FLUX-CORRECTED TRANSPORT (FCT) (1)

Basic Idea:

- High-Order Scheme $\Delta\mathbf{u}^h$ Desired
- Low-Order Scheme $\Delta\mathbf{u}^l$ Safe
- $\Rightarrow$ Re-Write $\Delta\mathbf{u}^h$ As:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta\mathbf{u}^h = \mathbf{u}^n + \Delta\mathbf{u}^l + (\Delta\mathbf{u}^h - \Delta\mathbf{u}^l)$$

- Limit Antidiffusive Flux $(\Delta\mathbf{u}^h - \Delta\mathbf{u}^l) \Rightarrow$

$$\mathbf{u}^{n+1} - \mathbf{u}^n = \Delta\mathbf{u} = \Delta\mathbf{u}^l + c \cdot (\Delta\mathbf{u}^h - \Delta\mathbf{u}^l)$$

- c: Limitor: Determined from Physics/$\mathbf{u}$
  - $c = 0 \Rightarrow$ Low-Order Scheme
  - $c = 1 \Rightarrow$ High-Order Scheme

FLUX-CORRECTED TRANSPORT (2)

Advection in 1-D:

$$u_{,t} + a u_{,x} = 0$$

Re-Write Basic FCT Scheme In Flux Form:

$$\Delta u_i = -\frac{\Delta t}{\Delta x} \left[\left(f^l_{i+\frac{1}{2}} - f^l_{i-\frac{1}{2}} \right) \right.$$
$$\left. + c_{i+\frac{1}{2}} \left(f^h_{i+\frac{1}{2}} - f^l_{i+\frac{1}{2}} \right) - c_{i-\frac{1}{2}} \left(f^h_{i-\frac{1}{2}} - f^l_{i-\frac{1}{2}} \right) \right]$$

Or:

$$\Delta u_i = \Delta u^l_i + \left[c_{i+\frac{1}{2}} g_{i+\frac{1}{2}} - c_{i-\frac{1}{2}} g_{i-\frac{1}{2}} \right]$$

Properties:

- Flux Form $\Rightarrow$ Fully Conservative
- c 'Lives Where The Fluxes Live'

In what follows: $g = g^h - g^l$

FLUX-CORRECTED TRANSPORT (3)

<u>Limiting Aim</u>: No New Maxima/Minima Formed by $\Delta u^h - \Delta u^l$

<u>Basic Ideas</u>:

- Evaluate Allowed Value for u_i^{n+1}
  Typically:

$$q_i^{\overset{+}{-}} = \begin{matrix} max \\ min \end{matrix} \left(u^n, u^n + \Delta u^l \right)_{i-1,i,i+1} - \left(u_i^n + \Delta u_i^l \right)$$

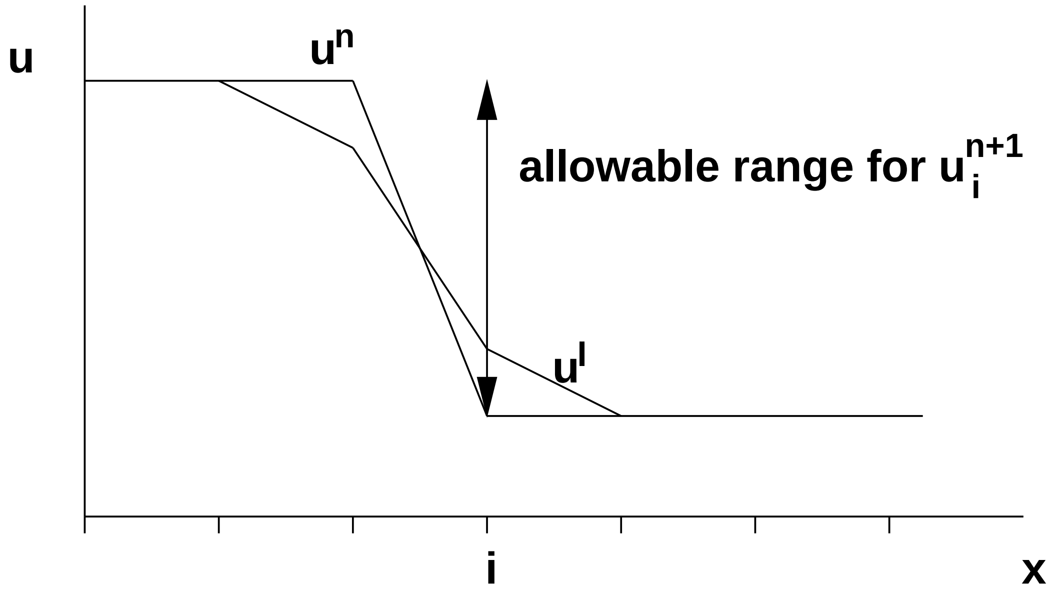
- Contributions to a Point: Mix of Positive and Negative
  $\Rightarrow$ Evaluate Worst Case Scenario:
  - All Positive Fluxes (Maxima) $\Rightarrow p_i^+$
  - All Negative Fluxes (Minima) $\Rightarrow p_i^-$
- Ratio of These Numbers

$$r_i^{\overset{+}{-}} := \begin{cases} min(1, q_i^{\overset{+}{-}} / p_i^{\overset{+}{-}}) \quad if \quad p_i^+ > 0, \ p_i^- < 0 \\ 0 \quad if \quad p_i^{\overset{+}{-}} = 0 \end{cases}$$

