - p^n) = \frac{\nabla \cdot \mathbf{v}^*}{\Delta t}$$

c) Velocity Correction: $\mathbf{v}^* \rightarrow \mathbf{v}^{n+1}$

$$\mathbf{v}^{n+1} = \mathbf{v}^* - \Delta t \nabla(p^{n+1} - p^n)$$

Remarks:

- Residuals of Pressure Correction Vanish for Steady-State
  $\Rightarrow$ Result Not Dependent on Projection Scheme
  $\Rightarrow$ Result Not Dependent on Δt

TEMPORAL DISCRETIZATION: PROJECTION METHOD

$$\mathbf{v}^{n+1} = \mathbf{v}^n + \Delta\mathbf{v}^a + \Delta\mathbf{v}^p = \mathbf{v}^{**} + \Delta\mathbf{v}^p$$

$\Rightarrow$

a) Advective/Diffusive Prediction: $\mathbf{v}^n \rightarrow \mathbf{v}^{**}$

$$\left[\frac{1}{\Delta t} - \frac{1}{Re}\nabla^2\right] \cdot (\mathbf{v}^{**} - \mathbf{v}^n) + \mathbf{v}^n \cdot \nabla\mathbf{v}^n = \frac{1}{Re}\nabla^2\mathbf{v}^n$$

b) Pressure Correction: $p^n \rightarrow p^{n+1}$

$$\nabla \cdot \mathbf{v}^{n+1} = 0$$
$$\mathbf{v}^{n+1} + \Delta t \nabla p^{n+1} = \mathbf{v}^{**}$$

$\Rightarrow$

$$\nabla^2 p^{n+1} = \frac{\nabla \cdot \mathbf{v}^{**}}{\Delta t}$$

c) Velocity Correction: $\mathbf{v}^{**} \rightarrow \mathbf{v}^{n+1}$

$$\mathbf{v}^{n+1} = \mathbf{v}^{**} - \Delta t \nabla p^{n+1}$$

Remarks:

- Residuals of Pressure Correction Do Not Vanish for Steady-State
  $\Rightarrow$ Result Dependent on Projection Scheme
  $\Rightarrow$ Result Dependent on Δt

COMPUTATION OF Δt

$$\Delta t = r(Re_c) \cdot \Delta t_{CFL}$$

Aim:

$$\Delta t_{CFL} = \frac{length\ covered}{velocity}$$

- Direction of normal: $\mathbf{n}_i = \frac{\nabla N_i}{|\nabla N_i|}$
- Length of normal: $h_i = \frac{1}{|\nabla N_i|}$

$\Rightarrow$

$$\Delta t_i = \frac{h_i}{|\mathbf{v} \cdot \mathbf{n}_i|} = \frac{1}{|\mathbf{v} \cdot \nabla N_i|}$$

<u>Take:</u> $\Delta t : min(\Delta t_i)$

COMPUTATION OF UPWINDING FACTOR α

Cell Re-nr. for normals

$$Re_i = h_i Re = \frac{|\mathbf{v} \cdot \nabla N_i|}{|\nabla N_i|^2} Re$$

$\Rightarrow$

$$\alpha_i = coth\left(\frac{Re_i}{2}\right) - \frac{2}{Re_i}$$

<u>Take:</u> $\alpha : max(\alpha_i)$

ACCELERATION TO STEADY-STATE

- Local timesteps
- Reduced pressure iterations
- Projective Prediction of Pressure Increments
- Substepping for advection terms
- Implicit treatment of advection terms
- Fully implicit treatment

REDUCED PRESSURE ITERATIONS

Most Time-Consuming Part: Pressure-Poisson

Steady-State $\Rightarrow$ $Div\mathbf{v} = 0$ Not Required Every Step

$\Rightarrow$

- Steps 1:n-1: Use Low Number of Iterations (e.g. 20)
- Step n: Converge to 10^{-3}

Re:

- Works Well for Euler
- Does Not Work Well for RANS Grids

PROJECTIVE PREDICTION OF PRESSURE INCREMENTS

Most Time-Consuming Part: Pressure-Poisson

$$\mathbf{K} \cdot \Delta\mathbf{p} = \mathbf{r}$$

Idea: Start With Good Estimate for $\Delta\mathbf{p}$

- $\Delta\mathbf{p} = 0$ (Steady State)
- $\Delta\mathbf{p}$ From Previous Timesteps

PROJECTIVE PREDICTION (1)

Basic Assumption: $\mathbf{K} \approx const.$

Timesteps/Iterations: $n-1, n-2, ..., n-i$:

$$\mathbf{K} \cdot \Delta \mathbf{p}^i = \mathbf{r}^i$$

Perform Least Squares Approximation of $\mathbf{r}$ With Basis $\mathbf{r}^i, i = 1, l$:

$$\left(\mathbf{r} - \alpha_i \mathbf{r}^i\right)^2 \rightarrow min$$

$\Rightarrow$

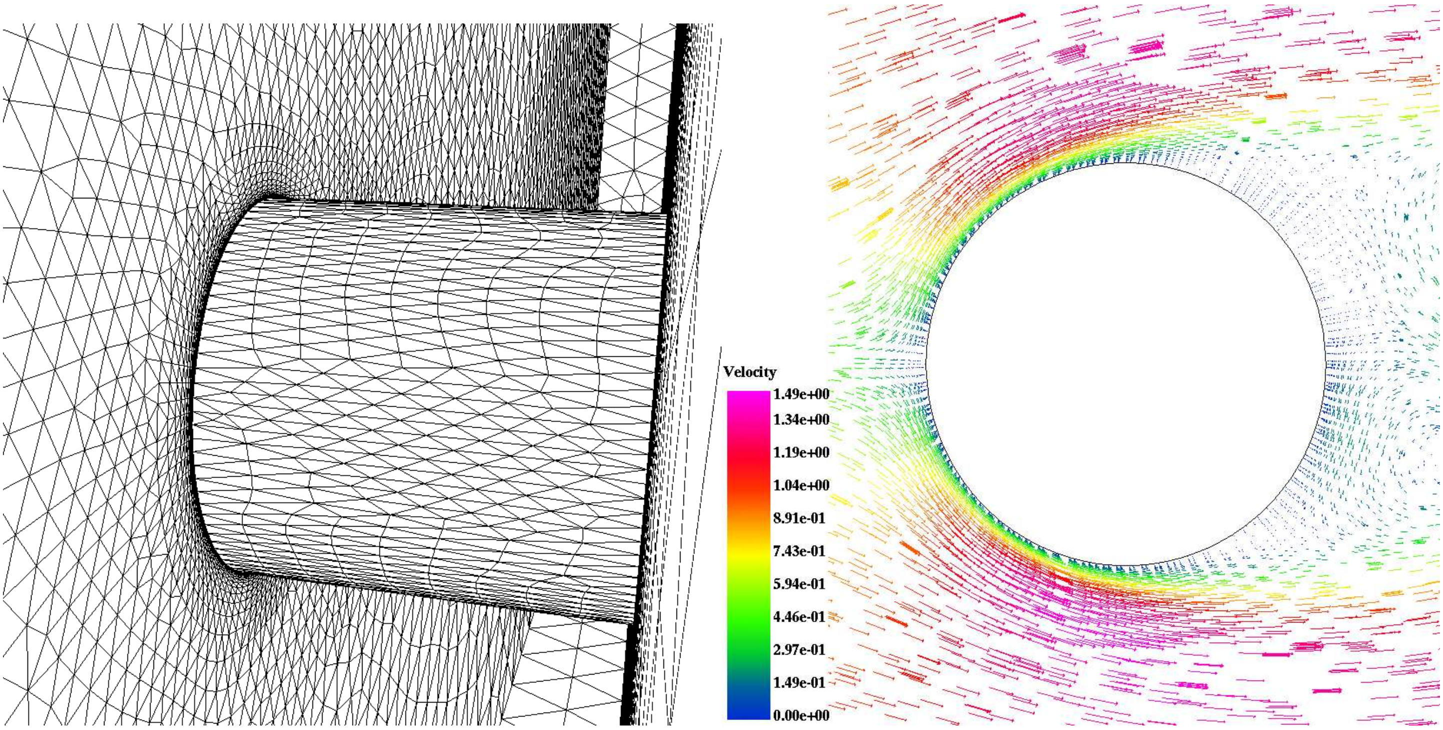
$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{s} \quad , \quad A^{ij} = \mathbf{r}^i \cdot \mathbf{r}^j \quad , \quad s^i = \mathbf{r}^i \cdot \mathbf{r}$$

With α_i:

$$\Delta \mathbf{p} = \alpha_i \Delta \mathbf{p}^i$$

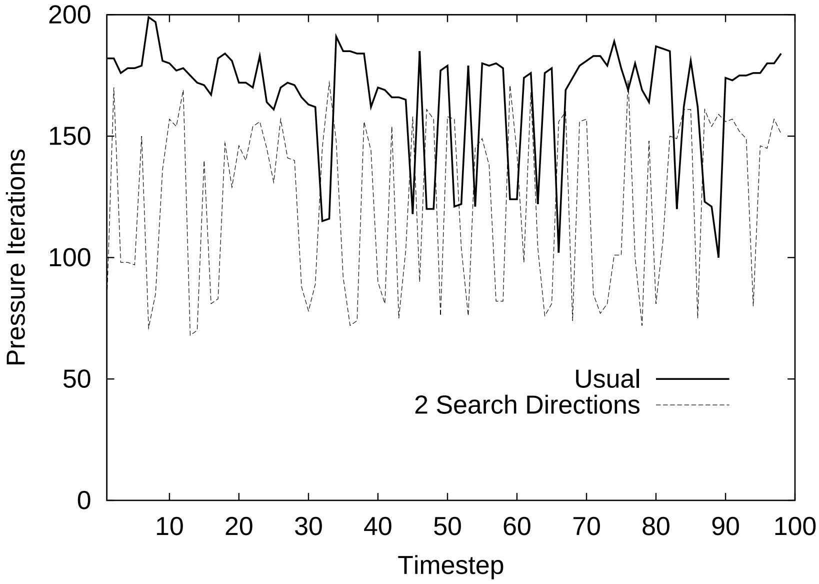
PPPI: CYLINDER (1)

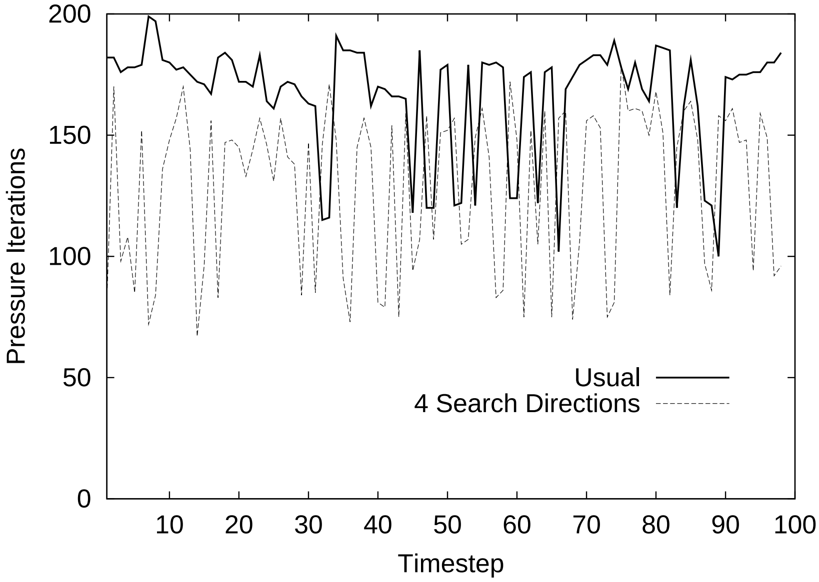
Cylinder: $Re = 200$



Cylinder: Mesh and Flowfield

PPPI: CYLINDER (2)





Cylinder: Iterations for the Pressure-Poisson System

Observed Gains: 25%

SUBSTEPPING OF ADVECTIVE TERMS (1)

Desired:

- Second order in space for steady state
- Second order in time
- Fourth order in phase for advection
- Steady state independent of timestep
- Simple (symmetric matrix) solver for systems of equations

Assume Large Variation of:

$$Re_h = \frac{\rho|\mathbf{v}|h}{\mu}$$

$Re_h < 1$ Time-Accuracy Not So Important;
$Re_h > 1$ $\Rightarrow$ Considerable Advantages of Higher Order Time-Marching Schemes (e.g. Vortex Transport)

SUBSTEPPING OF ADVECTIVE TERMS (2)

Dahlquist's Theorem: Unconditionally Stable (Implicit) Schemes:

$$O(\Delta t^p), p \leq 2$$

$\Rightarrow$ Use Explicit Runge-Kutta Schemes for Higher Order

Define:

$\mathbf{r}^a$: Advective Terms
$\mathbf{r}^v$: Viscous terms
$\mathbf{s}$: Remaining Source Terms
(Pressure, Particles, Buoyancy, etc.)

