SIMPLIFIED OR SPECIFIC FORMS OF THE GENERAL NAVIER-STOKES EQUATIONS

a) Limit: $Re >> 1$
- Problem : Resolution
- Neglect : Small Turbulent Modes/Scales
- Equation: Reynolds-Averaged Navier-Stokes (RANS)
- Model : Turbulent Stresses

b) Limit: $Ma \to 0$
- Problem : Pressure not from EOS
- Neglect : Speed-of-Sound Modes
- Equation: Incompressible NavSto

c) Limit: $Re \to \infty$
- Problem : Resolution
- Neglect : Viscous/Conductive Effects
- Equation: Euler

d) Limit: $\nabla \times \mathbf{v} \to 0$
- Problem : Unneccessary Effort
- Neglect : Rotation Effects
- Equation: Potential/Laplace

e) Limit: $c >> \overline{\mathbf{v}}$, $\nabla \times \mathbf{v} \to 0$
- Problem : Numerical Errors Due to $\overline{p} >> p'$
- Neglect : Mean Flow Effects
- Equation: Acoustics/Helmholtz

RANS (1)

<u>To Resolve</u>: All Turbulent Length Scales up to l

$\Rightarrow$ (Canuto, Hussaini, Quarteroni, Zhang (1988)):

$$\frac{L}{l} = Re^{\frac{3}{4}} \quad , \quad Re = \frac{uL}{\nu} \quad , \quad u = \left(\frac{1}{3}{u'}^2\right)$$

Assume: n_p Points per Length Scale l

$\Rightarrow$ For Cube L^3:

$$N_p = O(n_p^3 \cdot Re^{\frac{9}{4}})$$

Typical Re-Number in Aerodynamics: $Re = 10^6$

Best Case: $n_p = 1$: $N_p = O(Re^{\frac{9}{4}}) = O(10^{13.5})$

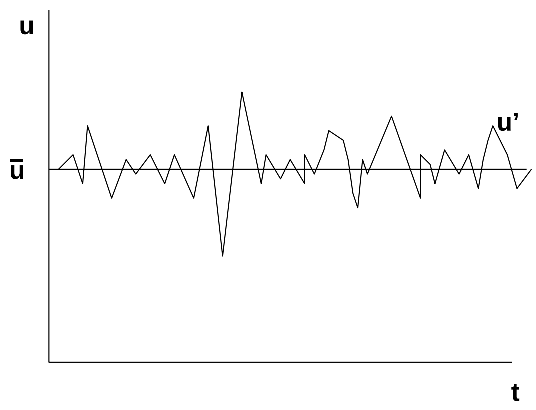
Reality : $n_p = 10$: $N_p = O(10^3 \cdot Re^{\frac{9}{4}}) = O(10^{16.5})$

$\Rightarrow$ **NEED TURBULENCE MODEL**

RANS (2)

<u>Basic Idea</u>:

a) **Split** Into Time-Averaged and Fluctuating Components



E.g.

$$p = \overline{p} + p' \quad , \quad \int p' \; dt = 0 \quad , \quad \int p \; dt = \overline{p}$$

Same for: $\rho, \mathbf{v}, T$, etc.

b) Write into NavSto-Eqns $\Rightarrow$ Higher Order Products

c) **Time-Average** NavSto-Eqns $\Rightarrow$ Removes Mixed Products

d) **Neglect** Higher Order Products

RANS (3)

a) Continuity Equation:

$$\overline{\rho}_{,t} + \rho'_{,t} + \nabla \cdot ((\overline{\rho} + \rho') \cdot (\overline{\mathbf{v}} + \mathbf{v}')) = 0$$

$\int .. \, dt$:

$$\overline{\rho}_{,t} + \nabla \cdot (\overline{\rho \mathbf{v}}) + \nabla \cdot (\rho' \mathbf{v}') = 0$$

Neglect: $\rho' \mathbf{v}'$:

- Incompressible: $\rho' = 0$
- Compressible: $|\rho'| << |\mathbf{v}'|$

$\Rightarrow$

$$\overline{\rho}_{,t} + \nabla \cdot (\overline{\rho \mathbf{v}}) = 0$$

RANS (4)

b) <u>Momentum Equation</u>:

$$((\overline{\rho}+\rho')\cdot(\overline{\mathbf{v}}+\mathbf{v}'))_{,t}+\nabla\cdot((\overline{\rho}+\rho')\cdot(\overline{\mathbf{v}}+\mathbf{v}')\otimes(\overline{\mathbf{v}}+\mathbf{v}'))+$$

$$\nabla(\overline{p}+p')=\nabla\cdot(\overline{\sigma}+\sigma')$$

$\int .. \, dt$: Terms That Vanish

- Linear in $\,'$: p', σ'
- First Order Products: $\overline{\rho}\mathbf{v}', \rho'\overline{\mathbf{v}}, \overline{\rho}\mathbf{v}'\overline{\mathbf{v}}, \rho'\overline{\mathbf{v}}\overline{\mathbf{v}}$

