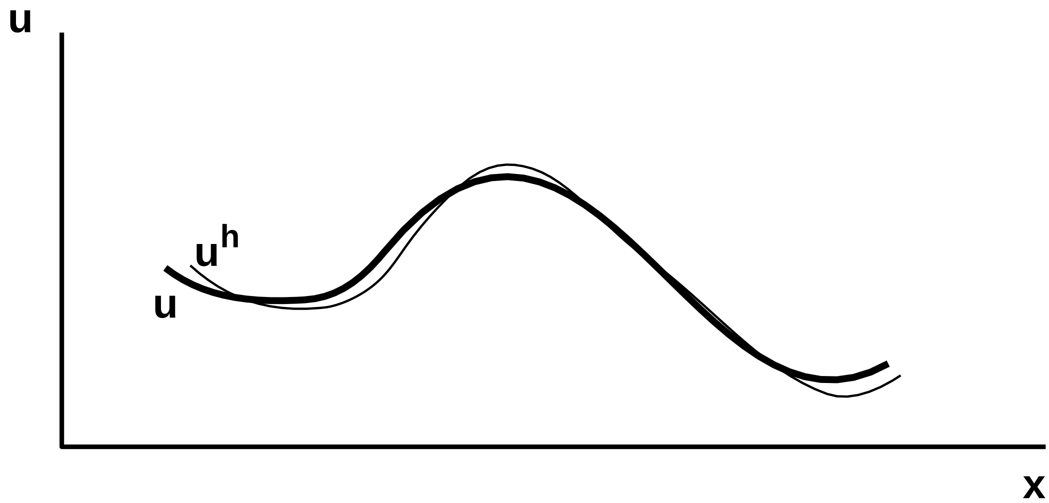
APPROXIMATION THEORY (1)

To do: given $u(x)$ in Ω, approximate by known functions

$$u(x) \approx u^h(x) = f^i(x)a_i = N^i(x)\hat{u}_i$$



Approximation of Functions

APPROXIMATION THEORY (2)

Examples:

- Truncated Taylor Series

$$u(x) \approx u^h(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$$

$$a_j = \frac{1}{j!} \left. \frac{d^j u}{dx^j} \right|_{x=0}$$

- Truncated Sine Series

$$u(x) \approx u^h(x) = a_j sin \frac{j\pi x}{L}$$

$$a_j = \frac{2}{L} \int_0^L u(x) sin \frac{j\pi x}{L} dx$$

- Legendre Polynomials
- Hermite Polynomials

In General: Choose complete set of trial functions N^j:

$$u(x) \approx u^h(x) = N^j a_j \quad ; \quad j = 1, 2, ...M$$

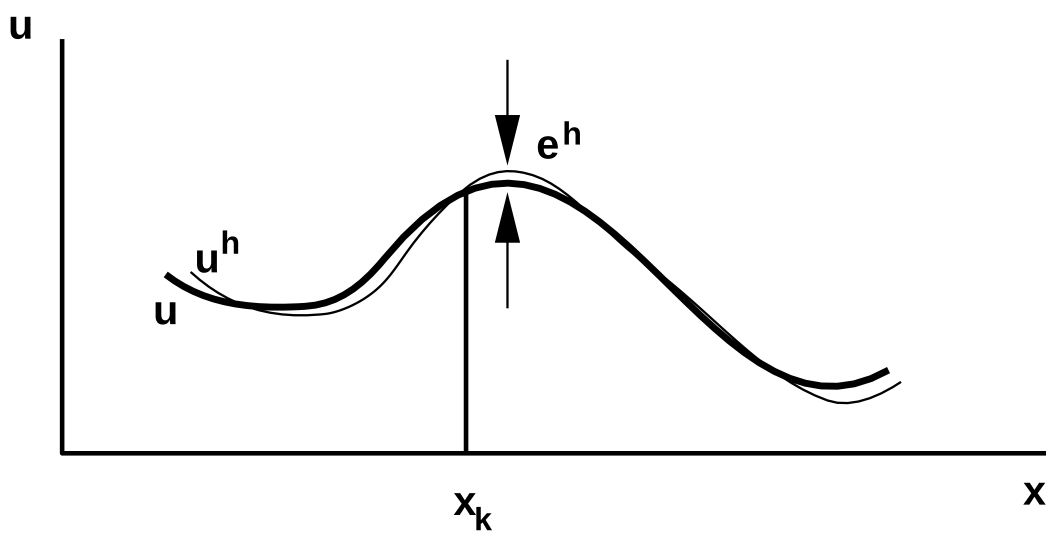
DETERMINATION OF CONSTANTS

1. Point Fitting: set $u^h = u$ at M selected points

$$u^h\big|_{x_k} = u\big|_{x_k} \quad k = 1, 2, ...M$$

$\Rightarrow$

$$\mathbf{K} \cdot \mathbf{a} = \mathbf{f}$$



Point Fitting

2. Weighted Residual Methods(WRM):

Define: $\epsilon^h = u - u^h$ - the error or residual

Require: $\epsilon^h \rightarrow 0$ in Ω

Introduce a set of weighting functions W^i ; $i = 1, 2, ...M$
Require that:

$$\int_\Omega W^i \epsilon^h d\Omega = 0 \quad , \quad i = 1, 2, ...M$$

Then, as $M \rightarrow \infty$, $\epsilon^h \rightarrow 0$ at all points in Ω

Insert expression for u^h:

$$\int_\Omega W^i \left(u - N^j a_j\right) d\Omega = 0$$

$\Rightarrow$

$$\mathbf{K} \cdot \mathbf{a} = \mathbf{r}$$

$$K^{ij} = \int_\Omega W^i N^j d\Omega \quad , \quad r^i = \int_\Omega W^i u d\Omega$$

Choice of W^i defines method !

2.1 Point Collocation

Choose: $W^i = \delta(x - x_i) \quad , \quad x_i \in \Omega$

WR statement

$$\int_\Omega \delta(x - x_i)\epsilon^h d\Omega = \epsilon^h(x_i) = 0 \quad , \quad i = 1, 2, ...M$$

$\Rightarrow$

$$N^j(x_i)a_j = u(x_i)$$

$\Rightarrow$ same as Point Fitting

2.2 Galerkin Method

Choose: $W^i = N^i$

WR statement

$$\int_\Omega N^i \epsilon^h d\Omega = \int_\Omega N^i \left(u - N^j a_j\right) d\Omega = 0 \quad , \quad i = 1, 2, ...M$$

$$\left[\int_\Omega N^i N^j d\Omega\right] a_j = \int_\Omega N^i u d\Omega \quad , \quad i = 1, 2, ...M$$

i.e.

$$\mathbf{M_c} \cdot \mathbf{a} = \mathbf{r}$$

Remarks:

- $\mathbf{M_c}$ is the consistent mass-matrix
- Lumping of $\mathbf{M_c}$ will not give point fitting, as $\mathbf{r}$ has integrals

LEAST SQUARES PROBLEM

$$I_{ls} = \int_\Omega (\epsilon^h)^2 d\Omega = \int_\Omega (u^h - u)^2 d\Omega$$

$$= \int_\Omega (N^k a_k - u)^2 d\Omega \to min$$

$\Rightarrow$

$$\delta I_{ls} = \delta a_k \int_\Omega N^k (N^l a_l - u) d\Omega = 0$$

$$\int_\Omega N^k N^l d\Omega \; a_l = \int_\Omega N^k u d\Omega$$

$\Rightarrow$ Equivalent to Galerkin WRM

$\Rightarrow$ Choice of W^i from same set as N^i optimal

DRAWBACKS OF GLOBAL TRIAL FUNCTIONS

1. Determining N^j's difficult for all but the simplest geometries in 2/3D
2. Matrix $\mathbf{K}$ is full
3. Matrix $\mathbf{K}$ can become ill-conditioned - even for simple problems (can use strongly orthogonal polynomials)
4. a_j's have no physical significance