2-Step Scheme:

$$\mathbf{M}_l(\mathbf{v}^1 - \mathbf{v}^0) = \frac{\Delta t}{2}\left(\mathbf{r}^a(\mathbf{v}^0) + \mathbf{r}^v(\mathbf{v}^0) + \mathbf{s}\right)$$

$$\mathbf{M}_c(\mathbf{v}^2 - \mathbf{v}^0) = \Delta t\left(\mathbf{r}^a(\mathbf{v}^1) + \mathbf{r}^v(\mathbf{v}^1) + \mathbf{s}\right)$$

- $\mathbf{M}_c \Rightarrow$ High Order in Time
- Boundary Layers: $\Delta t < h^2/\mu \Rightarrow$
  - Integrate Implicitly
  - Reduce for 1st Step

SUBSTEPPING OF ADVECTIVE TERMS (3)

$\Rightarrow$

$$\mathbf{M}_l(\mathbf{v}^1 - \mathbf{v}^0) = \gamma \frac{\Delta t}{2} \left(\mathbf{r}^a(\mathbf{v}^0) + \mathbf{r}^v(\mathbf{v}^0) + \mathbf{s} \right)$$

$$[\mathbf{M}_c + \Delta t \theta \mathbf{K}](\mathbf{v}^2 - \mathbf{v}^0) = \Delta t \left(\mathbf{r}^a(\mathbf{v}^1) + \mathbf{r}^v(\mathbf{v}^1) + \mathbf{s} \right)$$

- In Boundary Layer: $Re_h < 1 \Rightarrow \gamma \to 0$
- In Euler Region: $Re_h > 1 \Rightarrow \gamma \to 1$

SUBSTEPPING OF ADVECTIVE TERMS (4)

General Form: k-Step RK Scheme

For: $i = 1, k-1$:

$$\mathbf{v}^i = \mathbf{v}^0 + \alpha^i \gamma \Delta t \mathbf{M}_l^{-1} \left(\left(\mathbf{r}^a(\mathbf{v}^{i-1}) + \mathbf{r}^v(\mathbf{v}^0) + \mathbf{s} \right) \nabla \mu \nabla \mathbf{v}^{i-1} \right)$$

For $i = k$:

$$[\mathbf{M}_c + \Delta t \theta \mathbf{K}](\mathbf{v}^k - \mathbf{v}^0) = \Delta t \left(\mathbf{r}^a(\mathbf{v}^{k-1}) + \mathbf{r}^v(\mathbf{v}^0) + \mathbf{s} \right)$$

Determination of γ:

$$\Delta t_a \approx \frac{h}{|\mathbf{v}|} \quad ; \quad \Delta t_v \approx \frac{\rho h^2}{\mu}$$

$\Rightarrow$

$$\gamma = \frac{\Delta t_v}{\Delta t_a} \approx \frac{\rho |\mathbf{v}| h}{\mu} \approx Re_h \quad ,$$

$\Rightarrow$

$$\gamma = min(1, Re_h)$$

IMPLICIT TREATMENT OF ADVECTION (1)

Advective-Diffusive Predictor:

$$\left[\frac{1}{\Delta t} + \mathbf{v}^* \cdot \nabla - \nabla\mu\nabla\right](\mathbf{v}^* - \mathbf{v}^n) + \mathbf{v}^n \cdot \nabla\mathbf{v}^n + \nabla p^n = \nabla\mu\nabla\mathbf{v}$$

$\Rightarrow$ Non-Symmetric System:

$$\mathbf{A}\Delta\mathbf{v} = \mathbf{r} \quad .$$

Rewrite as:

$$\mathbf{A} \cdot \Delta\mathbf{v} = (\mathbf{L} + \mathbf{D} + \mathbf{U}) \cdot \Delta\mathbf{v} = \mathbf{r}$$

IMPLICIT TREATMENT OF ADVECTION (2)

Possible Relaxation Schemes:

a) Gauss-Seidel:

$$(\mathbf{L} + \mathbf{D}) \cdot \Delta \mathbf{v}^1 = \mathbf{r} - \mathbf{U} \cdot \Delta \mathbf{v}^0$$

$$(\mathbf{D} + \mathbf{U}) \cdot \Delta \mathbf{v} = \mathbf{r} - \mathbf{L} \cdot \Delta \mathbf{v}^1$$

b) Lower-Upper Symmetric Gauss-Seidel (LU-SGS):

$$(\mathbf{L} + \mathbf{D}) \cdot \mathbf{D}^{-1} \cdot (\mathbf{D} + \mathbf{U}) \cdot \Delta \mathbf{v} = \mathbf{r}$$

OPTIMIZATION OF RELAXATION SCHEMES (1)

Key ideas [Sharov, Luo]:

- Use Spectral Radius of **A** for Diagonal Entries:

$$\mathbf{D} = \left[\frac{1}{\Delta t} \mathbf{M}_l^i - 0.5 \sum \mathbf{C}^{ij} \rho_A \right] \mathbf{I}$$

- Replace:

$$\mathbf{A} \cdot \Delta \mathbf{u} \approx \Delta \mathbf{F}$$

$\Rightarrow$

$$\Delta \mathbf{F} = \mathbf{F}(\mathbf{u} + \Delta \mathbf{u}) - \mathbf{F}(\mathbf{u})$$

- Central Difference vs. Upwind
  - Luo'98: No Discernable Difference $\Rightarrow$
  - For Relaxation: Use Central Difference $\Delta \mathbf{F}$

OPTIMIZATION OF RELAXATION SCHEMES (2)

- Vectorization:
  - Half-Planes [Sharov]
  - Colouring of Planes

Combined Effect:

- Matrix Free
- No Extra Storage vis a vis Explicit
- Per Relaxation Sweep: Faster than Explicit

LU-SGS: FINAL FORM

a) Forward Sweep:

$$\Delta\hat{\mathbf{v}}^i = \mathbf{D}^{-1}\left[\mathbf{r}^i - 0.5\sum_{j<i}\mathbf{C}^{ij}\cdot(\Delta\hat{\mathbf{F}}_{ij} - |\mathbf{v}|_{ij}\Delta\hat{\mathbf{v}}_j)\right.$$

$$\left. + \sum_{j<i}\mathbf{k}^{ij}\Delta\hat{\mathbf{v}}_j\right]$$

b) Backward Sweep:

$$\mathbf{r} = \mathbf{D}\cdot\Delta\hat{\mathbf{v}}$$

$$\Delta\mathbf{v}^i = \mathbf{D}^{-1}\left[\mathbf{r}^i - 0.5\sum_{j>i}\mathbf{C}^{ij}\cdot(\Delta\mathbf{F}_{ij} - |\mathbf{v}|_{ij}\Delta\mathbf{v}_j)\right.$$

$$\left. + \sum_{j>i}\mathbf{k}^{ij}\Delta\hat{\mathbf{v}}_j\right]$$

GMRES

- Use LU-SGS As Preconditioners
- Central Matrix-Vector Product:

$$\mathbf{A} \cdot \Delta\mathbf{u} = (\mathbf{L} + \mathbf{D} + \mathbf{U}) \cdot \Delta\mathbf{u}$$

- Same Loop Structure $(\mathbf{L}, \mathbf{D}, \mathbf{U})$ for:
  - Gauss-Seidel
  - GMRES

  $\Rightarrow$ Single 'Sweep' Subroutine
- Initialize Gauss-Seidel Loop With LU-SGS

LU-SGS (4)

- Advancing Front Point Renumbering
  - ‘Hyperplane’ Structure
  - Minimizes Cache Misses
- Renumber Edges According to Points
- Renumber Points In Hyperplanes
  - Avoid Memory Contention
  - Allow Ordered Sweeps
- Renumber Edges According to Points
  - Ordered Edge Set
  - Minimizes Cache Misses
- Renumber Edges for Vectorization
  - Avoid Memory Contention
- Split Hyperplanes for SM Parallel

FULLY IMPLICIT INTEGRATION

Denote:

$$u^\theta = (1-\theta)u^n + \theta u^{n+1}$$

$\Rightarrow$

$$u^{n+1} - u^n = \frac{u^\theta - u^n}{\theta}$$

Implicit Timestepping Scheme:

$$\frac{\mathbf{v}^\theta - \mathbf{v}^n}{\theta \Delta t} + \mathbf{v}^\theta \nabla \mathbf{v}^\theta + \nabla p^\theta = \nabla \mu \nabla \mathbf{v}^\theta$$

$$\nabla \cdot \mathbf{v}^\theta = 0$$

Solve as Pseudo-Time System:

$$\mathbf{v}^\theta_{,\tau} + \mathbf{v}^\theta \nabla \mathbf{v}^\theta + \nabla p^\theta = \nabla \mu \nabla \mathbf{v}^\theta - \frac{\mathbf{v}^\theta - \mathbf{v}^n}{\theta \Delta t}$$

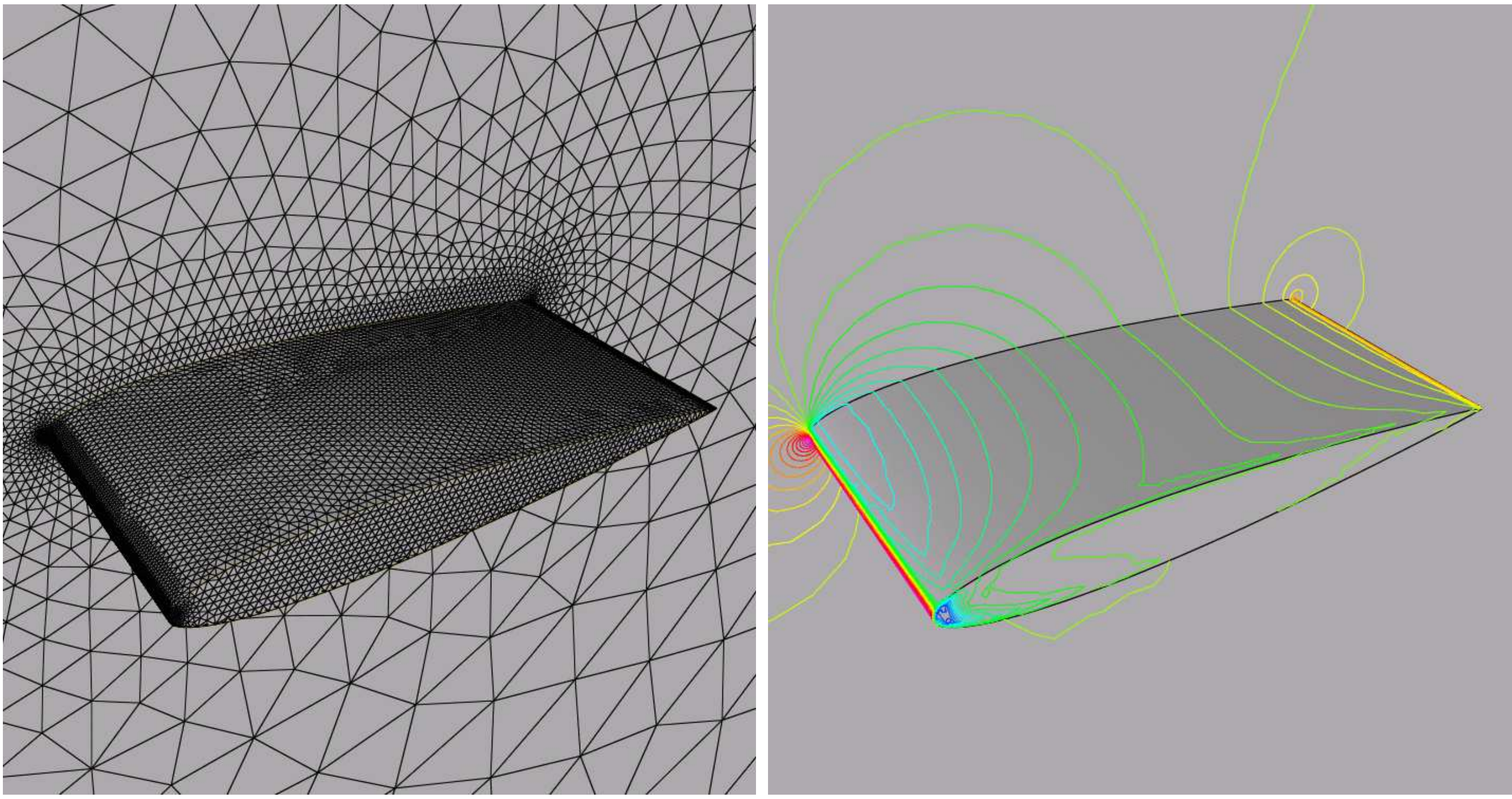
$$\nabla \cdot \mathbf{v}^\theta = 0$$

Re:

