OUTLINE

- Optimal Solvers
- Implicit Relaxation Schemes
- LU-SGS
- Numerical Examples
- Conclusions

OPTIMAL SOLVERS

Fastest Possible Convergence:

- Space-Marching/Blocking
- GMRES-LU-SGS
- Multigrid

IMPLICIT RELAXATION SCHEMES (1)

After Spatial Discretization:

$$\mathbf{M}_l^i \mathbf{u}_{,t}^i = \mathbf{r}^i = \sum \mathbf{C}^{ij} \cdot \mathbf{F}_{ij}$$

Time Integration:

- Backward Euler Scheme
- Linearization: $\mathbf{A}$: Jacobian of $\mathbf{F}$

$$\left[\frac{1}{\Delta t} \mathbf{M}_l^i - \sum \mathbf{C}^{ij} \cdot \mathbf{A}_{ij} \right] \Delta \mathbf{u}^i = \mathbf{r}^i$$

IMPLICIT RELAXATION SCHEMES (2)

Re-Write With Lower/Diagonal/Upper Entries As:

$$(\mathbf{L} + \mathbf{D} + \mathbf{U}) \cdot \Delta\mathbf{u} = \mathbf{r}$$

Variants:

- Gauss-Seidel (GS)

$$(\mathbf{L} + \mathbf{D}) \cdot \Delta\mathbf{u}^1 = \mathbf{r} - \mathbf{U} \cdot \Delta\mathbf{u}^0$$
$$(\mathbf{D} + \mathbf{U}) \cdot \Delta\mathbf{u} = \mathbf{r} - \mathbf{L} \cdot \Delta\mathbf{u}^1$$

- Lower-Upper Symmetric GS (LU-SGS)

$$(\mathbf{L} + \mathbf{D}) \cdot \mathbf{D}^{-1} \cdot (\mathbf{D} + \mathbf{U}) \cdot \Delta\mathbf{u} = \mathbf{r}$$

OPTIMIZATION OF RELAXATION SCHEMES (1)

Key ideas [Sharov, Luo]:

- Use Spectral Radius of $\mathbf{A}$ for Diagonal Entries:

$$\mathbf{D} = \left[\frac{1}{\Delta t} \mathbf{M}_l^i - 0.5 \sum \mathbf{C}^{ij} \rho_A \right] \mathbf{I}$$

- Replace:

$$\mathbf{A} \cdot \Delta \mathbf{u} \approx \Delta \mathbf{F}$$

$\Rightarrow$

$$\Delta \mathbf{F} = \mathbf{F}(\mathbf{u} + \Delta \mathbf{u}) - \mathbf{F}(\mathbf{u})$$

- Central Difference vs. Upwind
  - Luo'98: No Discernable Difference
  - $\Rightarrow$ For Relaxation: Use Central Difference $\Delta \mathbf{F}$

OPTIMIZATION OF RELAXATION SCHEMES (2)

- Vectorization:
  - Half-Planes [Sharov]
  - Colouring of Planes

Combined Effect:

- Matrix Free
- No Extra Storage vis a vis Explicit
- Per Relaxation Sweep: Faster than Explicit

LU-SGS: FINAL FORM

a) Forward Sweep:

$$\Delta\hat{\mathbf{u}}^i = \mathbf{D}^{-1}\left[\mathbf{r}^i - 0.5\sum_{j<i}\mathbf{C}^{ij}\cdot(\Delta\hat{\mathbf{F}}_{ij} - \rho_A\Delta\hat{\mathbf{u}}_j)\right]$$

b) Backward Sweep:

$$\mathbf{r} = \mathbf{D}\cdot\Delta\hat{\mathbf{u}}$$

$$\Delta\mathbf{u}^i = \mathbf{D}^{-1}\left[\mathbf{r}^i - 0.5\sum_{j>i}\mathbf{C}^{ij}\cdot(\Delta\mathbf{F}_{ij} - \rho_A\Delta\mathbf{u}_j)\right]$$

GMRES

- Use LU-SGS As Preconditioners
- Central Matrix-Vector Product:

$$\mathbf{A} \cdot \Delta\mathbf{u} = (\mathbf{L} + \mathbf{D} + \mathbf{U}) \cdot \Delta\mathbf{u}$$

- Same Loop Structure $(\mathbf{L}, \mathbf{D}, \mathbf{U})$ for:
  - Gauss-Seidel
  - GMRES

  $\Rightarrow$ Single 'Sweep' Subroutine
- Initialize Gauss-Seidel Loop With LU-SGS

LU-SGS (4)

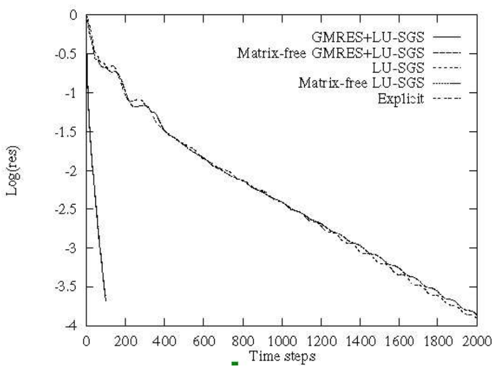
- Advancing Front Point Renumbering
    - ‘Hyperplane’ Structure
    - Minimizes Cache Misses
- Renumber Edges According to Points
- Renumber Points In Hyperplanes
    - Avoid Memory Contention
    - Allow Ordered Sweeps
- Renumber Edges According to Points
    - Ordered Edge Set
    - Minimizes Cache Misses
- Renumber Edges for Vectorization
    - Avoid Memory Contention
- Split Hyperplanes for SM Parallel