$\Rightarrow$ Use **LOCAL TRIAL FUNCTIONS**

LOCAL TRIAL FUNCTIONS

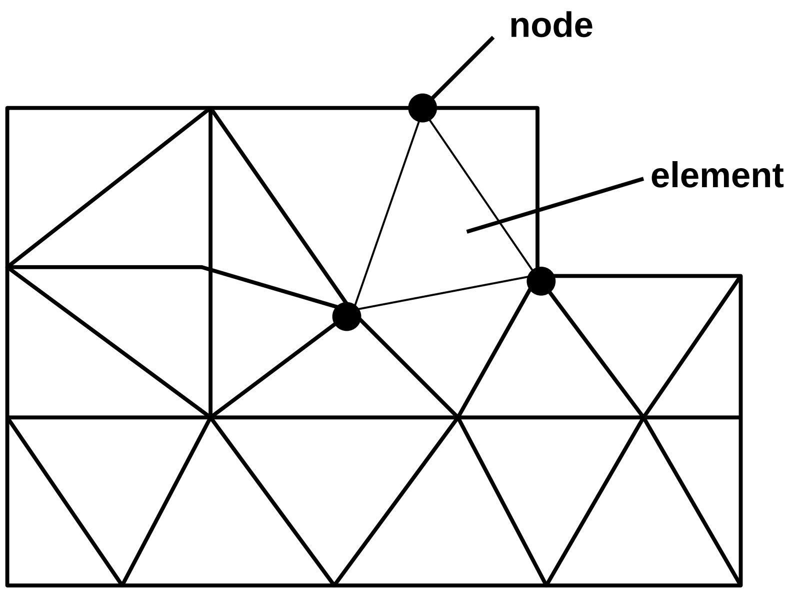
Given $u(x)$, $x \in \Omega$:

Divide Ω into a set of non-overlapping sub-intervals Ω_{el}

Define u^h in each sub-interval

Sub-intervals - ELEMENTS

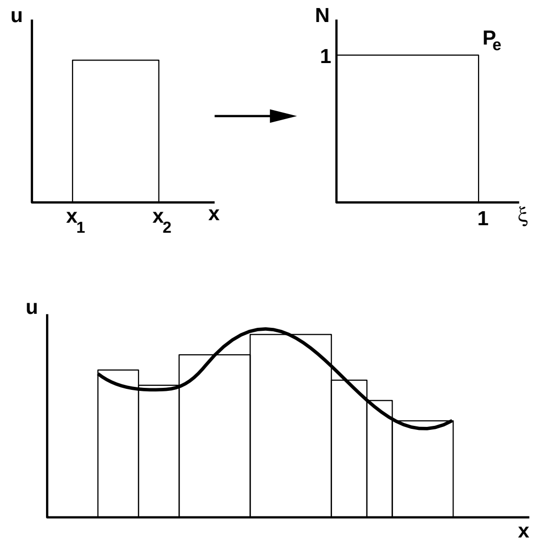
$x_1, x_2, ...$ - NODES



CONSTANT TRIAL FUNCTIONS

Define a piecewise constant function

$$P^E = \begin{cases} 1 & \text{in element } E \\ 0 & \text{in all other elements} \end{cases}$$



Constant Trial Functions

Then globally

$$u \approx u^h = P^E u_E$$

Locally, on element el,

$$u \approx u^h = u_{el}$$

LINEAR TRIAL FUNCTIONS

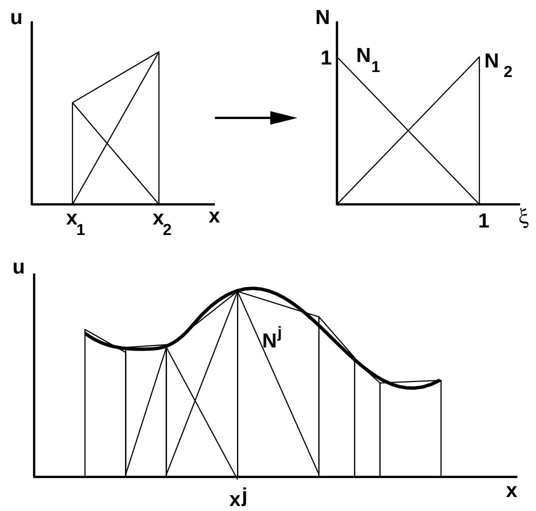
Better approximation: let u^h vary linearly

