EDGE-BASED SOLVERS (1)

Typical RHS:

$$\mathbf{r}^i = \int N^i \mathbf{r}(\mathbf{u}) d\Omega = \sum_{el} \int N^i \mathbf{r}(N^j \mathbf{u}_j) d\Omega_{el}$$

$\Rightarrow$ Two Sets of Data:

a) Point-data, for $\mathbf{r}^i, \mathbf{u}^i$, and
b) Element-data, for volumes, shape-functions, etc.

Information Flow:

1. `GATHER` From Points to Elements
2. Operate on Element Data
3. `SCATTER-ADD` Element RHS to Points

For Simple Flow Solvers: Cost Dominated by Steps 1,3

For **Linear Elements**: Use Edge Data Structures

EDGE-BASED SOLVERS (2)

For Linear Elements

Table 10.1: Gather/Scatter Overhead:

	Element-Based	Edge-Based
2-D	2*3*NELEM=12*NPOIN	2*2*NEDGE=12*NPOIN
3-D	2*4*NELEM=44*NPOIN	2*2*NEDGE=28*NPOIN

Table 10.2: Flops Overhead: Galerkin Fluxes

	Element-Based	Edge-Based
3-D	22*NELEM=121*NPOIN	7*NEDGE=49*NPOIN

LAPLACIAN OPERATOR (1)

Typical RHS:

$$r^i = -\int \nabla N^i \nabla N^j d\Omega \hat{u}_j = -\left[\sum_{el} \int \nabla N^i \nabla N^j d\Omega\right] \hat{u}_j$$

Split Into $j = i$, $j \neq i$:

$$r^i = -\sum_{j \neq i} \left[\sum_{el} \int \nabla N^i \nabla N^j d\Omega\right] \hat{u}_j - \sum_{el} \int \nabla N^i \nabla N^i d\Omega \hat{u}_i$$

Conservation Property of Shape-Functions:

$$N^i_{,k} = -\sum_{j \neq i} N^j_{,k}$$

$\Rightarrow$

$$r^i = -\sum_{j \neq i} \left[\sum_{el} \int \nabla N^i \nabla N^j d\Omega\right] \hat{u}_j$$

$$+ \left[\sum_{el} \int \nabla N^i \sum_{j \neq i} \nabla N^j d\Omega\right] \hat{u}_i$$

LAPLACIAN OPERATOR (2)

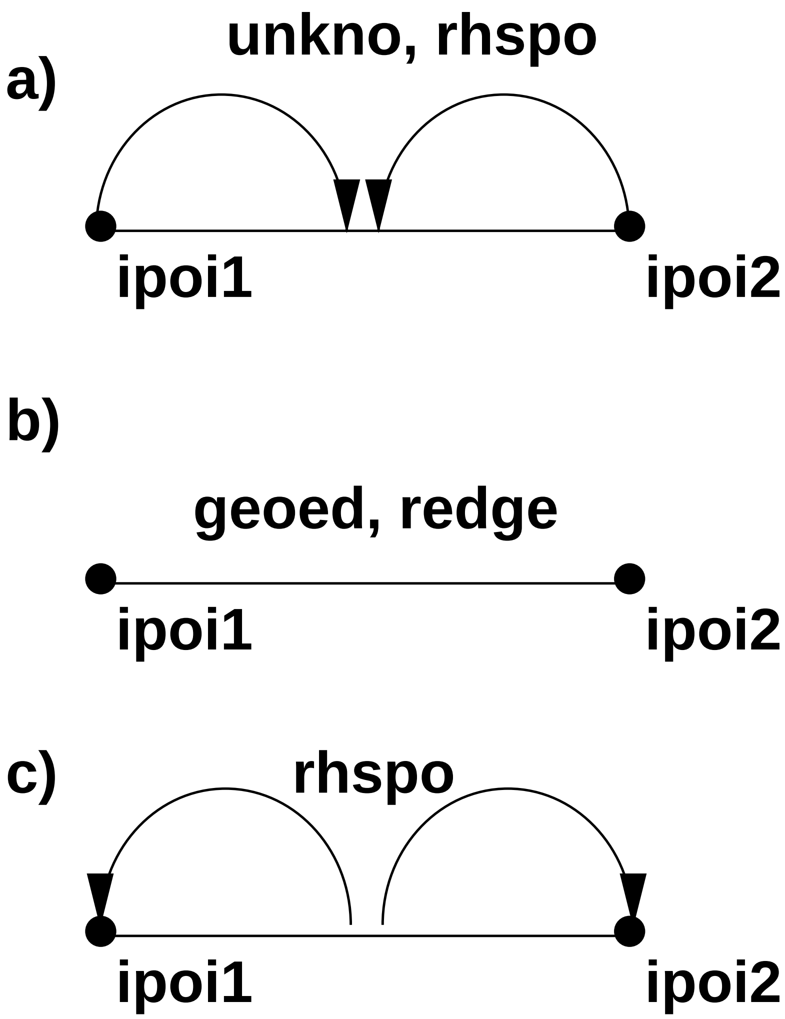
Same As:

$$r^i = k^{ij}\left(\hat{u}_i - \hat{u}_j\right) \quad k^{ij} = \sum_{el} \int \nabla N^i \nabla N^j d\Omega \quad j \neq i$$

Remarks:

- $k^{ji} = k^{ij}$; Expected for Symmetric Operator;
- Edge and Its Sense of Direction Have to be Defined:
  - **Connectivity Array** for Points of an Edge `INPOED(1:2,NEDGE):= IP1, IP2`, and
  - Taking `IP1` $<$ `IP2`;
- For k^{ij}: Need **Edges of an Element**: `INEDEL(NEDEL,NELEM):= IED1,IED2,..`

LAPLACIAN OPERATOR (3)



Edge-Based Laplacian

FIRST ORDER DERIVATIVES: FIRST FORM (1)

RHS:

$$\mathbf{r}^i = -\int N^i N^j_{,k} d\Omega \ \mathbf{F}^k_j$$

Split Into $j = i$, $j \neq i$:

$$\mathbf{r}^i = -\sum_{j \neq i} \left[\sum_{el} \int N^i N^j_{,k} d\Omega \right] \mathbf{F}^k_j - \sum_{el} \int N^i N^i_{,k} d\Omega \ \mathbf{F}^k_i$$

Again Use Conservation Property $\Rightarrow$

$$\mathbf{r}^i = -\sum_{j \neq i} \left[\sum_{el} \int N^i N^j_{,k} d\Omega \right] \mathbf{F}^k_j + \left[\sum_{el} \int N^i \sum_{j \neq i} N^j_{,k} d\Omega \right] \mathbf{F}^k_i$$

Or:

$$\mathbf{r}^i = d^{ij}_k \left(\mathbf{F}^k_i - \mathbf{F}^k_j \right) \quad d^{ij}_k = \sum_{el} N^i N^j_{,k} d\Omega \quad j \neq i$$

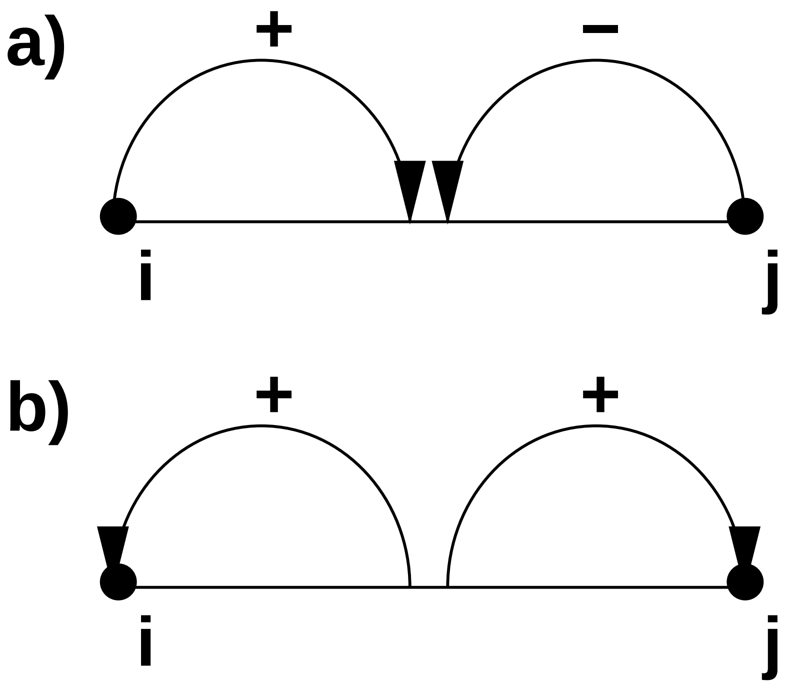
<u>Remark</u>:

- ij vs $ji \Rightarrow$

$$d^{ji}_k = -d^{ij}_k + \int_\Gamma N^j N^i n_k d\Gamma$$

Unsymmetric Operator...