- Same as Original NS + Source-Term
- Solve as Steady-State System

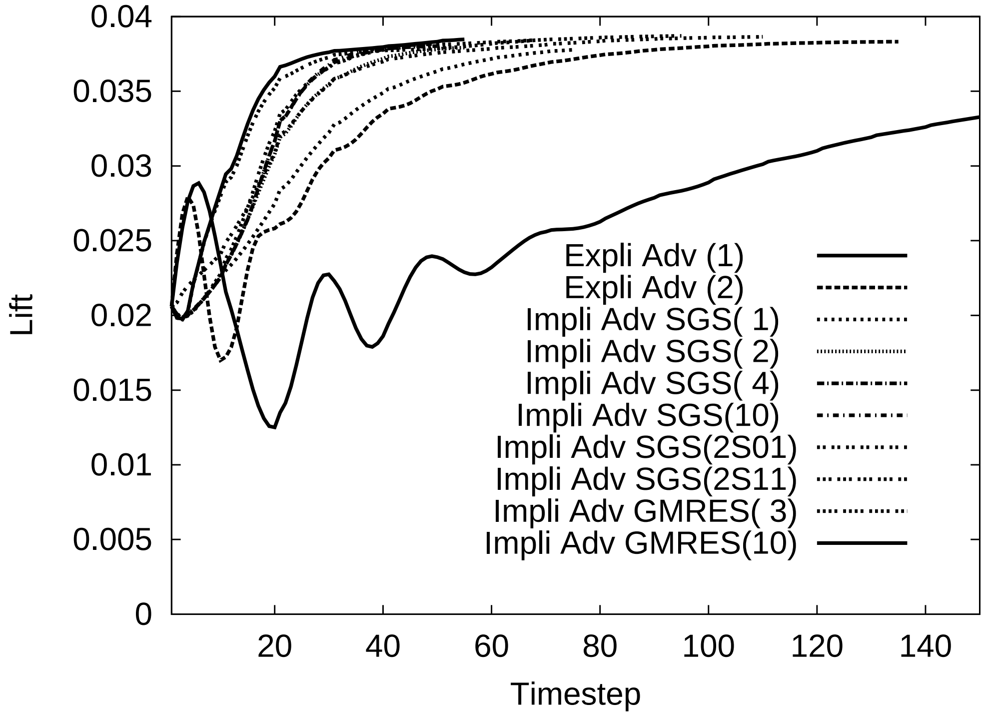
NACA0012 WING (1)

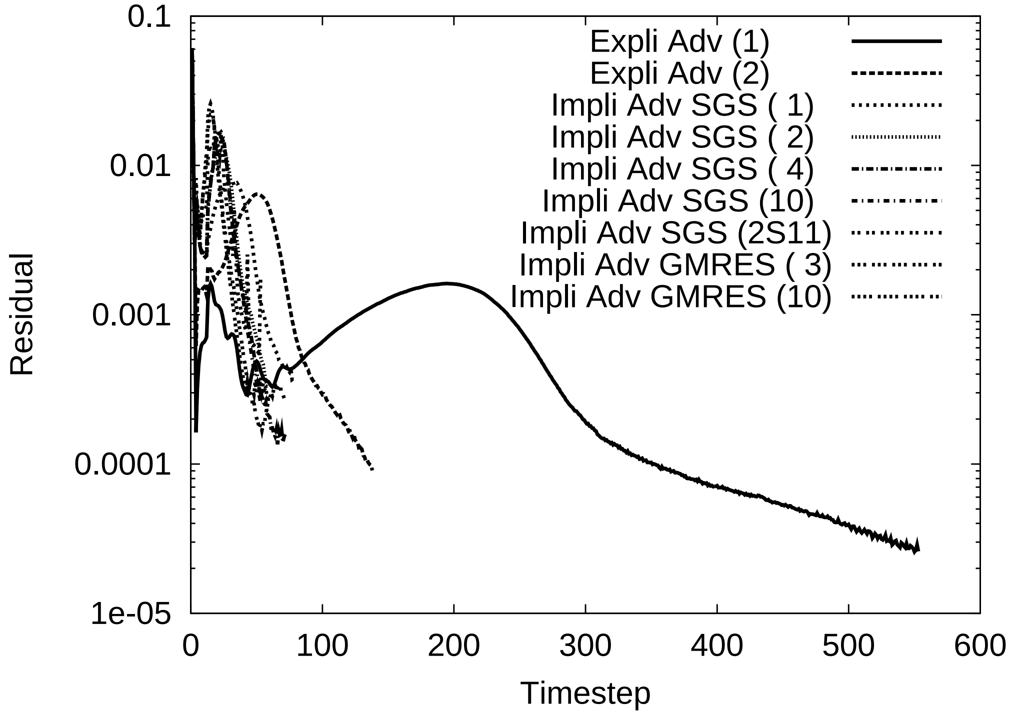
- Euler, Steady
- AOA: $\alpha = 5^o$
- `npoin= 68,321, nelem=368,872`
- Convergence Criterion: Lift $t_l = 10^{-3}$



NACA 0012: Surface Mesh and Pressure

NACA0012 WING (2)



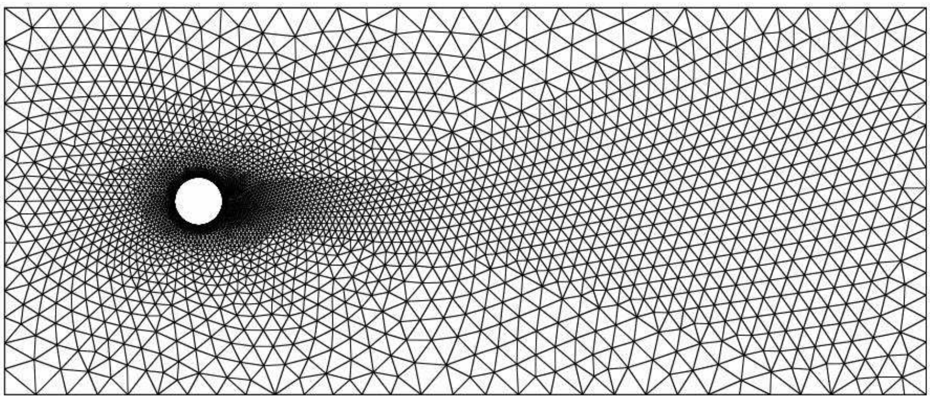


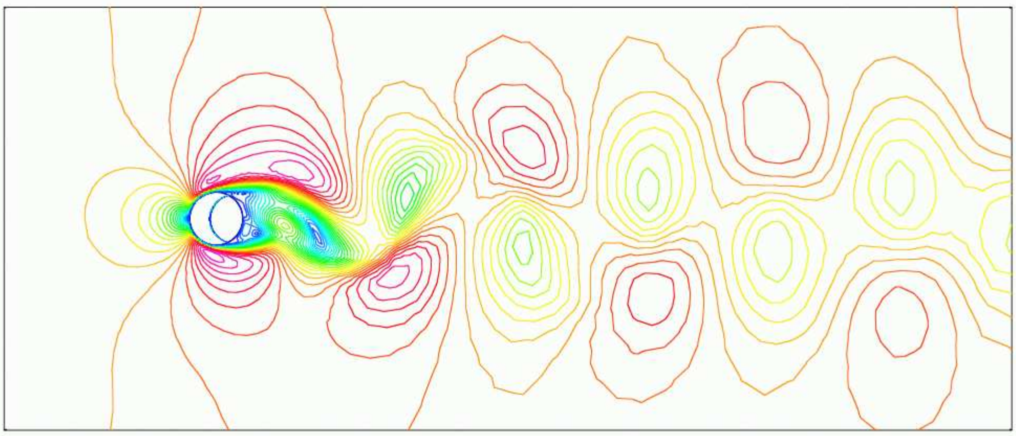
NACA0012 WING (3)

Scheme	`ntime`	CPU [sec]	Speedup
Expli 1	550	2,316	1.00
Expli 2	135	370	6.26
Expli 3	90	157	3.69
Expli 5	70	166	3.48
Impl-SGS (1)	75	172	13.46
Impl-SGS (2)	55	142	16.31
Impl-SGS (4)	70	188	12.30
Impl-SGS (10)	51	532	4.35
Impl-SGS 2 (01)	110	556	4.16
Impl-SGS 2 (11)	50	280	8.27
Impl-GMRES (3)	70	216	10.72
Impl-GMRES (10)	55	772	3.00

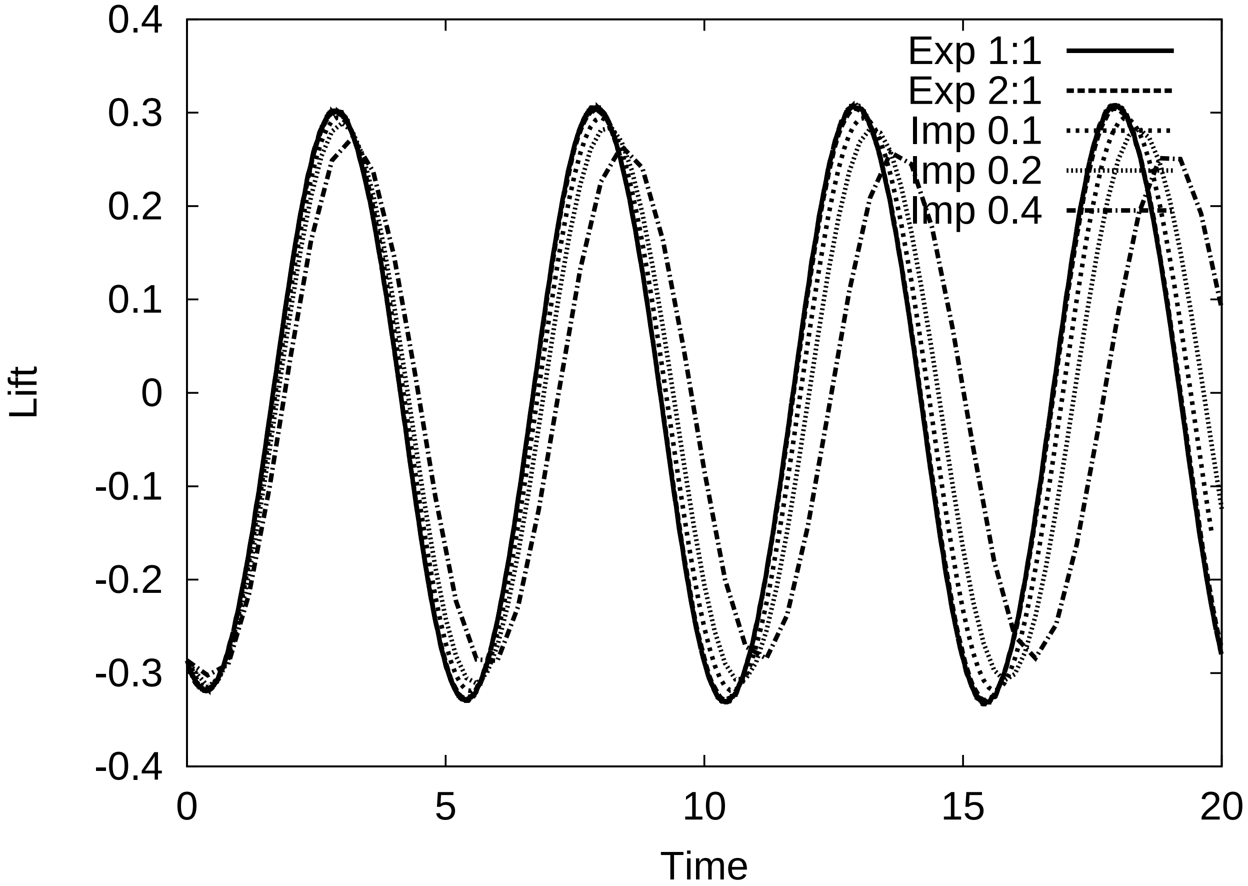
CYLINDER (1)

- $Re = 190$
- `npoin= 23,228, nelem=113,056`



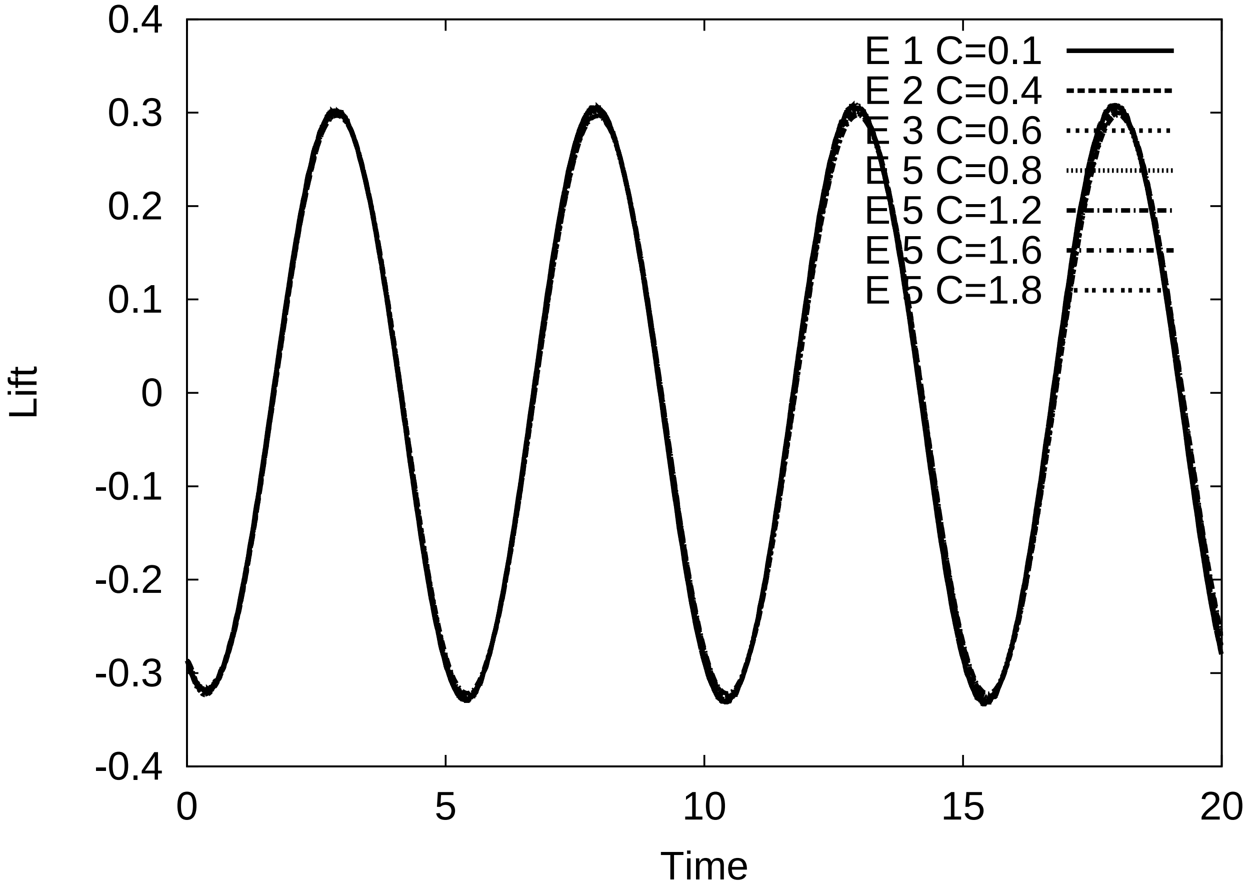


CYLINDER (2)



von Karman Vortex Street: Lift History

CYLINDER (3)



von Karman Vortex Street: Lift History

CYLINDER (4)