Place nodes at end of each element

Define a piecewise linear trial function

$$N^j = \begin{cases} 1 & \text{at node } j \\ 0 & \text{at all other nodes} \end{cases}$$

and N^j non-zero only on elements associated with node j



Linear Trial Functions

Globally

$$u \approx u^h = N^j(x)u(x_j) = N^j(x)\hat{u}_j$$

Locally over element el with nodes 1 and 2

$$u \approx u^h = N^1\hat{u}_1 + N^2\hat{u}_2$$

$\xi = (x - x_1)/(x_2 - x_1)$

$N^1_{el} = (x_2 - x)/h_{el} = 1 - \xi$

$N^2_{el} = (x - x_1)/h_{el} = \xi$

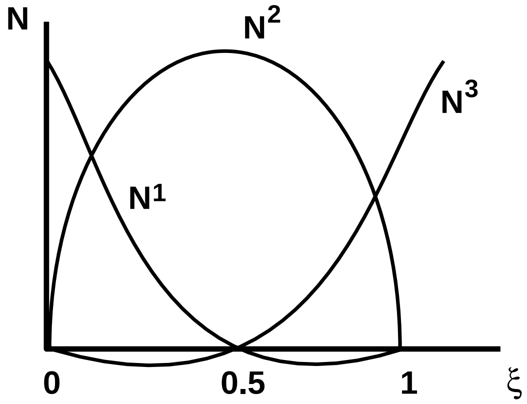
$u^h = \frac{(x_2-x)u_1+(x-x_1)u_2}{h_{el}} = (1-\xi)u_1 + \xi u_2$

Observe that: $x = (1-\xi)x_1 + \xi x_2 = N^1x_1 + N^2x_2$

QUADRATIC TRIAL FUNCTIONS

Better approximation: let u^h vary quadratically

Place nodes at end of each element, as well as the middle



Quadratic Trial Functions

$$N^1_{el} = (1-\xi)(1-2\xi)$$

$$N^2_{el} = 4\xi(1-\xi)$$

$$N^3_{el} = -\xi(1-2\xi)$$

GENERAL PROPERTIES OF SHAPE-FUNCTIONS

1. Interpolation Property:

$$u^h = N^i(x)\hat{u}_i$$

$\Rightarrow$

$$u^h(x_j) = N^i(x_j)\hat{u}_i = \hat{u}_j \Rightarrow N^i(x_j) = \delta^i_j$$

2. Constant Sum: Must be able to represent a constant

$$u = 1 \;\Rightarrow\; u^h = 1 = N^i(x)\hat{u}_i$$

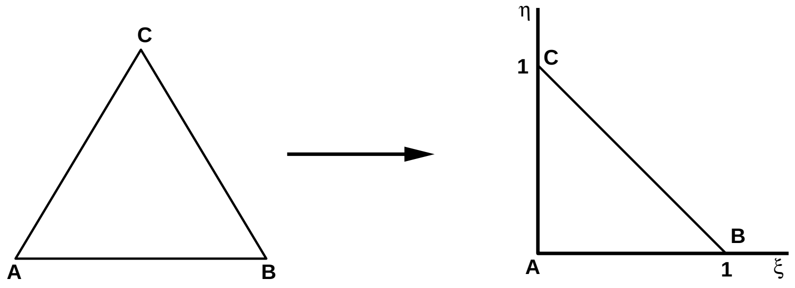
but interpolation property $\Rightarrow\; \hat{u}_i = 1 \;\Rightarrow$

$$\sum_i N^i(x) = 1 \quad , \quad \forall x \in \Omega \qquad (*)$$

3. Conservation property: from Eqn.$(*)$:

$$\sum_i N^i_{,k} = 0 \quad , \quad \forall x \in \Omega_{el}$$

LINEAR TRIANGLE (1)



Linear Triangle

1) <u>Shape Functions</u>

$$\mathbf{x} = \mathbf{x}_A + (\mathbf{x}_B - \mathbf{x}_A)\xi + (\mathbf{x}_C - \mathbf{x}_A)\eta$$

or

$$\mathbf{x} = N^i \mathbf{x}_i = (1 - \xi - \eta)\mathbf{x}_A + \xi \mathbf{x}_B + \eta \mathbf{x}_C$$

LINEAR TRIANGLE (2)

2) <u>Shape Function Derivatives</u>

$$N^i_{,x} = N^i_{,\xi}\xi_{,x} + N^i_{,\eta}\eta_{,x}$$

$$\mathbf{J} = \begin{pmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{pmatrix} = \begin{pmatrix} x_{BA} & x_{CA} \\ y_{BA} & y_{CA} \end{pmatrix}$$

$$det(\mathbf{J}) = 2A_{el} = x_{BA}\ y_{CA} - x_{CA}\ y_{BA}$$

$$\mathbf{J}^{-1} = \begin{pmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{pmatrix} = \frac{1}{2A}\begin{pmatrix} y_{CA} & -x_{CA} \\ -y_{BA} & x_{BA} \end{pmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} N^1 \\ N^2 \\ N^3 \end{bmatrix}_{,x} = \frac{1}{2A}\begin{bmatrix} -y_{CA} + y_{BA} \\ y_{CA} \\ -y_{BA} \end{bmatrix}$$

$$\begin{bmatrix} N^1 \\ N^2 \\ N^3 \end{bmatrix}_{,y} = \frac{1}{2A}\begin{bmatrix} x_{CA} - x_{BA} \\ -x_{CA} \\ x_{BA} \end{bmatrix}$$