Neglect: $\rho'\mathbf{v}', \rho'\mathbf{v}'\overline{\mathbf{v}}$:

- Incompressible: $\rho'=0$
- Compressible: $|\rho'| << |\mathbf{v}'|$

$\Rightarrow$

$$\overline{\rho\mathbf{v}}_{,t}+\nabla\cdot\overline{\rho\cdot\mathbf{v}\otimes\mathbf{v}}+\nabla\cdot\overline{\rho\cdot\mathbf{v}'\otimes\mathbf{v}'}+\nabla\overline{p}=\nabla\cdot\overline{\sigma}$$

RANS (5)

Final Form (Overbar Omitted):

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = \nabla \cdot \mathbf{F}^v + \mathbf{S}$$

$$\mathbf{u} = \left\{ \begin{array}{r} \rho \\ \rho v_i \\ \rho e \end{array} \right\} \ , \ \mathbf{F}^a_j = \left\{ \begin{array}{r} \rho v_j \\ \rho v_i v_j + p\delta_{ij} \\ v_j(\rho e + p) \end{array} \right\} \ ,$$

$$\mathbf{F}^v_j = \left\{ \begin{array}{r} 0 \\ \sigma_{ij} + \sigma^t_{ij} \\ v_l(\sigma_{lj} + \sigma^t_{lj}) - q_j \end{array} \right\}$$

RANS (6)

Remarks:

a) Almost No Change

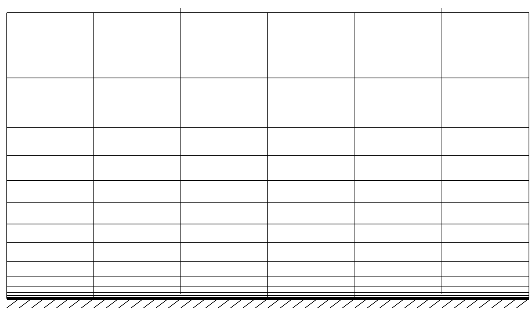
b) New ‘Virtual/Turbulent Stress’ or ‘Eddy Viscosity’:

$$\sigma_{ij}^t = -\rho \overline{v_i' v_j'}$$

c) Need Model for σ_{ij}^t

d) Only Need Fine Grid Normal to Wall $\Rightarrow$ Can Use Stretched Grids

- Savings of 10^4 - 10^6 in Number of Gridpoints



INCOMPRESSIBLE NAVIER-STOKES EQUATIONS (1)

Limit:

$$c^2 = \frac{\partial p}{\partial \rho}|_s \to \infty$$

Approximations:

$$\rho = const.$$

Final Form:

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = \nabla \cdot \mathbf{F}^v + \mathbf{S}$$

$$\mathbf{u} = \left\{ \begin{matrix} \\ \rho v_i \\ \rho c_p T \end{matrix} \right\} \ , \ \mathbf{F}_j^a = \left\{ \begin{matrix} v_j \\ \rho c_p v_i v_j + p\delta_{ij} \\ v_j \rho c_p T \end{matrix} \right\} \ ,$$

$$\mathbf{F}_j^v = \left\{ \begin{matrix} 0 \\ \sigma_{ij} + \sigma_{ij}^t \\ v_l(\sigma_{lj} + \sigma_{lj}^t) - q_j \end{matrix} \right\}$$

Remarks:

- Pressure Not From Equation of State
  - From $\nabla \cdot \mathbf{v} = 0$
  - 'Lagrange Multiplier'
- Energy Equation Decoupled

INCOMPRESSIBLE NAVIER-STOKES EQUATIONS (2)

$\nabla \cdot \sigma$: For $\nabla \cdot \mathbf{v} = 0$

$$\nabla \cdot \sigma = (\sigma_{ij})_{,i} = \left(\mu (v_{i,j} + v_{j,i} - \frac{2}{3} v_{k,k}) \right)_{,i} = \mu v_{j,ii} = \mu \nabla^2 \mathbf{v}$$

Common Form (Obtained After Algebraic Simplifications):

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \mathbf{v}_{,t} + \rho \mathbf{v} \nabla \mathbf{v} + \nabla p = \frac{1}{Re} \nabla^2 \mathbf{v}$$

$$\rho c_p T_{,t} + \rho c_p \mathbf{v} \nabla T = \frac{1}{RePr} \nabla^2 T$$

<u>Pressure Poisson Equation</u>:

Take: Div(Momentum) $\Rightarrow$

$$\nabla^2 p = -\nabla \cdot \rho \mathbf{v} \nabla \mathbf{v}$$

<u>Remarks</u>:

- Pressure Adjusts Immediately to Flowfield
- Consequence of Infinite Speed of Sound Assumption
- Trade-Off: Many Small Hyperbolic Timesteps vs. Elliptic Pressure Solve
- Advantageous for $Ma < 0.1$

EULER EQUATIONS (1)

Limit:

$$\mu \rightarrow 0 \quad , \quad k \rightarrow 0$$

$\Rightarrow$

- Neglect all Viscosity Effects:

$$\sigma^{ij} = 0$$

- Neglect all Conductivity Effects:

$$q_j = 0$$

- Set $\mathbf{F}^v = 0$ in the Navier-Stokes Equations

- $\Rightarrow$ Euler Equations (1785)

Final Form:

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = 0$$

EULER EQUATIONS (2)

Compressible:

$$\mathbf{u} = \left\{ \begin{array}{r} \rho \\ \rho v_i \\ \rho e \end{array} \right\} \ , \ \mathbf{F}_j^a = \left\{ \begin{array}{r} \rho v_j \\ \rho v_i v_j + p\delta_{ij} \\ v_j(\rho e + p) \end{array} \right\}$$

Incompressible:

$$\mathbf{u} = \left\{ \begin{array}{r} \\ v_i \end{array} \right\} \ , \ \mathbf{F}_j^a = \left\{ \begin{array}{r} v_j \\ v_i v_j + \dfrac{p}{\rho}\delta_{ij} \end{array} \right\}$$

Remarks:

a) Hyperbolic System ⇒ Different Boundary Conditions

b) System Less Stiff than NavSto ⇒ Faster Convergence

c) Can Use Equilateral 'Coarse' Grid Up to Wall

- Savings of 2 - 20 in Number of Gridpoints

POTENTIAL FLOW (1)

<u>Basic Idea</u>:

- Inviscid ‘Smooth’ Flow Irrotational
- $\Rightarrow \nabla \times \mathbf{v} \rightarrow 0$
- $\Rightarrow$ Introduce **Potential** Φ

$$\mathbf{v} = \nabla\Phi \quad , \quad \Rightarrow \quad \nabla \times \nabla\Phi = 0$$

For Steady Flows:

$$\nabla \cdot \rho\mathbf{v} = \nabla \cdot (\rho\nabla\Phi) = 0$$

Density: From Isentropic Assumption and Bernoulli Equation

$$\frac{\rho}{\rho_0} = \left(1 - \frac{(\nabla\Phi)^2}{H_0}\right)^{\frac{1}{\gamma-1}}$$

POTENTIAL FLOW (2)

For **Incompressible** Flows: $\rho = \rho_0$

$\Rightarrow$

$$\nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0$$

$\Rightarrow$ **Laplace Equation**

<u>Remarks</u>:

- Vector Unknown (Euler) $\rightarrow$ Scalar Unknown (Laplace)
- Very Fast Solution Techniques Available
- Solution Sequence:
  - Laplace: $\rightarrow \Phi$
  - $\nabla\Phi$: $\rightarrow \mathbf{v}$
  - Bernoulli: $\rightarrow p$

ACOUSTIC LIMIT (1)

Basic Idea:

- Separate Out the High Frequency Components From the Mean Flow

$$p = \overline{p} + p' \quad , \quad \mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}' \quad , \quad \rho = \overline{\rho} + \rho'$$

- Keep Only High Frequency Terms (Opposite to RANS)
- Assume 'Average' Quantities Already Satisfy NAVSTO, RANS or EULER

Take Limit:

- $c \;>> |\mathbf{v}|$
- $p_{,t} >> \rho_{,t}$
- $p' << p$

Final Form (Overbar Omitted):

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = 0$$

$$\mathbf{u} = \left\{ \begin{array}{r} p \\ \rho v_i \end{array} \right\} \; , \; \mathbf{F}^a_j = \left\{ \begin{array}{r} \rho c^2 v_j \\ p\delta_{ij} \end{array} \right\}$$

ACOUSTIC LIMIT (2)

Or:

$$p_{,t} + c^2 \rho \nabla \cdot \mathbf{v} = 0$$

$$\mathbf{v}_{,t} + \frac{1}{\rho} \nabla p = 0$$

Combining the Equations:

$$p_{,tt} - c^2 \nabla^2 p = 0$$

$\Rightarrow$ **Helmholtz Equation**

Frequency-Domain Form:

$$p = p_s(\mathbf{x}) \cdot p_t(t) = p_s(\mathbf{x}) \cdot e^{i\omega t}$$

$\Rightarrow$

$$\left(\frac{\omega}{c}\right)^2 p_s + \nabla^2 p_s = 0$$