Scheme	Δt	`ntime`	CPU [sec]	Speedup
Expli 1	$O(0.002)$	9961	12,929	1.00
Expli 2	$O(0.008)$	2490	4,194	3.08
Expli 3	$O(0.012)$	1660	3,296	3.92
Expli 5	$O(0.016)$	1245	3,201	4.03
Expli 5	$O(0.025)$	830	1,546	8.36
Expli 5	$O(0.033)$	623	1,114	11.60
Expli 5	$O(0.037)$	554	995	12.99
Impl-SGS (5)	0.1	200	3,189	4.05
Impl-SGS (5)	0.2	100	1,612	8.02
Impl-SGS (5)	0.4	50	962	13.43

FREE SURFACE HYDRO: REFERENCE FRAME

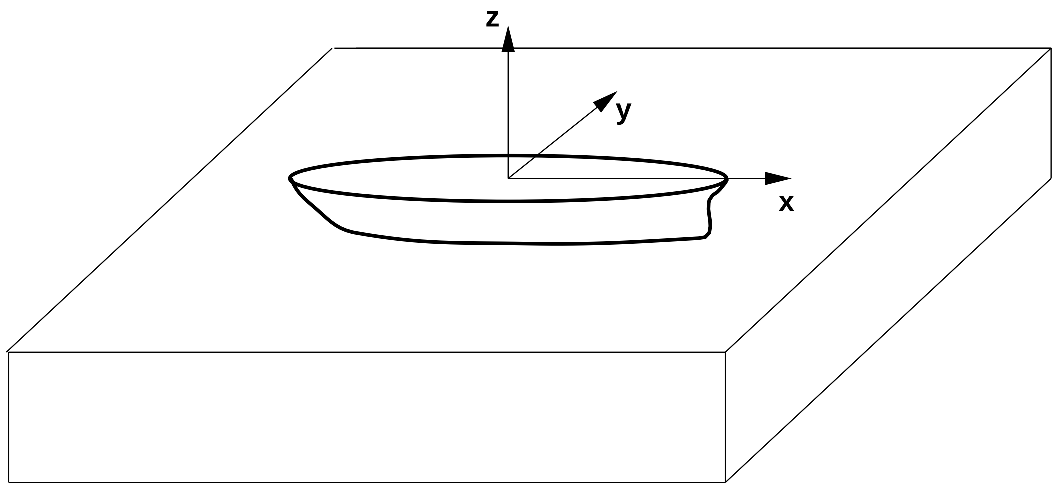


Figure 1: Reference Frame and Ship Position

FREE SURFACE HYDRO: EQUATIONS SOLVED

Conservation of Mass and Momentum:

$$\nabla \cdot \mathbf{v} = 0$$

$$\mathbf{v}_{,t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \Psi = 0$$

$$\Psi = p + \frac{z}{Fr^2} \quad , \quad Fr = \frac{|\mathbf{v}_\infty|}{\sqrt{g \cdot L}}$$

Free Surface:

$$\beta_{,t} + u\beta_{,x} + v\beta_{,y} = w$$

BOUNDARY CONDITIONS

a) Inflow Plane:

$$\mathbf{v} = (1, 0, 0) \quad , \quad \Psi = 0 \quad , \quad \beta = 0$$

b) Exit Plane: Extrapolation

c) Free Surface: $p = 0 \Rightarrow$

$$\Psi = \beta Fr^{-2}$$

d) Hard Bottom: $\mathbf{v} \cdot \mathbf{n} = 0$

d) Infinite Depth Bottom: $p = \rho g z_b \Rightarrow \Psi = 0$

e) Ship Hull: $\mathbf{v} \cdot \mathbf{n} = 0$

f) Side Walls: $\mathbf{v} \cdot \mathbf{n} = 0$

FREE SURFACE DISCRETIZATION

Standard Scalar Advection With Source Terms $\Rightarrow$

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j \quad , \quad \mathbf{f}_i = S_x^{ij} u_i + S_y^{ij} v_i$$

With Fourth Order Damping:

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j - |\lambda^{ij}|(\beta_i - \beta_j + \frac{l^{ij}}{2}(\nabla\beta_i + \nabla\beta_j))$$

FREE SURFACE: DAMPING TERMS (1)

Hino Wave Height Damping Near Inflow and Outflow:

$$\beta_{,t} + u\beta_{,x} + v\beta_{,y} = w - d_h(\mathbf{x})\beta$$

For Outflow Boundary:

$$d_h = c_1\xi^2 \quad , \quad \xi = max\left(0, \frac{x - x_{dmax}}{x_{max} - x_{dmax}}\right) \quad ,$$

$$x_{dmax} = x_{max} - 2\pi Fr^2$$

- $c_1 = O(1)$

FREE SURFACE: DAMPING TERMS (2)

w-Velocity Damping Near Outflow:

$$\beta_{,t} + u\beta_{,x} + v\beta_{,y} = d_w(\mathbf{x})w - d_h(\mathbf{x})\beta$$

$$d_w = 1 - 3\xi^2 + 2\xi^3$$

Final Semi-Discrete Scheme:

$$\mathbf{M}_l\beta_{,t} = \mathbf{r} = \mathbf{r}_a(u, v, \beta) + \mathbf{r}_s(d_w, w) + \mathbf{r}_d(d_h, \beta)$$

Integrate Using 5-Stage RK Scheme

OVERALL SCHEME

One Complete Timestep:

- Given B.C. for Ψ: Update 3-D Flowfield
- Extract Velocity Vector $\mathbf{v} = (u, v, w)$ at Free Surface $\mathbf{v}_\beta$
- Given $\mathbf{v}_\beta$: Update Free Surface $\Rightarrow \beta$;
- Given β: Impose New Boundary Conditions for Ψ

<u>Steady State</u>:

- Use Local Timesteps
- N-Fluid vs. M-Free Surface Steps
  - Current Preference: Equivalent 'Time-Interval' Ratio 1:8

MESH UPDATE STRATEGY (1)

Time-Accurate: For Each Timestep

- Update Positions
- Interrogate CAD-Data and Reposition
- Recompute Geometry-Parameters (e.g. Volumes)

Steady-State: Timesteps 1:N-1 , N=O(100)

- Do Not Move the Mesh
- Impose Geometrically Linearized B.C. for β

Steady-State: Timestep N

- Update Positions
- Interrogate CAD-Data and Reposition
- Recompute Geometry-Parameters (e.g. Volumes)
- Remesh if Negative Elements Appear

MESH UPDATE STRATEGY (2)

- From β: Obtain New Elevation for Free Surface Points
  $\Rightarrow \mathbf{d}_\Gamma = (0, 0, d_\Gamma)$
- Apply Boundary Conditions for Points on Waterline
  $\Rightarrow$ Additional Horizontal (x,y-Direction) Displacements
- Smooth Displacement Field:

$$\nabla \cdot k \nabla \mathbf{d} = 0$$

- Reposition Points on Hull (CAD-Data)

BOUNDARY CONDITIONS FOR MESH MOVEMENT (1)

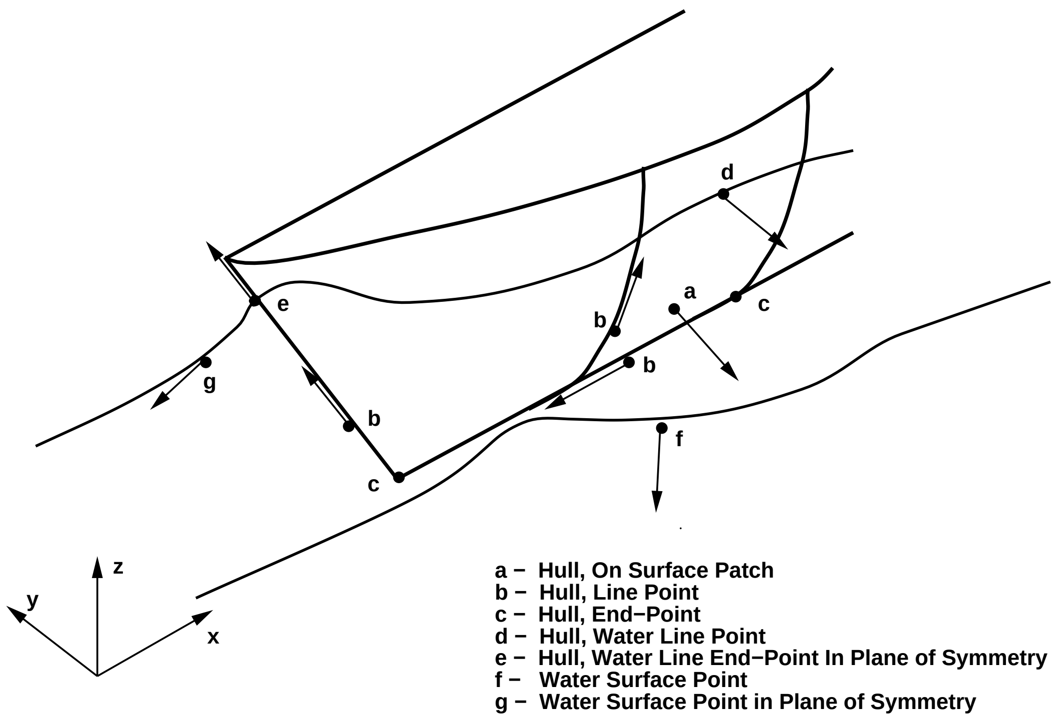


Figure 2: Boundary Conditions for Mesh Movement

BOUNDARY CONDITIONS FOR MESH MOVEMENT (1)

Define:

$\mathbf{d}_0$: Initial Displacement from Free Surface $\mathbf{d}_*$: Predicted Displacement

Δz: Change in Surface Elevation in z

a) Hull, On Surface Patch: No Normal $\mathbf{d}$

$$\mathbf{d} = \mathbf{d}_* - (\mathbf{d}_* \cdot \mathbf{n})\ \mathbf{n}$$

b) Hull, Line Point: Tangential $\mathbf{d}$

$$\mathbf{d} = (\mathbf{d}_* \cdot \mathbf{t})\ \mathbf{t}$$

c) Hull, End-Point: No Movement

$$\mathbf{d} = 0$$

d) Hull, Water Line Point or Water Line End-Point:

$$\mathbf{d} = \frac{\mathbf{d}_0 - (\mathbf{d}_0 \cdot \mathbf{n})\ \mathbf{n}}{1 - n_z^2}$$

BOUNDARY CONDITIONS FOR MESH MOVEMENT (3)

e) Hull, Water Line End-Point in Plane of Symmetry:

$$\mathbf{d} = \frac{(\mathbf{d}_0 \cdot \mathbf{t})\ \mathbf{t}}{1 - n_z^2}$$

f) Water Surface Points:

$$\mathbf{d} = \mathbf{d}_* - (\mathbf{d}_* \cdot \mathbf{n})\ \mathbf{n}$$

g) Water Surface Points in Plane of Symmetry:

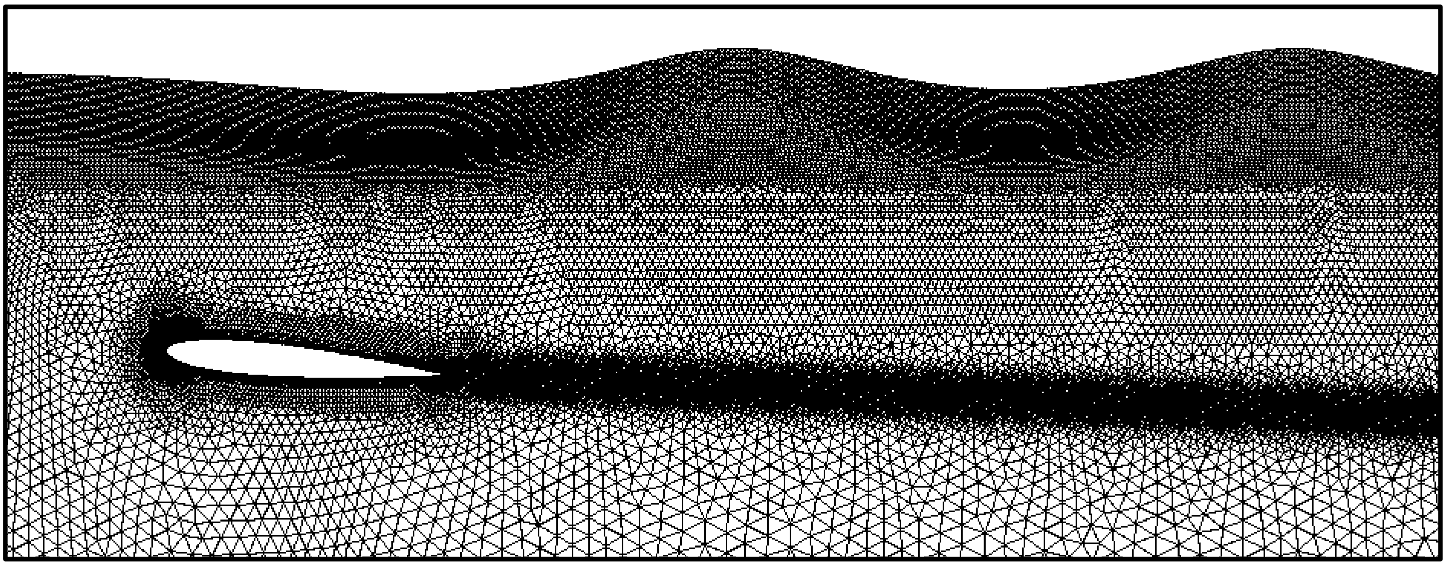
$$\mathbf{d} = (\mathbf{d}_* \cdot \mathbf{t})\ \mathbf{t}$$

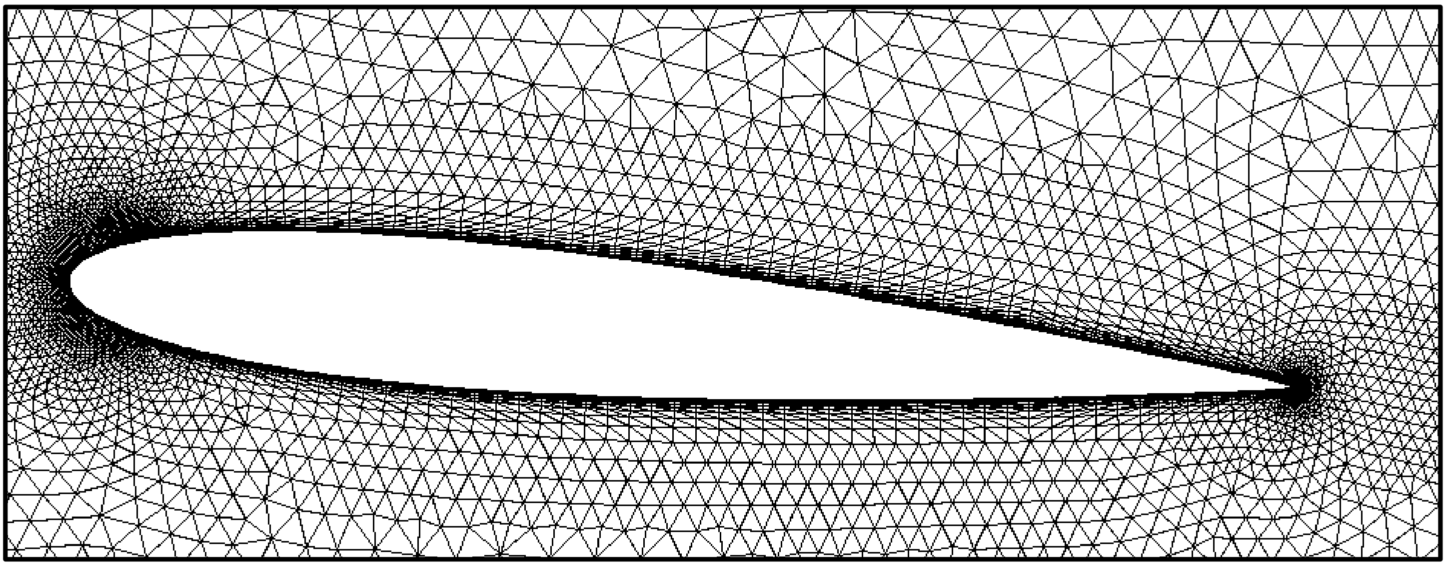
h) Transum Sterns:

$$\mathbf{d} = 0$$

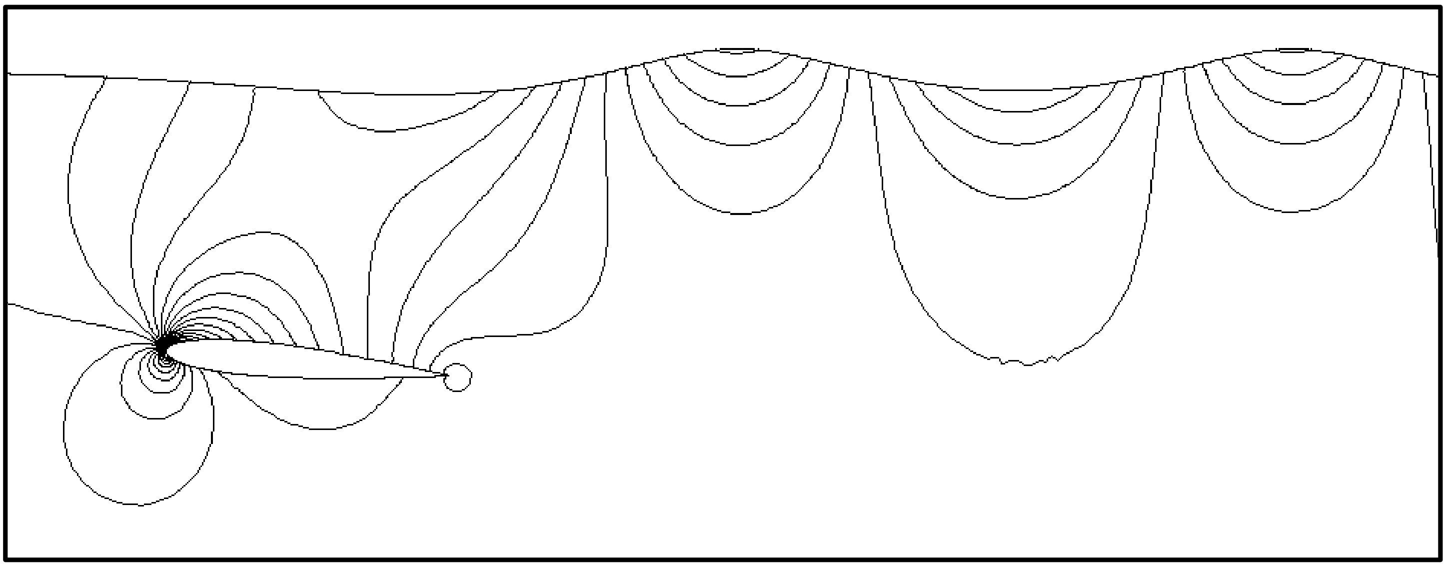
SUBMERGED NACA0012 (1)

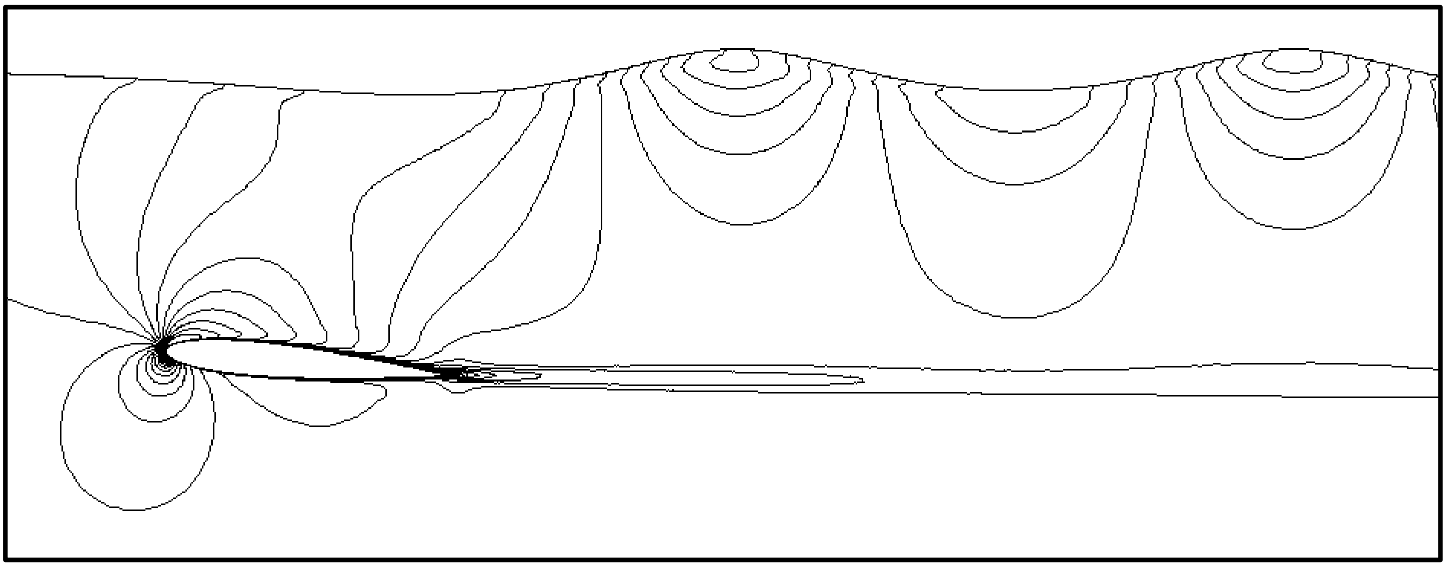
- $\alpha = 5^o$
- $Fr = 0.5672$, Euler & RANS [$Re = 10^6$, BL]
- Duncan (1983), Hino (1993; 1997)
- `npoin=465,752, nelem=2,409,720`





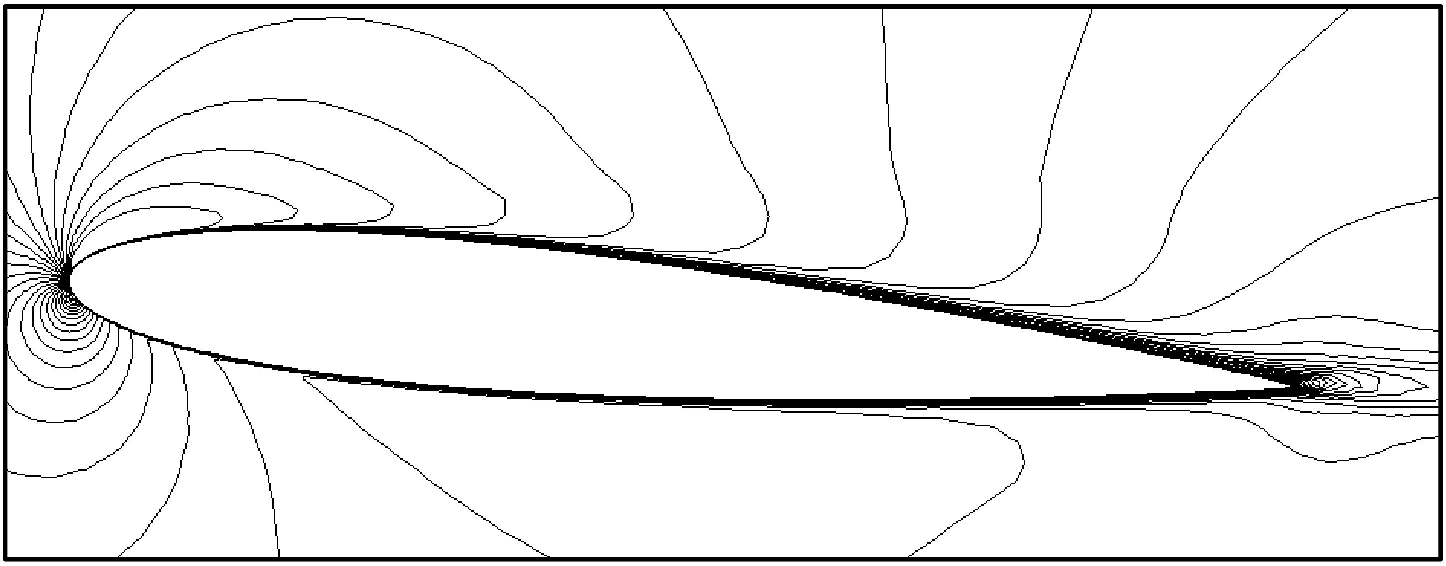
SUBMERGED NACA0012 (2)