LINEAR TRIANGLE (3)

3) Normals

3.1 Direction:

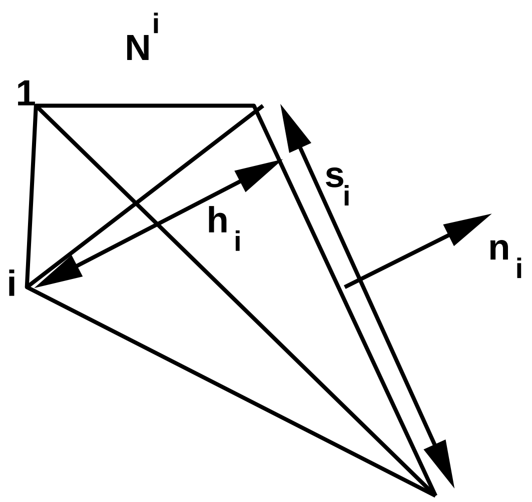
$$\mathbf{n}^i = -\frac{\nabla N^i}{|\nabla N^i|}$$

3.2 Height:

$$|\nabla N^i| = \frac{1}{h_i} \quad \Rightarrow \quad h_i = \frac{1}{|\nabla N^i|}$$

3.3 Face-Normals:

$$(s\mathbf{n})^i = -s^i \frac{\nabla N^i}{|\nabla N^i|} = -s^i h_i \nabla N^i = -2A\nabla N^i$$



LINEAR TRIANGLE (4)

4) <u>Basic Integrals</u>

$$\int_{el} N^i d\Omega = \frac{A}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{M}_{el} = \int N^i N^j d\Omega = \frac{\Delta_{el}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

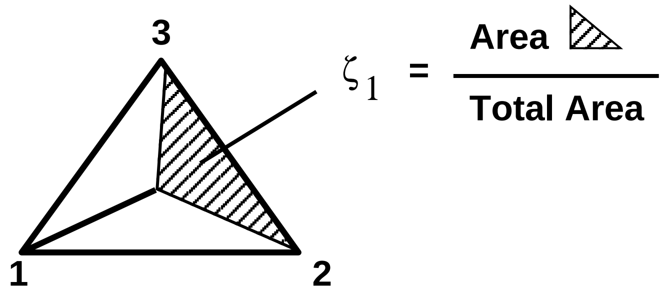
SHAPE-FUNCTIONS FOR TRIANGLES (1)

1) Linear Triangle

$N^1 = \zeta_1 = 1 - \xi - \eta$

$N^2 = \zeta_2 = \xi$

$N^3 = \zeta_3 = \eta$



Area Coordinates

SHAPE-FUNCTIONS FOR TRIANGLES (2)

2) Quadratic Triangle

$$N^1 = \zeta_1(2\zeta_1 - 1) = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$$

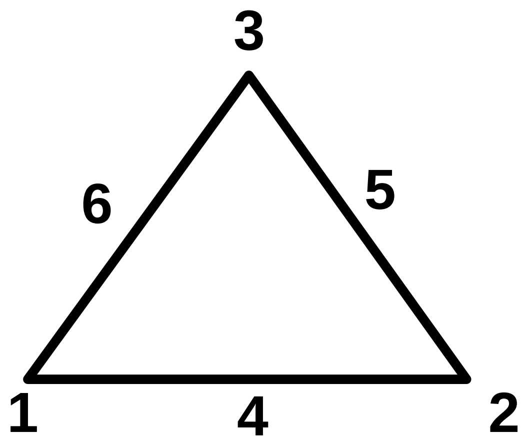
$$N^2 = \zeta_2(2\zeta_2 - 1) = \xi(2\xi - 1)$$

$$N^3 = \zeta_3(2\zeta_3 - 1) = \eta(2\eta - 1)$$

$$N^4 = 4\zeta_1\zeta_2 = 4\xi(1 - \xi - \eta)$$

$$N^5 = 4\zeta_2\zeta_3 = 4\xi\eta$$

$$N^6 = 4\zeta_1\zeta_3 = 4\eta(1 - \xi - \eta)$$



Degrees of Freedom of Quadratic Triangle

SHAPE-FUNCTIONS FOR TRIANGLES (2)