- Take The Most Conservative Ratio for $c_{i+\frac{1}{2}}$:

$$c_{i+\frac{1}{2}} = min \left(\begin{cases} r_{i+1}^+ \ if \ g_{i+\frac{1}{2}} > 0 \\ r_{i+1}^- \ if \ g_{i+\frac{1}{2}} < 0 \end{cases} , \begin{cases} r_i^- \ if \ g_{i+\frac{1}{2}} < 0 \\ r_i^+ \ if \ g_{i+\frac{1}{2}} > 0 \end{cases} \right)$$

FLUX-CORRECTED TRANSPORT (4)



FLUX-CORRECTED TRANSPORT (5)

Multi-Dimensional FCT:

1) Compute LEC: 'Low-Order Element/Edge Contr.'
   - Any Monotonic (Low-Order) Scheme

2) Compute HEC: 'High-Order Element/Edge Contr.'
   - Desired High-Order Scheme

3) Define 'Antidiffusive Element/Edge Contributions' :

$$AEC = HEC - LEC$$

4) Compute Updated Low-Order Solution U^l :

$$U_i^l = U_i^n + \sum_{el} LEC, \quad i = 1, ..., npoin$$

5) Limit (Correct) the AEC:
   - U^{n+1} Should be Free of Extrema Not Also Found in U^l or U^n :

$$AEC^c = C_{el} * AEC, \quad 0 \le C_{el} \le 1$$

6) Apply the Limited AEC :

$$U_i^{n+1} = U_i^l + \sum_{el} AEC^c$$

The Limiting Procedure (1)

Define:

$P_i^{\pm}$: Sum of All Positive (Negative) Element/Edge Contributions to Node i

$$P_i^{\pm} = \sum_{el} \left\{ \begin{matrix} max \\ min \end{matrix} \right\} (0, AEC_{el})$$

$Q_i^{\pm}$: Maximum (Minimum) Increment Node i is Allowed to Achieve in Step 6 Above

$$Q_i^{\pm} = U_i^{\substack{max \\ min}} - U^l$$

Then:

$$R^{\pm} := \begin{cases} min(1, Q^{\pm}/P^{\pm}) \quad if \quad P^{+} > 0,\ P^{-} < 0 \\ 0 \quad if \quad P^{\pm} = 0 \end{cases}$$

Take :

$$C_{el} = min(element\ nodes) \begin{cases} R^{+}\ if\ AEC > 0 \\ R^{-}\ if\ AEC < 0 \end{cases}$$

The Limiting Procedure (2)

Obtain $U_i^{\substack{max\\min}}$ in Three Steps :

a) Maximum (Minimum) Nodal U of U^n and U^l :

$$U_i^* = \begin{Bmatrix} max \\ min \end{Bmatrix} (U_i^l, U_i^n)$$

b) Maximum (Minimum) Nodal Value of Element/Edge:

$$U_{el}^* = \begin{Bmatrix} max \\ min \end{Bmatrix} (U_A^*, U_B^*, ..., U_C^*)$$

c) Maximum (Minimum) U of All Elements/Edges Surrounding Node i:

$$U_i^{\substack{max\\min}} = \begin{Bmatrix} max \\ min \end{Bmatrix} (U_1^*, U_2^*, ..., U_m^*)$$

<u>Remarks:</u>

- 'Clipping Limitor': In a), Set: $U_i^* = \left\{ \substack{max \\ min} \right\} U_i^l$
- Limiting Based on U, **Not** on Ratio of Differences (TVD)

For Explicit Taylor-Galerkin:

- High-order : Consistent Mass Taylor Galerkin

$$\mathbf{M}_c \cdot \Delta\mathbf{u}^h = \mathbf{r} \quad \Rightarrow \quad \mathbf{M}_l \cdot \Delta\mathbf{u}^h = \mathbf{r} + (\mathbf{M}_l - \mathbf{M}_c) \cdot \Delta\mathbf{u}^h$$

- Low-order : Lumped Mass Taylor Galerkin + Diffusion

$$\mathbf{M}_l \cdot \Delta\mathbf{u}^l = \mathbf{r} + \mathbf{d} = \mathbf{r} - (\mathbf{M}_l - \mathbf{M}_c) \cdot \mathbf{u}$$

$\Rightarrow$

$$AEC = \left[(\mathbf{M}_l - \mathbf{M}_c) \cdot (\mathbf{u} + \Delta\mathbf{u}^h)\right]_{el}$$

Note:

- No Physical Flux-Terms Appear in $AEC \Rightarrow$
    - Fast
    - General

Limiting for Systems of Equations

No Obvious Way; Possibilities Are:

- Equation Splitting [LCPFCT]
- Synchronization
  - c(ia)=min(c(ia), ia=1, nr. of equations)
  - c(ia)=c(density)
  - c(ia)=min(c(density), c(energy))
  - c(ia)=c(derived quantity, e.g. pressure)

....connection to Riemann invariants ?

SOD SHOCK TUBE PROBLEM