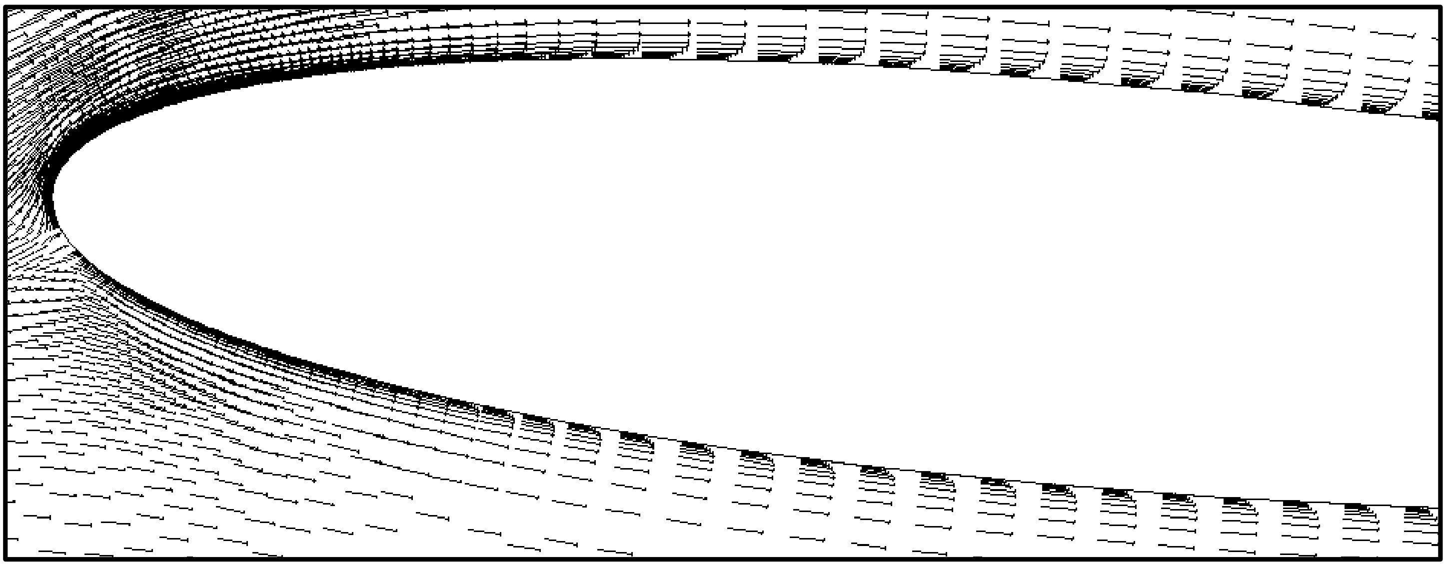




Submerged NACA0012: Pressure and Velocity Fields

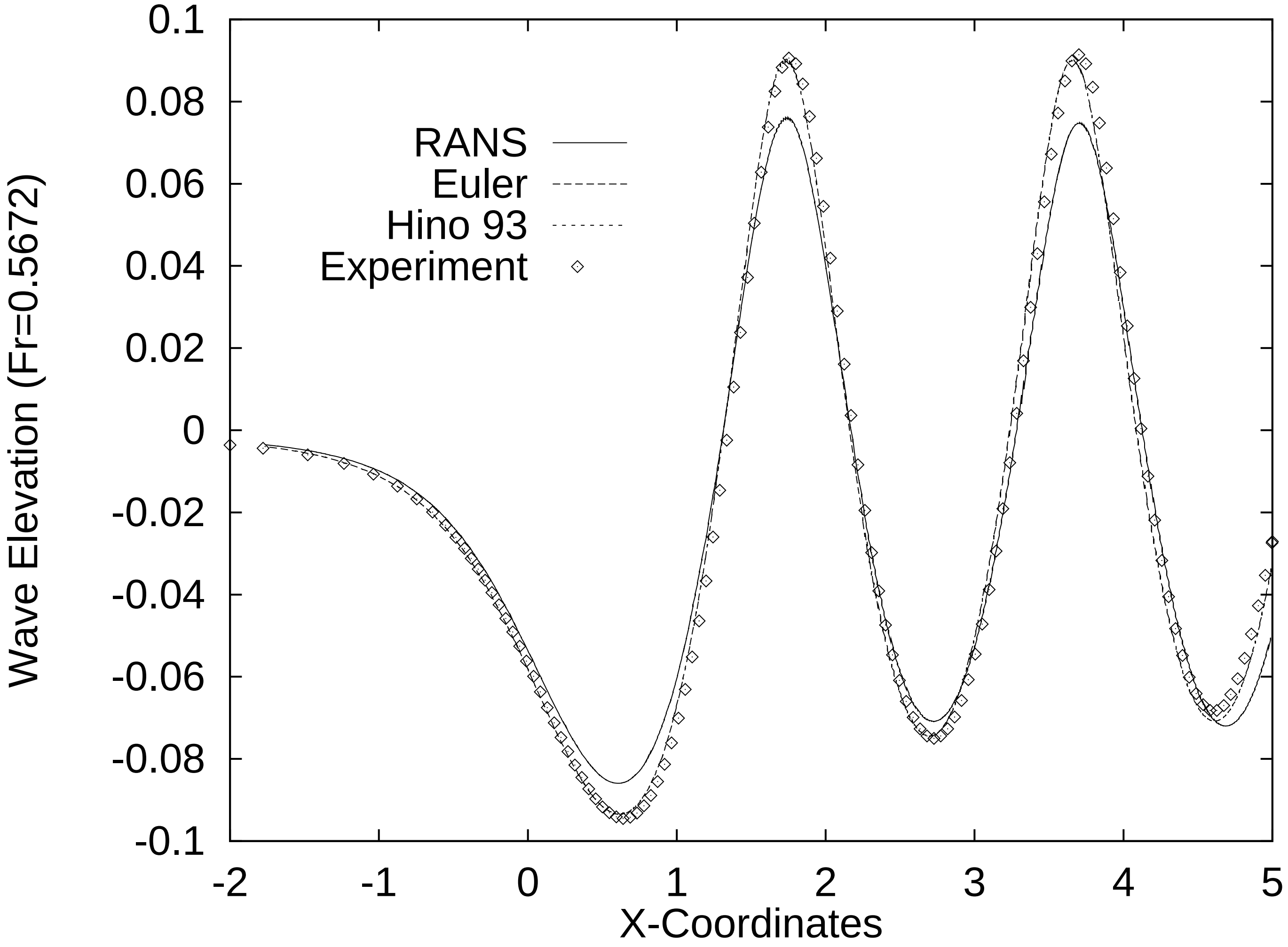
SUBMERGED NACA0012 (3)





Submerged NACA0012: Velocity Field (Zoom)

SUBMERGED NACA0012 (4)

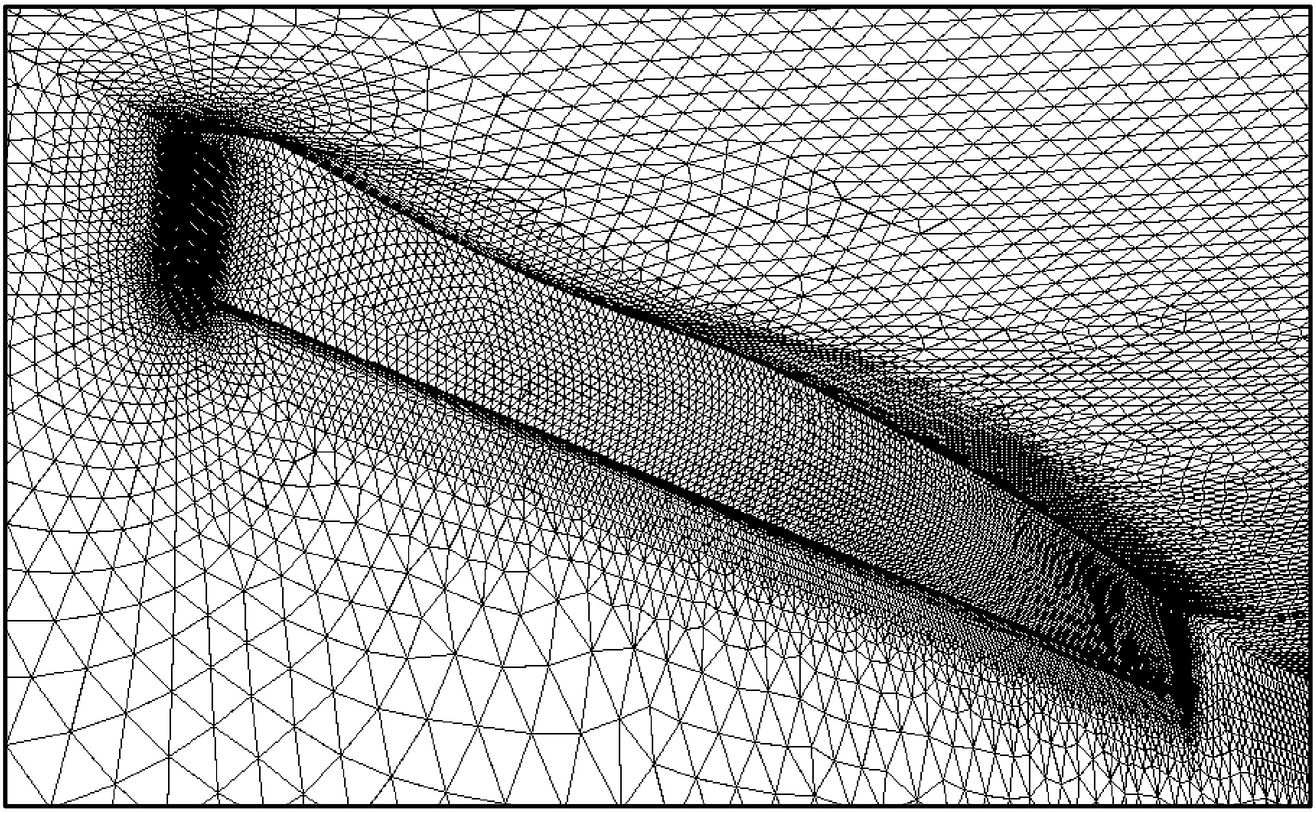


Submerged NACA0012: Wave Profiles

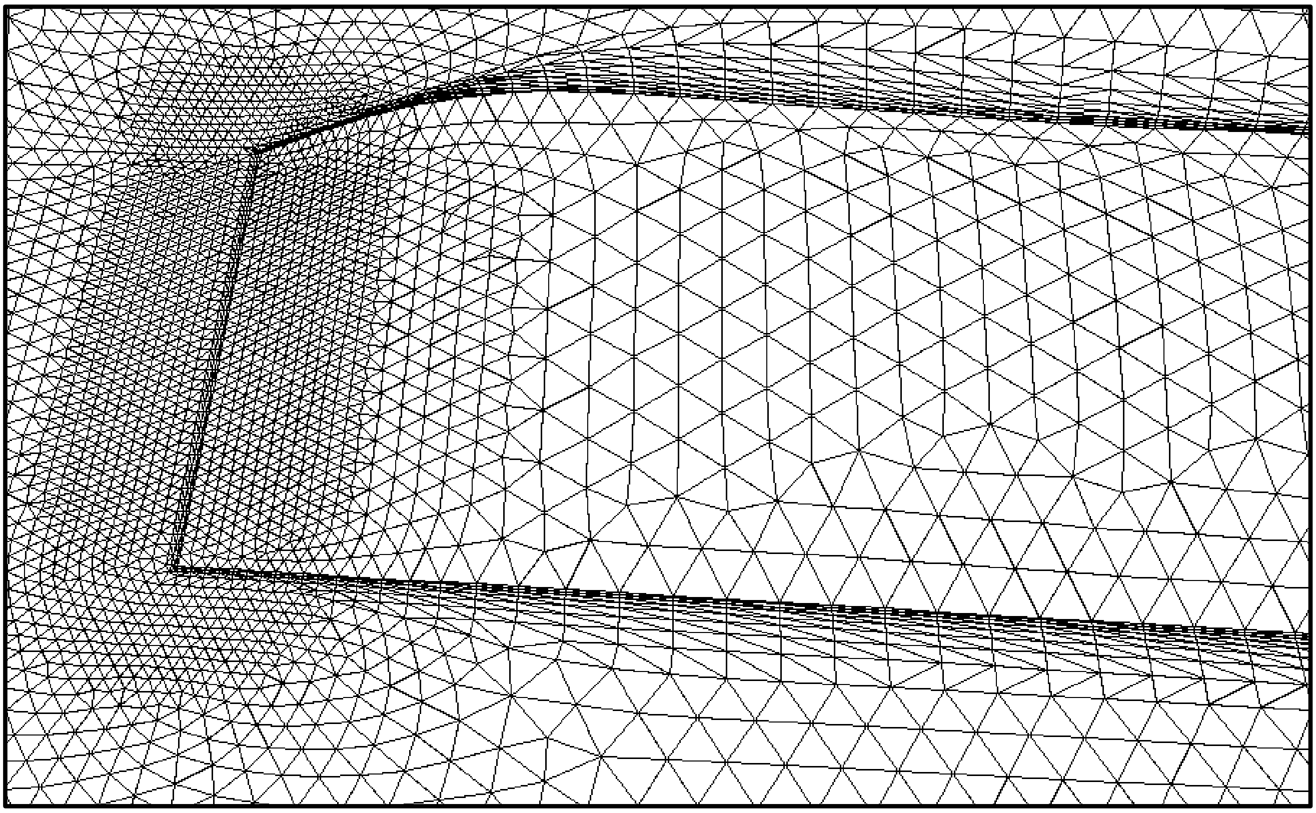
WIGLEY HULL (1)

$$y = 0.5 \cdot B \cdot \left[1 - 4x^2\right] \cdot \left[1 - \left(\frac{z}{D}\right)^2\right]$$

