ROE SOLVER (1)

Re-Interpret Upwind Scheme Flux

$$\mathbf{f}^*_{i+1/2} = \mathbf{f}_i^+ + \mathbf{f}_{i+1}^- = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) - \frac{1}{2}|\mathbf{A}|(\mathbf{u}_{i+1} - \mathbf{u}_i)$$

Consider Non-Conserved Variables $\mathbf{w}$:

$$\Delta\mathbf{u} = \mathbf{P}\Delta\mathbf{w}$$

Can Be Written As:

$$\Delta\mathbf{u} = \sum_j \Delta\mathbf{w}_{(j)}\mathbf{r}^{(j)} = \sum_j \Delta\mathbf{u}^{(j)}$$

ROE SOLVER (2)

1-D Euler:

$$\Delta \mathbf{u} = \Delta w_1 \begin{bmatrix} 1 \\ u \\ u^2/2 \end{bmatrix} + \frac{\rho}{2c} \Delta w_2 \begin{bmatrix} 1 \\ u+c \\ H+uc \end{bmatrix}$$

$$+ \frac{\rho}{2c} \Delta w_3 \begin{bmatrix} 1 \\ u-c \\ H-uc \end{bmatrix}$$

$$\Delta \mathbf{w} = \begin{bmatrix} \Delta\rho - \dfrac{\Delta p}{c^2} \\ \Delta u + \dfrac{\Delta p}{\rho c} \\ \Delta u - \dfrac{\Delta p}{\rho c} \end{bmatrix}$$

ROE SOLVER (3)

Consider Just Linearized Equations

$$\Delta \mathbf{f} = \mathbf{A}\Delta \mathbf{u} = \mathbf{A}\sum_j \Delta \mathbf{u}^{(j)} = \mathbf{A}\sum_j \Delta \mathbf{w}_{(j)}\mathbf{r}^{(j)}$$

$$= \sum_j \lambda_{(j)} \Delta \mathbf{w}_{(j)}\mathbf{r}^{(j)}$$

$$\mathbf{f}^*_{i+1/2} = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) - \frac{1}{2}\sum_j |\lambda_{(j)}| \Delta \mathbf{w}_{(j)}\mathbf{r}^{(j)}$$

Generalize to Nonlinear Case

ROE SOLVER (4)

Desired:

1) $\forall$ Pair $\mathbf{u}_i, \mathbf{u}_{i+1}$: **Exactly**

$$\mathbf{f}_{i+1} - \mathbf{f}_i = \mathbf{A}(\mathbf{u}_i, \mathbf{u}_{i+1}) \cdot (\mathbf{u}_{i+1} - \mathbf{u}_i)$$

- Garantees Rankine-Hugoniot
- Recognizes Only, But Exactly, Discontinuities
- Admits Expansion Shocks

2) For: $\mathbf{u}_i = \mathbf{u}_{i+1} = \mathbf{u}$:

$$\mathbf{A}(\mathbf{u}, \mathbf{u}) = \mathbf{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

3) Eigenvectors of $\mathbf{A}$: Real, Linearly Independent

ROE SOLVER (5)

Roe's Observation (1981):

Can Express **Both $\mathbf{f}, \mathbf{u}$** As **Quadratic** Functions Of:

$$\mathbf{z} = \sqrt{\rho} \begin{bmatrix} 1 \\ u \\ H \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} z_1^2 \\ z_1 z_2 \\ \dfrac{z_1 z_3}{\gamma} + \dfrac{(\gamma-1) z_2^2}{2\gamma} \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} z_1 z_2 \\ \dfrac{(\gamma-1) z_1 z_3}{\gamma} + \dfrac{(\gamma-1) z_2^2}{2\gamma} \\ z_2 z_3 \end{bmatrix}$$

ROE SOLVER (6)

$\Rightarrow$ Can Use Identity

$$\Delta(ab) = \overline{a}\Delta b + \overline{b}\Delta a$$

$\Rightarrow$

$$\Delta \mathbf{u} = \overline{\mathbf{B}}\Delta \mathbf{z}$$

$$\overline{\mathbf{B}} = \left[\begin{array}{ccc} 2z_1 & 0 & 0 \\ z_2 & z_1 & 0 \\ \dfrac{z_3}{\gamma} & \dfrac{\gamma-1}{\gamma}z_2 & \dfrac{z_1}{\gamma} \end{array}\right\}$$

$$\Delta \mathbf{f} = \overline{\mathbf{C}}\Delta \mathbf{z}$$

$$\overline{\mathbf{C}} = \left[\begin{array}{ccc} z_2 & z_1 & 0 \\ \dfrac{\gamma-1}{\gamma}z_3 & \dfrac{\gamma+1}{\gamma}z_2 & \dfrac{\gamma-1}{\gamma}z_1 \\ 0 & z_3 & z_2 \end{array}\right\}$$