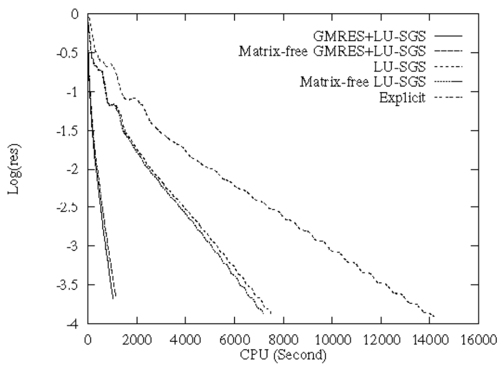
ONERA M6 WING (1)

Grid:

- `Nelem=740K`
- `Npoin=136K`

Machine: Cray C-90, 1 Processor





ONERA M6 WING (2)

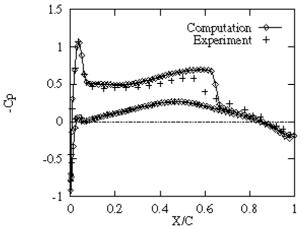


Fig. 2c: Comparison between computed and experimental surface pressure coefficient for wing section at 20% semispan.

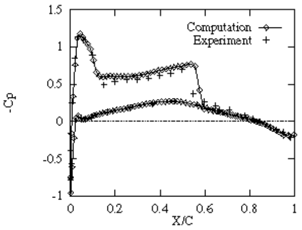


Fig. 2d: Comparison between computed and experimental surface pressure coefficient for wing section at 44% semispan.

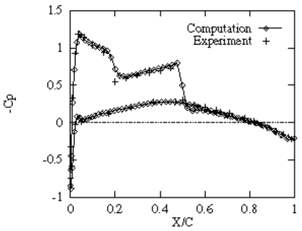


Fig. 2e: Comparison between computed and experimental surface pressure coefficient for wing section at 65% semispan.

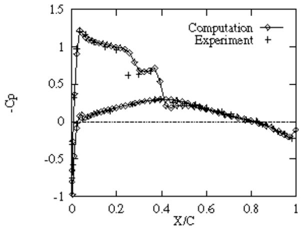


Fig. 2f: Comparison between computed and experimental surface pressure coefficient for wing section at 80% semispan.

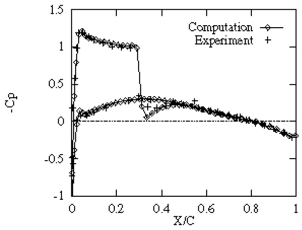


Fig. 2g: Comparison between computed and experimental surface pressure coefficient for wing section at 90% semispan.

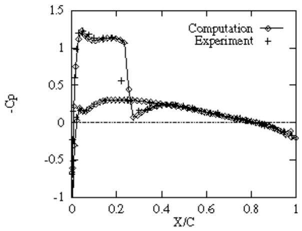


Fig. 2h: Comparison between computed and experimental surface pressure coefficient for wing section at 95% semispan.

FLOW PAST TRANSONIC SPHERE

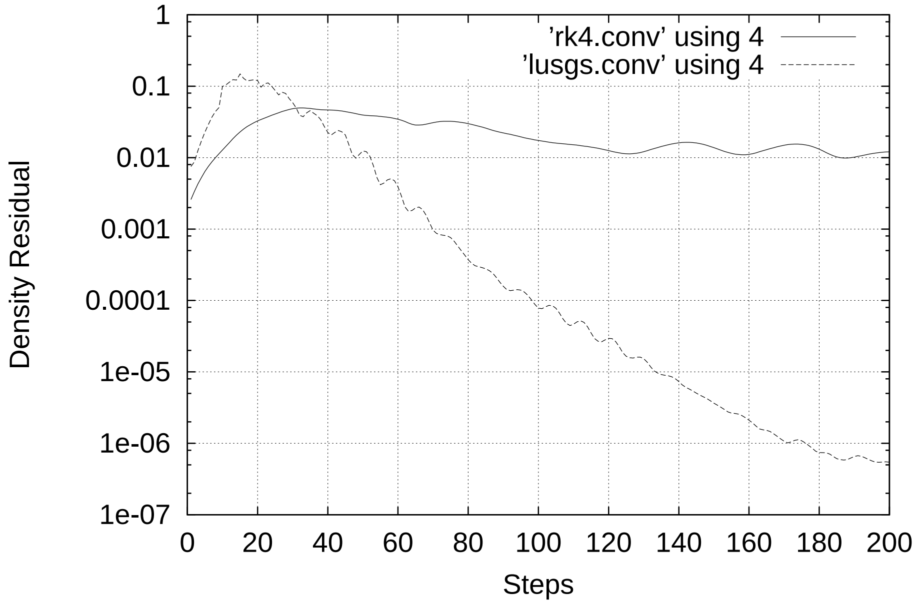


Figure 1 LUSGS Convergence results

CONCLUSIONS

- LU-SGS (Sweep) Implemented in FEFLO
- GMRES + Gauss-Seidel Option
- Vectorized and Parallelized
- Unclear:
  Optimal Renumbering for RANS Grids