UPWINDING (1)

1-D Advection, constant mesh:

$$u_{,t} + f_{,x} = 0$$

$\Rightarrow$

$$\Delta x \frac{\Delta \mathbf{u}_i}{\Delta t} + \left[F_{i+1/2} - F_{i-1/2}\right] = 0$$

Central Differencing:

$$F_{i+1/2} = \frac{1}{2}\left(f_i + f_{i+1}\right)$$

$\Rightarrow$ Simple Average $\Rightarrow$ No Account of Physics

<u>Physics</u>: $\Rightarrow$ Information Only one-Sided

$$a > 0 \ : \ F_{i+1/2} = F_{i+1/2}(u_i, u_{i-1}, u_{i-2}, ...)$$

$$a < 0 \ : \ F_{i+1/2} = F_{i+1/2}(..., u_{i+3}, u_{i+2}, u_{i+1})$$

<u>General Euler Case</u>:

$$M > 1 \ : \ F_{i+1/2} = F_{i+1/2}(u_i, u_{i-1}, u_{i-2}, ...)$$

$$0 < M < 1 \ : \ F_{i+1/2} = F_{i+1/2}(..., u_{i+1}, u_i, u_{i-1}, ...)$$

UPWINDING (2)

Simplest Example: Advection

$$u_{,t} + a u_{,x} = 0$$

$$a > 0 \; : \; \Delta \mathbf{u}_i + \frac{a\Delta t}{\Delta x}\left(u_i - u_{i-1}\right) = 0$$

$$a < 0 \; : \; \Delta \mathbf{u}_i + \frac{a\Delta t}{\Delta x}\left(u_{i+1} - u_i\right) = 0$$

Stable for $\frac{|a|\Delta t}{\Delta x} \leq 1$ (Homework)

UPWINDING (3)

Can be Re-Written As:

$$\Delta \mathbf{u}_i + \frac{a^+ \Delta t}{\Delta x}\left(u_i - u_{i-1}\right) + \frac{a^- \Delta t}{\Delta x}\left(u_{i+1} - u_i\right) = 0 \qquad (*)$$

Where:

$$a^+ = \frac{1}{2}\left(a + |a|\right) \quad , \quad a^- = \frac{1}{2}\left(a - |a|\right)$$

$\Rightarrow$ (*):

$$\Delta \mathbf{u}_i + \frac{\Delta t}{\Delta x}\left[\left(a^- \cdot u_{i+1} + a^+ \cdot u_i\right) - \left(a^- \cdot u_i + a^+ \cdot u_{i-1}\right)\right]$$

Or:

$$\Delta \mathbf{u}_i + \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) = 0$$

With:

$$F_{i+1/2} = a^+ u_i + a^- u_{i+1}$$

So-Called **Flux Vector Splitting Form**

UPWINDING (4)

May be Re-Written as:

$$F_{i+1/2} = \frac{a}{2}\left(u_i + u_{i+1}\right) - \frac{|a|}{2}\left(u_{i+1} - u_i\right)$$

$$F_{i+1/2} = \frac{1}{2}\left[f(u_i) + f(u_{i+1})\right] - \frac{|a|}{2}\left(u_{i+1} - u_i\right)$$

So-Called **Flux Difference Splitting Form**

Assembled:

$$\Delta \mathbf{u}_i = -\frac{\Delta t}{2\Delta x}\left[f(u_{i+1}) - f(u_{i-1})\right]$$
$$+\frac{1}{2}\frac{|a|\Delta t}{\Delta x}\left[u_{i+1} - 2u_i + u_{i-1}\right]$$

$\Rightarrow$ The Equivalent Modified Equation Is:

$$u_{,t} + au_{,x} = \frac{1}{2}|a|\Delta x u_{,xx}$$

UPWINDING (5)

<u>System of Linear Equations</u>

$$\mathbf{u}_{,t} + \mathbf{A}\mathbf{u}_{,x} = 0$$

Diagonalize:

$$\mathbf{v}_{,t} + \Lambda \mathbf{v}_{,x} = 0 \quad , \quad \mathbf{v} = \mathbf{T}^{-1}\mathbf{u} \quad , \quad \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \Lambda$$

Use Explicit Upwinding for Each Characteristic:

$$\Delta \mathbf{v} = -\frac{\Delta t}{\Delta x}\Lambda^{+}\left(\mathbf{v}_i - \mathbf{v}_{i-1}\right) - \frac{\Delta t}{\Delta x}\Lambda^{-}\left(\mathbf{v}_{i+1} - \mathbf{v}_i\right)$$

...

$$\Delta \mathbf{v} = -\frac{\Delta t}{2\Delta x}\Lambda\left[\mathbf{v}_{i+1} - \mathbf{v}_{i-1}\right] + \frac{\Delta t}{2\Delta x}|\Lambda|\left[\mathbf{v}_{i+1} - 2\mathbf{v}_i + \mathbf{v}_{i-1}\right]$$

UPWINDING (6)