FIRST ORDER DERIVATIVES: FIRST FORM (2)



First Order Derivatives: First Form

FIRST ORDER DERIVATIVES: SECOND FORM (1)

Desired RHS:

$$\mathbf{r}^i = e_k^{ij}\left(\mathbf{F}_j^k + \mathbf{F}_i^k\right)$$

Possible for Linear Elements

$$\mathbf{r}^i = -\sum_{j \neq i}\left[\sum_{el}\int N^i N^j_{,k} d\Omega\right]\mathbf{F}_j^k - \sum_{el}\int N^i N^i_{,k} d\Omega\ \mathbf{F}_i^k$$

In the Sequel

- Sum Over Indices $i, j \Rightarrow i \neq j$
- $\mathbf{F}_{ij}^k := \mathbf{F}_i^k + \mathbf{F}_j^k$

First Integral, Integration by Parts:

$$\mathbf{r}^i = \int_\Omega N^i_{,k} N^j d\Omega\ \mathbf{F}_j^k - \int_\Gamma N^i N^j n_k d\Gamma\ \mathbf{F}_j^k - \int_\Omega N^i N^i_{,k} d\Omega\ \mathbf{F}_i^k$$

FIRST ORDER DERIVATIVES: SECOND FORM (2)

Use:

$$\int_\Omega N^i N^i_{,k} d\Omega \; \mathbf{F}^k_i = \int_\Gamma N^i N^i n_k d\Gamma \; \mathbf{F}^k_i - \int_\Omega N^i N^i_{,k} d\Omega \; \mathbf{F}^k_i$$

$\Rightarrow$

$$\int_\Omega N^i N^i_{,k} d\Omega \; \mathbf{F}^k_i = \frac{1}{2} \int_\Gamma N^i N^i n_k d\Gamma \; \mathbf{F}^k_i$$

$\Rightarrow$

$$\mathbf{r}^i = \int_\Omega N^i_{,k} N^j d\Omega \; \mathbf{F}^k_{ij} - \int_\Omega N^i_{,k} N^j d\Omega \; \mathbf{F}^k_i - \int_\Gamma N^i N^j n_k d\Gamma \; \mathbf{F}^k_j$$
$$- \frac{1}{2} \int_\Gamma N^i N^i n_k d\Gamma \; \mathbf{F}^k_i$$

But

$$\int_\Omega N^i_{,k} N^j d\Omega \; \mathbf{F}^k_{ij} = \int_\Gamma N^i N^j n_k d\Gamma \; \mathbf{F}^k_{ij} - \int_\Omega N^j_{,k} N^i d\Omega \; \mathbf{F}^k_{ij}$$

$\Rightarrow$ Split 1st Integral to Form

$$\mathbf{r}^i = \frac{1}{2} \int_\Omega (N^i_{,k} N^j - N^j_{,k} N^i) \; d\Omega \; \mathbf{F}^k_{ij} + \frac{1}{2} \int_\Gamma N^i N^j n_k d\Gamma \; \mathbf{F}^k_{ij}$$
$$- \int_\Omega N^i_{,k} N^j d\Omega \; \mathbf{F}^k_i - \int_\Gamma N^i N^j n_k d\Gamma \; \mathbf{F}^k_j - \frac{1}{2} \int_\Gamma N^i N^i n_k d\Gamma \; \mathbf{F}^k_i$$

FIRST ORDER DERIVATIVES: SECOND FORM (3)

For Linear Elements: $N^i_{,k} = const.$ in Element $\Rightarrow$

$$\int_\Omega N^i_{,k} N^j d\Omega \ \mathbf{F}^k_i = (nn-1) \int_\Omega N^i_{,k} N^i d\Omega \ \mathbf{F}^k_i =$$

$$\frac{nn-1}{2} \int_\Gamma N^i N^i n_k d\Gamma \ \mathbf{F}^k_i$$

nn : Number of Nodes in Element

$\Rightarrow$

$$\mathbf{r}^i = \frac{1}{2} \int_\Omega (N^i_{,k} N^j - N^j_{,k} N^i) \ d\Omega \ \mathbf{F}^k_{ij} - \frac{1}{2} \int_\Gamma N^i N^j n_k \ d\Gamma \ \mathbf{F}^k_{ij}$$

$$+ \int_\Gamma N^i N^j n_k \ d\Gamma \ \mathbf{F}^k_i - \frac{nn-2}{2} \int_\Gamma N^i N^i n_k \ d\Gamma \ \mathbf{F}^k_i$$

FIRST ORDER DERIVATIVES: SECOND FORM (4)