- $D = 0.0625$, $B = 0.1$
- $Fr = 0.25$, Euler & RANS [$Re = 10^6$, KE]
- Tokyo (1983), Farmer (1994), Raven (1996), etc.
- `npoin=204,155, nelem=1,119,703`

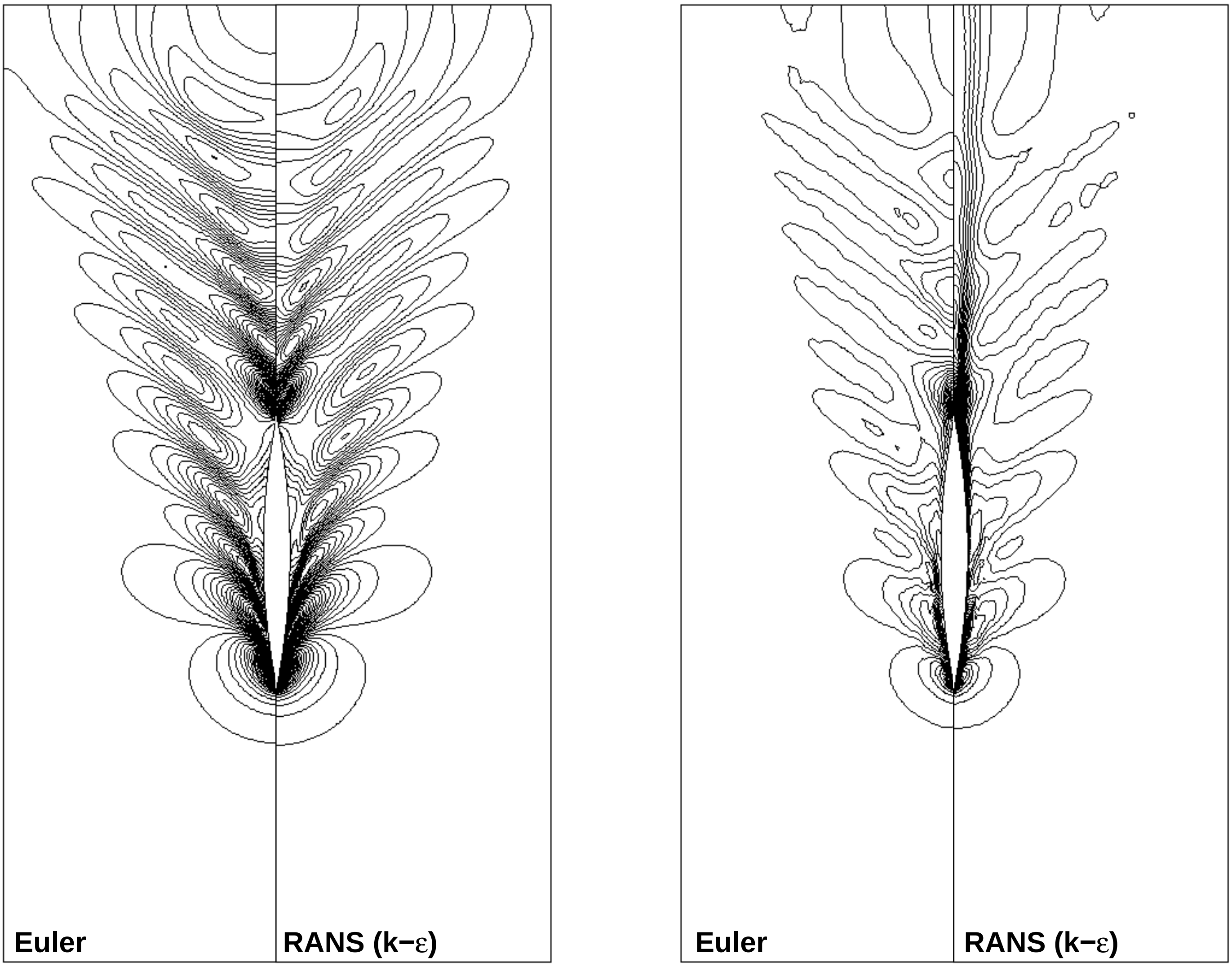


WIGLEY HULL (2)



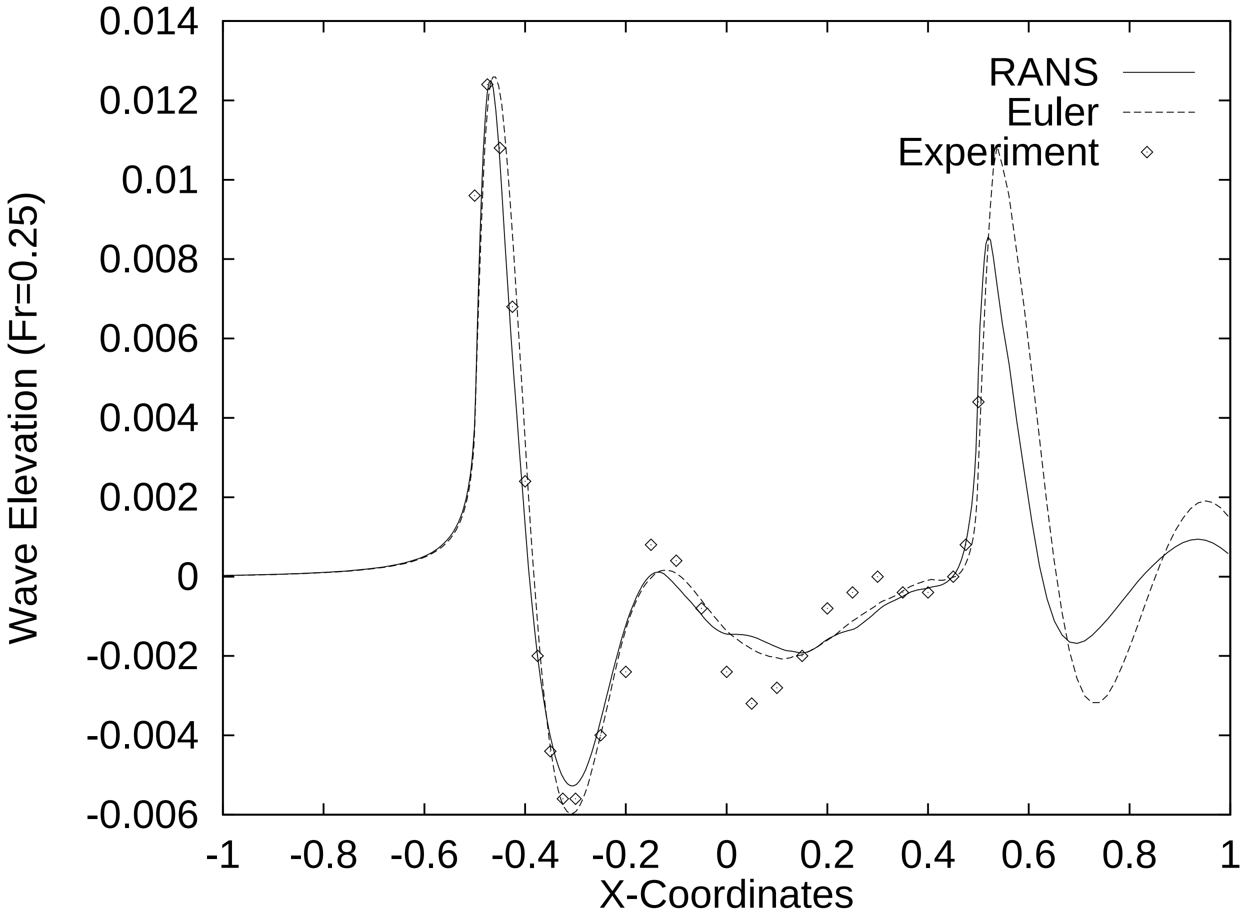
Wigley Hull: Surface of Mesh

WIGLEY HULL (3)



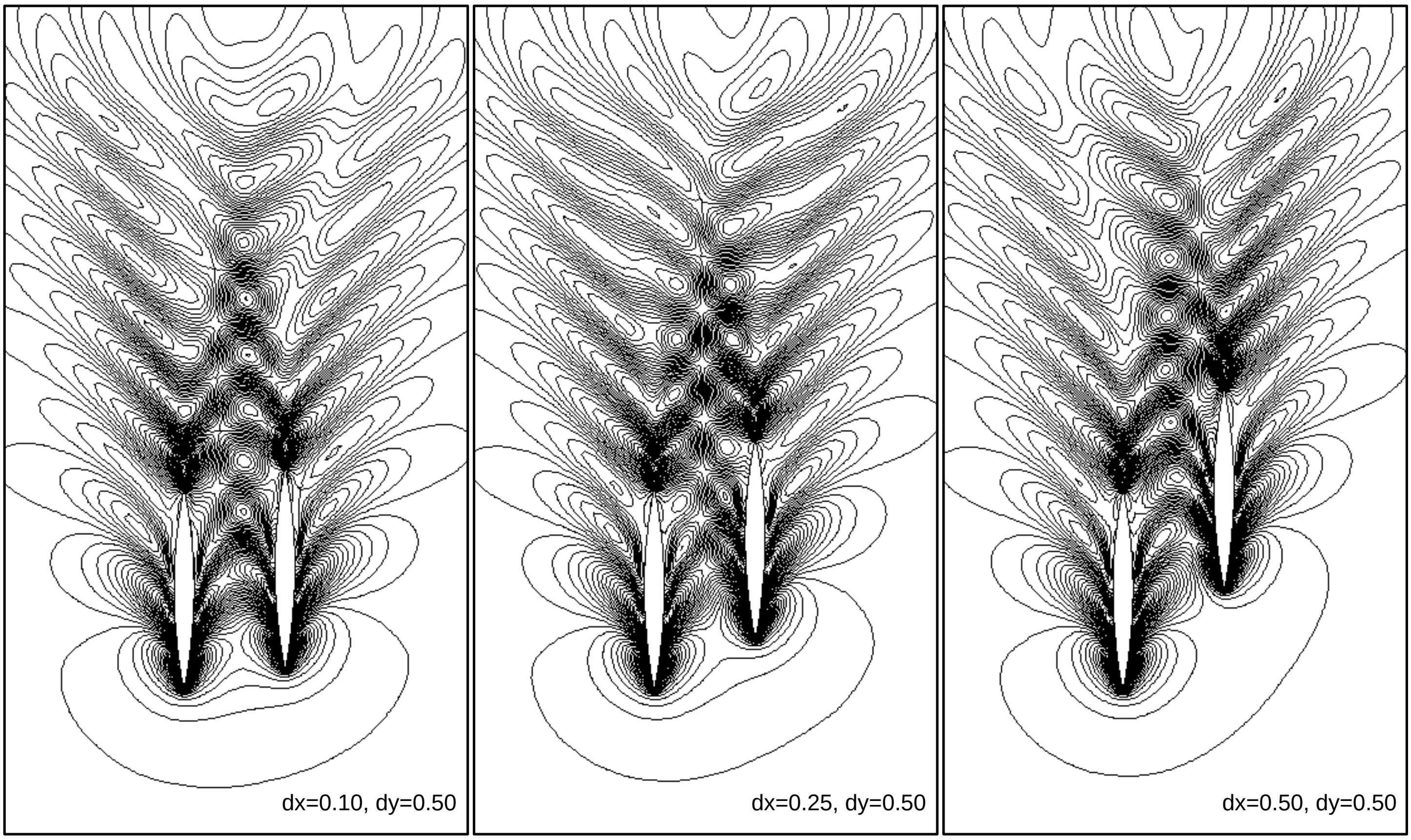
Wigley Hull: Wave Elevation and Surface Velocity

WIGLEY HULL (4)



