EDGE-BASED EULER SOLVERS (1)

Edge-Based RHS Obtained Was of the Form:

$$r^i = d_k^{ij} (\mathbf{F}_i^k + \mathbf{F}_j^k)$$

Inner Product Over Dimensions k:

$$r^i = D^{ij} \mathcal{F}_{ij} = D^{ij} (\mathbf{f}_i + \mathbf{f}_j)$$

$\mathbf{f}_i$: 'Fluxes Along Edges'

$$\mathbf{f}_i = S_k^{ij} \mathbf{F}_i^k \quad , \quad S_k^{ij} = \frac{d_k^{ij}}{D^{ij}} \quad , \quad D^{ij} = \sqrt{d_k^{ij} d_k^{ij}}$$

And

$$d_k^{ij} = \frac{1}{2} \int_\Omega (N_{,k}^i N^j - N_{,k}^j N^i) \, d\Omega \quad , \quad d_k^{ij} = -d_k^{ji}$$

Galerkin:

$$\mathcal{F}_{ij} = \mathbf{f}_i + \mathbf{f}_j$$

HIGHER-ORDER GODUNOV

Idea: Use Riemann-Problem as the Basic Model

- Set: $\mathbf{u}_r = \mathbf{u}_i$, $\mathbf{u}_l = \mathbf{u}_j$
- Solve Riemann Problem: $\mathbf{u}_{lr}^R = Rie(\mathbf{u}_l, \mathbf{u}_r)$
- First-order RHS:

$$r^i = D^{ij} f(\mathbf{u}_{ij}^R)$$

Higher Order:

- Better Approximation to $\mathbf{u}_r, \mathbf{u}_l$:
  - From Gradients
  - From Interpolation
- Limiting

APPROXIMATE RIEMANN-SOLVERS

Idea: Replace Costly Exact Riemann-Sover

- Set: $\mathbf{u}_r = \mathbf{u}_i$, $\mathbf{u}_l = \mathbf{u}_j$
- Roe-Solver

$$r^i = D^{ij} \left(f_i + f_j - |\mathbf{A}|(u_r - u_l) \right)$$

Higher Order:

- Better Approximation to $\mathbf{u}_r, \mathbf{u}_l$:
  - From Gradients
  - From Interpolation
- Limiting

SCALAR DISSIPATION

Idea: Replace Costly Roe-Solver

- Set: $\mathbf{u}_r = \mathbf{u}_i$, $\mathbf{u}_l = \mathbf{u}_j$
- Set: $|\mathbf{A}| \approx \lambda_{max}$