Use Interpolation Property:

$$\sum_i N^i(\mathbf{x}) = 1 \quad \forall \mathbf{x}$$

$\Rightarrow$

$$\sum_{j \neq i} N^j = 1 - N^i$$

$\Rightarrow N^i N^j$ Part:

$$\int_\Gamma N^i N^j n_k \; d\Gamma \; \mathbf{F}_i^k = \int_\Gamma N^i (1 - N^i) \; n_k \; d\Gamma \; \mathbf{F}_i^k$$

$\Rightarrow$ Final Form of the Residual:

$$\mathbf{r}^i = \frac{1}{2} \int_\Omega (N^i_{,k} N^j - N^j_{,k} N^i) \; d\Omega \; \mathbf{F}_{ij}^k - \frac{1}{2} \int_\Gamma N^i N^j n_k \; d\Gamma \; \mathbf{F}_{ij}^k$$

$$+ \int_\Gamma N^i (1 - \frac{nn}{2} N^i) \; n_k \; d\Gamma \; \mathbf{F}_i^k$$

FIRST ORDER DERIVATIVES: SECOND FORM (5)

Therefore:

$$\mathbf{r}^i = \mathbf{r}^i_{\Omega_{edg}} + \mathbf{r}^i_{\Gamma_{edg}} + \mathbf{r}^i_{\Gamma_{pts}}$$

Where

$$\mathbf{r}^i_{\Omega_{edg}} = \frac{1}{2}\int_{\Omega} (N^i_{,k} N^j - N^j_{,k} N^i)\ d\Omega\ (\mathbf{F}^k_j + \mathbf{F}^k_i) = d^{ij}_k (\mathbf{F}^k_i + \mathbf{F}^k_j)$$

$$\mathbf{r}^i_{\Gamma_{edg}} = -\frac{1}{2}\int_{\Gamma} N^i N^j n_k\ d\Gamma\ (\mathbf{F}^k_j + \mathbf{F}^k_i) = b^{ij}_k (\mathbf{F}^k_i + \mathbf{F}^k_j)$$

$$\mathbf{r}^i_{\Gamma_{pts}} = \int_{\Gamma} N^i (1 - \frac{nn}{2} N^i)\ n_k\ d\Gamma\ \mathbf{F}^k_i = b^i_k \mathbf{F}^k_i$$

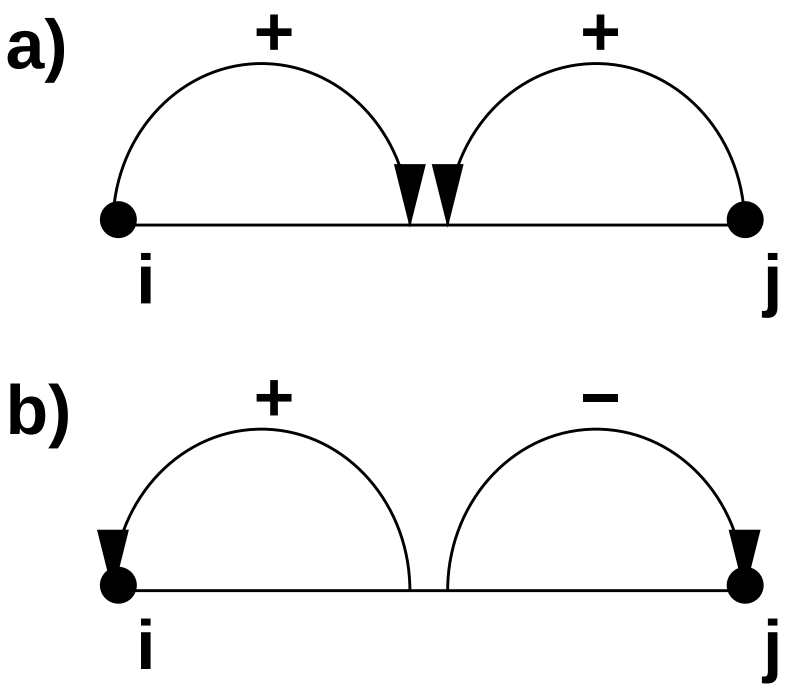
Or:

$$\mathbf{r}^i = d^{ij}_k (\mathbf{F}^k_j + \mathbf{F}^k_i) + b^{ij}_k (\mathbf{F}^k_j + \mathbf{F}^k_i) + b^i_k \mathbf{F}^k_i$$

FIRST ORDER DERIVATIVES: SECOND FORM (6)

Remarks:

- Fully Antisymmetric for d_k^{ij}
- Fully Symmetric for for b_k^{ij}
- Additive Flux Form Achieved
- Possible Only for Linear Elements



First Order Derivatives: Second Form