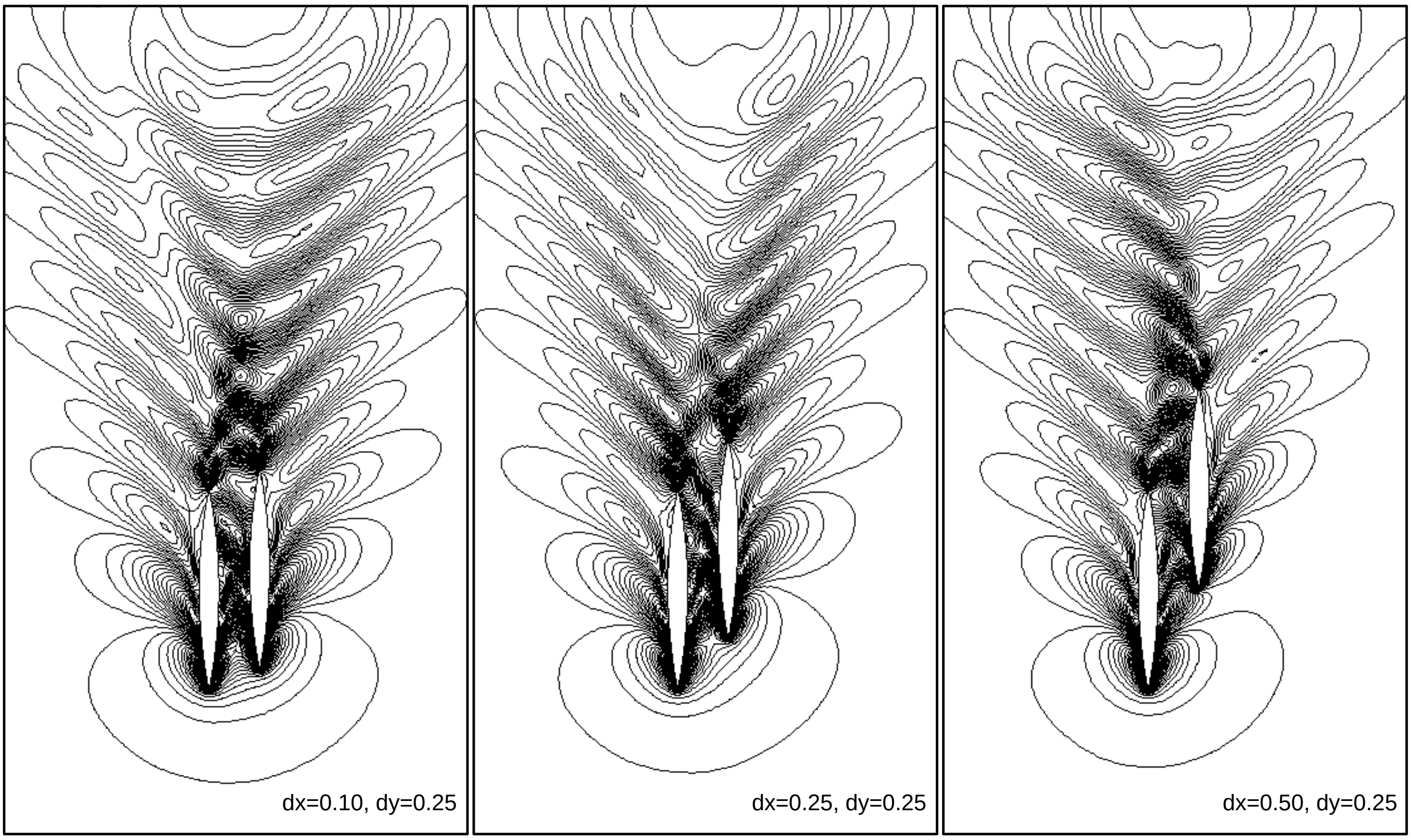
Wave Elevation at the Hull

TWO WIGLEY HULLS (1)



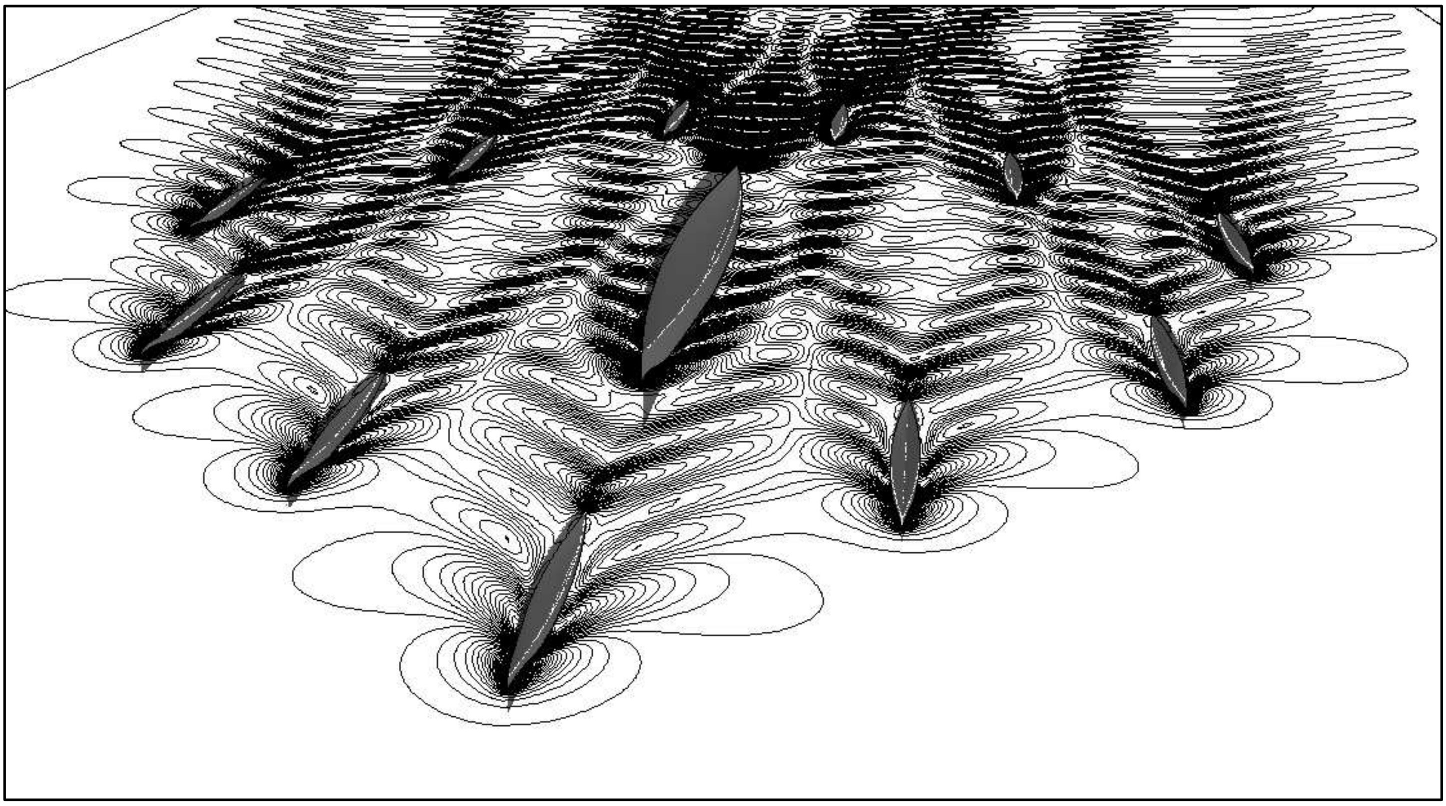
Wave Elevation for Two Wigley Hulls (Fr=0.316)

TWO WIGLEY HULLS (2)



Wave Elevation for Two Wigley Hulls (Fr=0.316)

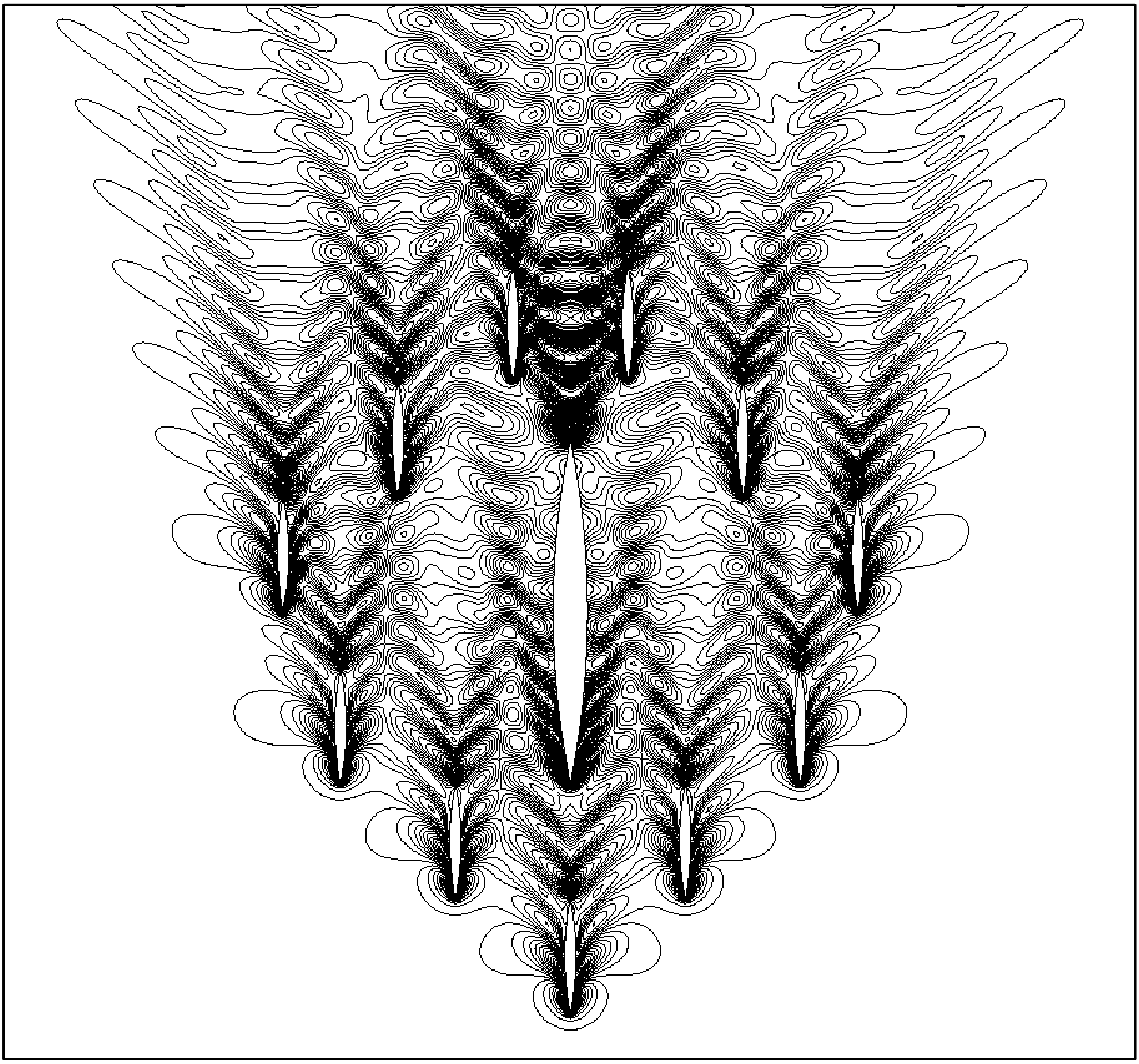
WIGLEY CARRIER GROUP (1)



Wigley Carrier Group

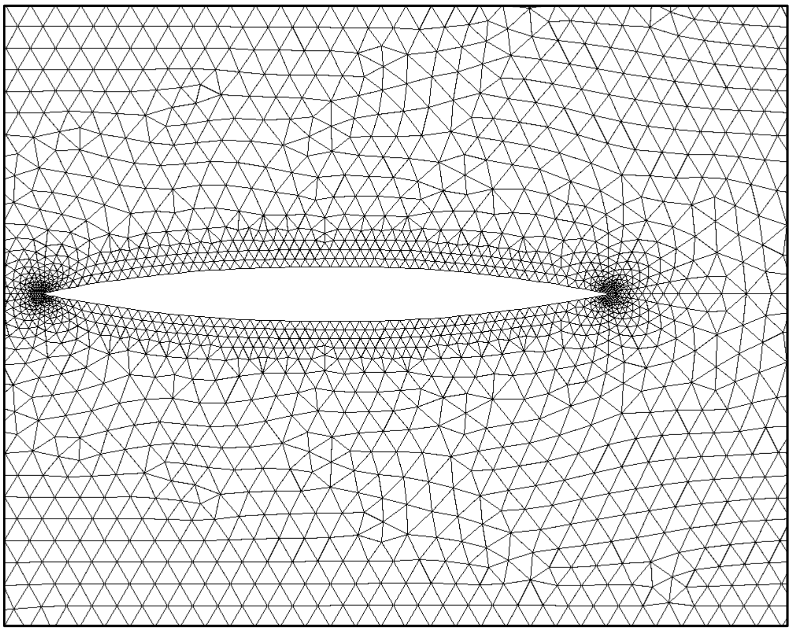
vfill

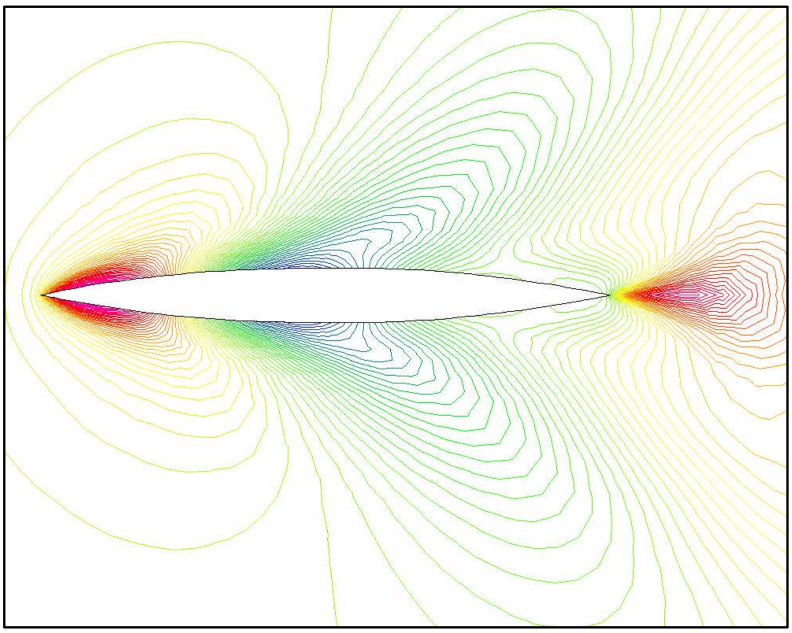
WIGLEY CARRIER GROUP (2)



Wave Elevation (Fr=0.316)

WIGLEY + SINK + TRIM (1)





WIGLEY + SINK + TRIM (2)