$$r^i = D^{ij} \left(f_i + f_j - \lambda_{max}(u_i - u_j) \right)$$

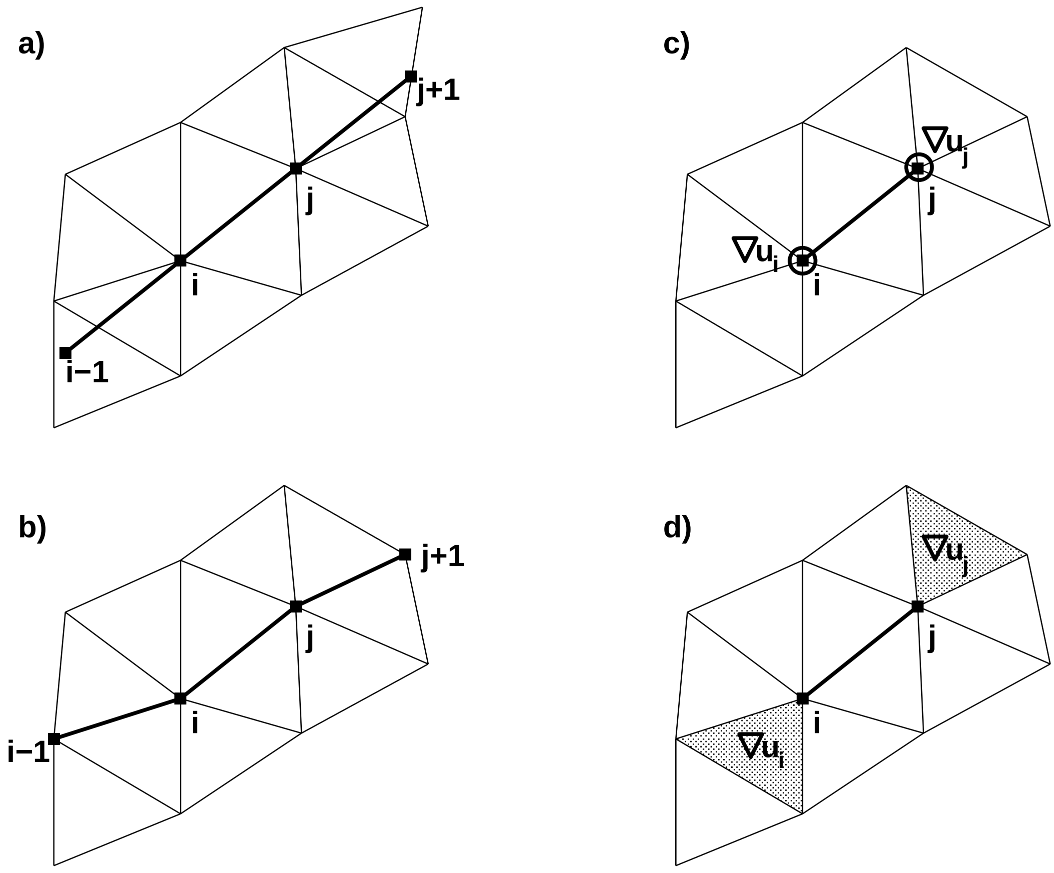
Higher Order:

- Better Approximation to $\mathbf{u}_r, \mathbf{u}_l$:
  - From Gradients
  - From Interpolation
- Limiting

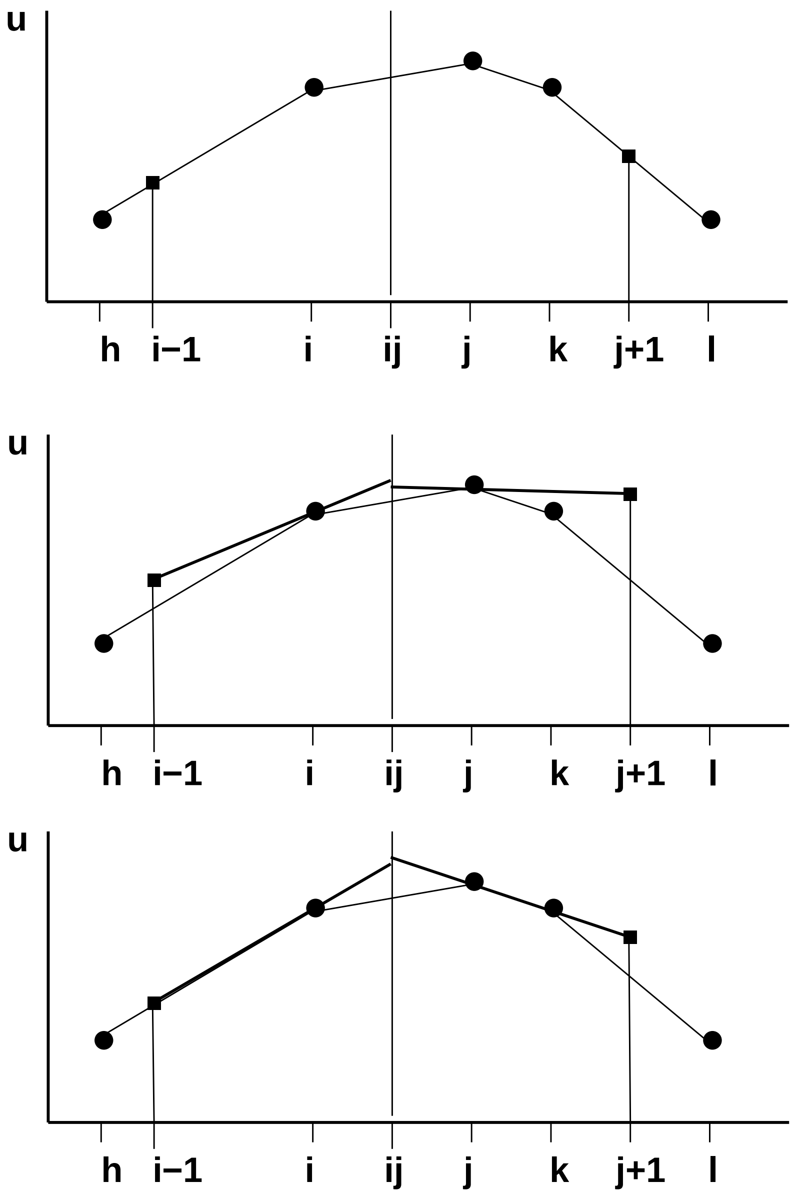
HIGHER ORDER APPROXIMATIONS (1)