$\Rightarrow$

$$\Delta \mathbf{f} = \overline{\mathbf{C}}\Delta \mathbf{z} = \overline{\mathbf{C}\mathbf{B}}^{-1}\Delta \mathbf{u}$$

$\Rightarrow$

$$\mathbf{A} = \overline{\mathbf{C}\mathbf{B}}^{-1}$$

ROE SOLVER (7)

It Gets Even Better:

The Matrix **A** is **Identical to the Local Jacobian** When Expressed As a Function of ρ, u, H **If These Variables Are Replaced By an Average Weighted by the Square Root of the Densities** [Roe Averaging].

Define:

$$R_{i+1/2} = \sqrt{\frac{\rho_{i+1}}{\rho_i}}$$

$$\rho_{i+1/2} = \sqrt{\rho_{i+1}\rho_i} = R_{i+1/2}\rho_i$$

$$u_{i+1/2} = \frac{(u\sqrt{\rho})_{i+1} + (u\sqrt{\rho})_i}{\sqrt{\rho_{i+1}} + \sqrt{\rho_i}} = \frac{R_{i+1/2}u_{i+1} + u_i}{R_{i+1/2} + 1}$$

$$H_{i+1/2} = \frac{(H\sqrt{\rho})_{i+1} + (H\sqrt{\rho})_i}{\sqrt{\rho_{i+1}} + \sqrt{\rho_i}} = \frac{R_{i+1/2}H_{i+1} + H_i}{R_{i+1/2} + 1}$$

Take Eigenvalues/Eigenvectors of The Jacobian **A** With These Values (So-Called **Roe Matrix**)

ROE SOLVER (8)

Step 1: Roe-Average State

$$\mathbf{u}_i, \mathbf{u}_{i+1} \rightarrow R_{i+1/2}, \rho_{i+1/2}, u_{i+1/2}, H_{i+1/2}$$

$$c_{i+1/2} = (\gamma - 1)\left(H_{i+1/2} - \frac{u_{i+1/2}^2}{2}\right)$$

Step 2: Eigenvalues and Eigenvectors of Roe-Average State

$$\lambda_1 = u \quad , \quad \lambda_2 = u + c \quad , \quad \lambda_3 = u - c$$

$$r^{(1)} = \begin{bmatrix} 1 \\ u \\ \dfrac{u^2}{2} \end{bmatrix} \;,\; r^{(2)} = \frac{\rho}{2c}\begin{bmatrix} 1 \\ u + c \\ H + uc \end{bmatrix} \;,\; r^{(3)} = -\frac{\rho}{2c}\begin{bmatrix} 1 \\ u - c \\ H - uc \end{bmatrix}$$

ROE SOLVER (9)

Step 3: Wave Amplitudes

$$\Delta \mathbf{w} = \begin{bmatrix} \Delta\rho - \dfrac{\Delta p}{c_{i+1/2}^2} \\ \Delta u + \dfrac{\Delta p}{\rho_{i+1/2} c_{i+1/2}} \\ \Delta u - \dfrac{\Delta p}{\rho_{i+1/2} c_{i+1/2}} \end{bmatrix}$$

Step 4: Numerical Flux

$$\mathbf{f}_{i+1/2}^* = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) - \frac{1}{2}\sum_j |\lambda_{(j)}| \Delta\mathbf{w}_{(j)} \mathbf{r}^{(j)}$$

ROE SOLVER (10)

Removal of Expansion Shocks: Harten and Hyman (1983)

Option 1:

$$|\lambda|_{mod} = \begin{cases} |\lambda|_{i+1/2} & if \quad |\lambda|_{i+1/2} \geq \epsilon \\ \epsilon & if \quad |\lambda|_{i+1/2} < \epsilon \end{cases}$$

$$\epsilon = max[0, (\lambda_{i+1/2} - \lambda_i), (\lambda_{i+1} - \lambda_{i+1/2})]$$

Option 2:

$$|\lambda|_{mod} = \begin{cases} |\lambda|_{i+1/2} & if \quad |\lambda|_{i+1/2} \geq \epsilon \\ \frac{1}{2}\left(\frac{\lambda^2_{i+1/2}}{\epsilon} + \epsilon\right) & if \quad |\lambda|_{i+1/2} < \epsilon \end{cases}$$