Transforming Back To Original Variables:

$$\Delta \mathbf{u} = -\frac{\Delta t}{\Delta x}\mathbf{A}^{+}\left(\mathbf{u}_i - \mathbf{u}_{i-1}\right) - \frac{\Delta t}{\Delta x}\mathbf{A}^{-}\left(\mathbf{u}_{i+1} - \mathbf{u}_i\right)$$

$$\Delta \mathbf{u} = -\frac{\Delta t}{\Delta x}\left[\left(\mathbf{A}^{-}\mathbf{u}_{i+1} + \mathbf{A}^{+}\mathbf{u}_i\right) - \left(\mathbf{A}^{-}\mathbf{u}_i + \mathbf{A}^{+}\mathbf{u}_{i-1}\right)\right]$$

$$\Delta \mathbf{u} = -\frac{\Delta t}{2\Delta x}\mathbf{A}\left[\mathbf{u}_{i+1} - \mathbf{u}_{i-1}\right] + \frac{\Delta t}{2\Delta x}|\mathbf{A}|\left[\mathbf{u}_{i+1} - 2\mathbf{u}_i + \mathbf{u}_{i-1}\right]$$

Where:

$$\mathbf{A}^{\substack{+\\-}} = \mathbf{T}\Lambda^{\substack{+\\-}}\mathbf{T}^{-1} \quad , \quad |\mathbf{A}| = \mathbf{T}|\Lambda|\mathbf{T}^{-1}$$

Stable for: $max_k(\Delta t|a_k|/\Delta x) \leq 1$

UPWINDING (7)

This Shows How Much Smoothing/Diffusion is Required for Each Mode

Simple 2nd Difference Smoothing:

$$\Delta\mathbf{u} = -\frac{\Delta t}{\Delta x}\mathbf{A}\left[\mathbf{u}_{i+1} - \mathbf{u}_{i-1}\right] + \frac{\Delta t}{2\Delta x} max(|a_k|)$$

$$\left[\mathbf{u}_{i+1} - 2\mathbf{u}_i + \mathbf{u}_{i-1}\right]$$

Compared to Upwinding via Characteristic Variables:

If $|a_k|$ **Vary Greatly** $\Rightarrow$ **Excessive Damping**

Typical Case: Transonic Flows

UPWINDING (8)

Non-Linear Systems:

$$\mathbf{u}_{,t} + \mathbf{f}_{,x} = 0$$

$$\Delta \mathbf{u} = -\frac{\Delta t}{\Delta x}\left[\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}\right]$$

Flux Vector Splitting:

$$\mathbf{F}_{i+1/2} = \mathbf{F}^{+}(\mathbf{u}_i) + \mathbf{F}^{-}(\mathbf{u}_{i+1})$$

Steger and Warming: JCP 40, 263-293
vanLeer: Lecture Notes in Physics 170, 507, Springer (1982)
Liou-Steffen: JCP 107 (1993)

Flux Difference Splitting:

$$\mathbf{F}_{i+1/2} = \frac{1}{2}\left[\mathbf{F}(\mathbf{u}_i) + \mathbf{F}(\mathbf{u}_{i+1})\right] - \frac{1}{2}|\mathbf{A}|\left[\mathbf{u}_{i+1} - \mathbf{u}_i\right]$$

Roe: Ann. Rev. Fluid Mech. 86, 337; JCP 43 (1981)
Osher and Solomon: Math. & Comp. 38, 158, 339-374 (1982)

VAN LEER FVS (1)

$$\mathbf{F}_{i+1/2} = \mathbf{F}^{+}(\mathbf{u}_i) + \mathbf{F}^{-}(\mathbf{u}_{i+1})$$

$$\mathbf{F}^{\pm} = \left\{ \begin{array}{r} f^{\pm} \\ f^{\pm}\left[(\gamma-1)v \pm 2c\right]/\gamma \\ f^{\pm}\left[(\gamma-1)v \pm 2c\right]^2/2(\gamma^2-1) \end{array} \right\}$$

Mass-Flux f:

$$f^{\pm} = \pm\rho c\left[\frac{1}{2}(M \pm 1)\right]^2$$

$$c = \sqrt{\frac{\gamma p}{\rho}} \quad , \quad M = \frac{v}{c}$$

VAN LEER FVS (2)

Properties:

- $\mathbf{F}^{+}(\mathbf{u}) + \mathbf{F}^{-}(\mathbf{u}) = \mathbf{F}(\mathbf{u})$
- Eigenvalues of $\mathbf{F}^{+}_{,\mathbf{u}} > 0, \mathbf{F}^{-}_{,\mathbf{u}} < 0$
- $\mathbf{F}^{\pm}$ Are Continuous
- $\mathbf{F}^{+}(\mathbf{u}) = \mathbf{F}(\mathbf{u})$ for $M > 1$
- $\mathbf{F}^{\pm}_{,\mathbf{u}}$ Are Continuous
- $\mathbf{F}^{\pm}_{,\mathbf{u}}$ Have One Zero Eigenvalue for $|M| < 1$
- Stable Provided That

$$\frac{\Delta t}{\Delta x}\left(|\mathbf{v}| + c\right) \leq \frac{1}{\gamma + 3}\left[2\gamma + |M|(3 - \gamma)\right] \quad , \quad |M| \leq 1$$

$$\frac{\Delta t}{\Delta x}\left(|\mathbf{v}| + c\right) \leq 1 \quad , \quad |M| > 1$$

Ref: vanLeer, Anderson, Thomas - AIAA-85-0122, AIAA-85-1680