Key Idea: Reduce $|u_r - u_l|$

- Continuation/Interpolation
- Extension/Most Aligned Edge
- Gradients



HIGHER ORDER APPROXIMATIONS (2)



SCALAR DISSIPATION + FCT LIMITING

Idea: Replace Costly Grad-Limiting

$$r_l^i = D^{ij}\left(f_i + f_j - \lambda_{max}(u_i - u_j)\right)$$

$$r_h^i = D^{ij}\left(f_i + f_j\right))$$

- Limit Based on $\mathbf{u}_l^{n+1}, \mathbf{u}^n$
- $\Rightarrow$ No Gradients

SCALAR DISSIPATION + NO LIMITING

Idea: Replace Costly Limiting Procedure

$$r^i = D^{ij}\left(f_i + f_j - \lambda_{max}\left[u_i - u_j + \frac{\beta}{2}\mathbf{l}^{ij}(\nabla u_i + \nabla u_j)\right]\right)$$

- $0 < \beta < 1$: Pressure Sensor ('Limitor')
- $\mathbf{l}^{ij}$ Unit Vector Along Edge

SCALAR DISSIPATION + NO LIMITING/GRADIENTS

Idea: Replace Costly Gradients

- Separate Second and Fourth Order Damping

$$d_2 = \lambda_{max}(1-\beta)\left[u_i - u_j\right]$$

$$d_4 = \lambda_{max}\beta\left[u_i - u_j + \frac{\mathbf{l}^{ij}}{2}(\nabla u_i + \nabla u_j)\right]$$

- Taylor-series Expansion

$$u_i - u_j + \frac{\mathbf{l}^{ij}}{2}(\nabla u_i + \nabla u_j) \approx \frac{l^2}{4}\left[\frac{\partial^2 u}{\partial l^2}_j - \frac{\partial^2 u}{\partial l^2}_i\right]$$

- Simplification

$$\frac{l^2}{4}\left[\frac{\partial^2 u}{\partial l^2}_j - \frac{\partial^2 u}{\partial l^2}_i\right] \approx \frac{l^2}{4}\left[\nabla^2 u_j - \nabla^2 u_i\right]$$

- RHS for Advective Fluxes

$$r^i = D^{ij}(f_i + f_j - \lambda_{max}(1-\beta)\left[u_i - u_j\right]$$
$$-\lambda_{max}\beta\frac{l^2}{4}\left[\nabla^2 u_j - \nabla^2 u_i\right])$$

Table 1: Comparison of Edge-Based Euler Solvers

Solver	`EIGENV`	`LIMIT`	`GRADU`
Higher-Order Godunov	YES	YES	YES
Roe-Solver	YES	YES	YES
Scalar Dissipation	NO	YES	YES
Sc.Diss.+Edge-Based FCT	NO	YES	NO
Sc.Diss.+Edge-Based 2+4	NO	NO	YES
Sc.Diss.+PDE-Based 2+4	NO	NO	NO