3) Cubic Triangle

$$N^1 = \frac{1}{2}\zeta_1(3\zeta_1 - 1)(3\zeta_1 - 2)$$

$$N^2 = \frac{1}{2}\zeta_2(3\zeta_2 - 1)(3\zeta_2 - 2)$$

$$N^3 = \frac{1}{2}\zeta_3(3\zeta_3 - 1)(3\zeta_3 - 2)$$

$$N^4 = \frac{9}{2}\zeta_1\zeta_2(3\zeta_1 - 1)$$

$$N^5 = \frac{9}{2}\zeta_1\zeta_2(3\zeta_2 - 1)$$

$$N^6 = \frac{9}{2}\zeta_2\zeta_3(3\zeta_2 - 1)$$

$$N^7 = \frac{9}{2}\zeta_2\zeta_3(3\zeta_3 - 1)$$

$$N^8 = \frac{9}{2}\zeta_3\zeta_1(3\zeta_3 - 1)$$

$$N^9 = \frac{9}{2}\zeta_3\zeta_1(3\zeta_1 - 1)$$

$$N^{10} = 27\zeta_1\zeta_2\zeta_3$$

SHAPE-FUNCTIONS FOR QUADS

1) Bilinear Quad

$$N^1 = (1-\xi)(1-\eta)$$

$$N^2 = \xi(1-\eta)$$

$$N^3 = \xi\eta$$

$$N^4 = (1-\xi)\eta$$

2.1) Quadratic Serendipity Quad

$$N^1 = (1-\xi)(1-\eta)(1-2\xi-2\eta)$$

$$N^2 = -\xi(1-\eta)(1-2\xi+2\eta)$$

$$N^3 = -\xi\eta(3-2\xi-2\eta)$$

$$N^4 = -(1-\xi)\eta(1+2\xi-2\eta)$$

$$N^5 = 4\xi(1-\xi)(1-\eta)$$

$$N^6 = 4\eta\xi(1-\eta)$$

$$N^7 = 4(1-\xi)\xi\eta$$

$$N^8 = 4(1-\eta)(1-\xi)\eta$$

WRM OF APPROXIMATION WITH LOCAL FUNCTIONS

Basic Idea:

$$\int_{\Omega} \ldots = \sum_{el} \int_{\Omega_{el}} \ldots \qquad (*)$$

$\Rightarrow$

- build integrals on element level
- gather info from global point-arrays to local element-arrays
- scatter-add resulting integrands to global rhs/matrix locations

$$K^{ij} u_j = \left[\sum_{el} K^{ij}_{el} \right] [u_j]_{el} = \sum_{el} r^i_{el} = r^i$$

Note: for Eqn.($*$) only need info: nodes belonging to an element

$\Rightarrow$ drastic simplification of data structures/logic

EFFORT vs ACCURACY (1)

Optimal Effort:

Assume even error distribution (optimal mesh) initially; then

$$Eff \geq c_1 h^{-d}$$

d: dimensionality of problem

Error:

$$\|u - u^h\| = c_2 h^{p+1} |u|_{p+1}$$

p: order of approximation for the elements

Desired: attain $\|u - u^h\| \to 0$ **without** $Eff \to \infty$

$$Eff \cdot \|u - u^h\| = c_3 h^{p+1-d} |u|_{p+1}$$

Worst Case Scenario (e.g. Turbulence)

Dimension	Eff * Err	Decrease with $h \to 0$
1-D	h^p	$p \geq 1$
2-D	h^{p-1}	$p \geq 2$
3-D	h^{p-2}	$p \geq 3$

$\Rightarrow$ strive for elements of higher order

EFFORT vs ACCURACY (2)

However: Redeeming factors:

- 1-D features in 2/3-D: boundary layers, shocks, contact discontinuities
- Engineering Accuracy $O(1\%)$
- Unknown Physics

REFERENCES

[1] O.C. Zienkiewicz and K. Morgan - *Finite Elements and Approximation*; J. Wiley & Sons (1983).