EIGENVALUE DECOMPOSITION (1)

1-D Euler:

$$\mathbf{u}_{,t} + \mathbf{F}_{,x} = 0$$

$$\mathbf{u} = \left\{ \begin{array}{r} \rho \\ \rho v \\ \rho e \end{array} \right\} \quad , \quad \mathbf{F} = \left\{ \begin{array}{r} \rho v \\ \rho v^2 + p \\ v(\rho e + p) \end{array} \right\}$$

$$p = (\gamma - 1) \left[\rho e - \frac{1}{2} \rho v^2 \right]$$

Introduce Jacobian: $\mathbf{A}$:

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}}$$

$\Rightarrow$ Can Write As:

$$\mathbf{u}_{,t} + \mathbf{A}\mathbf{u}_{,x} = 0$$

EIGENVALUE DECOMPOSITION (2)

Write **F** in terms of Conserved Variables:

$$\mathbf{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \quad , \quad \mathbf{F} = \left\{ \begin{array}{c} u_2 \\ \dfrac{u_2^2}{u_1} + p \\ \dfrac{u_2}{u_1}(u_3 + p) \end{array} \right\} ,$$

$$p = (\gamma - 1)\left[u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right]$$

The Jacobian is:

$$\mathbf{A} = \left\{ \begin{array}{ccc} 0 & 1 & 0 \\ -(3-\gamma)\dfrac{v^2}{2} & (3-\gamma)v & \gamma - 1 \\ (\gamma-1)v^3 - \gamma v e & \gamma e - 3\dfrac{(\gamma-1)}{2}v^2 & \gamma v \end{array} \right\}$$

EIGENVALUE DECOMPOSITION (3)

Idea: Decompose $\mathbf{A}$ into Eigenmodes

$$\mathbf{A} \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$$

$\Rightarrow$

$$\mathbf{A} \cdot \mathbf{V} = \mathbf{V} \cdot \Lambda \quad , \quad \Lambda = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

Then, Taking:

$$d\mathbf{u} = \mathbf{V} \cdot d\mathbf{w}$$

$$\mathbf{V} \cdot \mathbf{w}_{,t} + \mathbf{A} \cdot \mathbf{V} \cdot \mathbf{w}_{,x} = 0$$

Or:

$$\mathbf{w}_{,t} + \Lambda \cdot \mathbf{w}_{,x} = 0$$

$\Rightarrow$ **A Set of Independent Advection Systems !**

EIGENVALUE DECOMPOSITION (4)

<u>Exercise</u>: Determination of Eigenvalues/Eigenvectors

$$c^2 = \frac{\gamma p}{\rho} = \gamma(\gamma - 1)(e - \frac{v^2}{2})$$

$$\Lambda = \left\{ \begin{array}{ccc} v & 0 & 0 \\ 0 & v+c & 0 \\ 0 & 0 & v-c \end{array} \right\}$$

$$\mathbf{V} = \left\{ \begin{array}{ccc} 1 & \frac{\rho}{2c} & -\frac{\rho}{2c} \\ v & \frac{\rho}{2c}(v+c) & -\frac{\rho}{2c}(v-c) \\ \frac{v^2}{2} & \frac{\rho}{2c}(\frac{v^2}{2} + vc + \frac{c^2}{\gamma - 1} & -\frac{\rho}{2c}(\frac{v^2}{2} - vc + \frac{c^2}{\gamma - 1} \end{array} \right\}$$

$$\mathbf{V}^{-1} = \left\{ \begin{array}{ccc} 1 - \frac{\gamma - 1}{2}\frac{v^2}{c^2} & (\gamma - 1)\frac{v}{c^2} & -\frac{\gamma - 1}{c^2} \\ \frac{1}{\rho c}(\frac{\gamma - 1}{2}v^2 - uc) & \frac{1}{\rho c}(c - (\gamma - 1)v) & \frac{1}{\rho c}(\gamma - 1) \\ -\frac{1}{\rho c}(\frac{\gamma - 1}{2}v^2 + uc) & \frac{1}{\rho c}(c + (\gamma - 1)v) & -\frac{1}{\rho c}(\gamma - 1) \end{array} \right\}$$

EIGENVALUE DECOMPOSITION (5)

$\Rightarrow$

$$\lambda_1 = v \quad : dw_1 = d\rho - \frac{1}{c^2} dp$$

$$\lambda_2 = v + c : dw_2 = dv + \frac{1}{\rho c} dp$$

$$\lambda_3 = v - c : dw_3 = dv - \frac{1}{\rho c} dp$$

BOUNDARY CONDITIONS: FAR FIELD (1)

Idea:

- Define Predicted State by '$*$', e.g. ρ_*
- Define **Inward** Normal $\mathbf{n}$
- Perform Linearized Characteristics Analysis to Correct Predicted ($*$) Values

2-D Euler Linearized Characteristics:

$$\lambda = \begin{pmatrix} v_n \\ v_n \\ v_n + c \\ v_n - c \end{pmatrix} \quad , \quad \mathbf{W} = \begin{pmatrix} \rho - p/\overline{c}^2 \\ v_t \\ [\;\; v_n + p/(\overline{\rho c})]/\sqrt{2} \\ [-v_n + p/(\overline{\rho c})]/\sqrt{2} \end{pmatrix}$$

BOUNDARY CONDITIONS: FAR FIELD (2)

a) Supersonic Inflow:

- No Info Can Go Upstream

$\Rightarrow$

- All Quantities: From "∞"

b) Subsonic Inflow:

- Incoming Characteristics: λ_{1-3}
- Outgoing Characteristic : λ_4

$\Rightarrow$

- W_1, W_2, W_3: From "∞"
- W_4: From Predicted State

$$\rho - p/\overline{c}^2 = \rho_\infty - p_\infty/\overline{c}^2$$

$$v_t = v_{t\infty}$$

$$v_n + p/(\overline{\rho c}) = \quad v_{n\infty} + p_\infty/(\overline{\rho c})$$

$$-v_n + p/(\overline{\rho c}) = -v_{n*} + p_*/(\overline{\rho c})$$

$\Rightarrow$

$$v_t = v_{t\infty} \quad ; \quad p = 0.5\,[\; p_\infty + p_* + \overline{\rho c}(v_{n\infty} - v_{n*})]$$

$$\rightarrow \rho = \rho_\infty + (p - p_\infty)/\overline{c}^2 \quad ; \quad v_n = v_{n\infty} + (p_\infty - p)/(\overline{\rho c})$$

BOUNDARY CONDITIONS: FAR FIELD (3)

<u>If</u>: **Total Pressure** Given (Engines):

- Given: $\mathbf{v}$
- Assume: Constant Total Pressure, Entropy Along Streamline

Bernoulli $\Rightarrow$

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} + \frac{\gamma - 1}{2\gamma}\left[v_0^2 - v^2\right]$$

+ Isentropic Relation:

$$\rho = \left[\frac{\rho_0^\gamma}{p_0}\frac{p}{\rho}\right]^{\frac{1}{\gamma-1}}$$

$\Rightarrow$

$$p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma$$

<u>Set</u>: $v_\infty^2 = v_*^2 \Rightarrow \rightarrow \rho_\infty \quad , \quad \rightarrow p_\infty$
$\Rightarrow$

$$v_t = v_{t\infty} \quad ; \quad v_n = v_{n*}$$

$$\rightarrow p = 0.5\,(p_\infty + p_*) \quad ; \quad \rightarrow \rho = \rho_\infty + (p - p_\infty)/\overline{c}^2$$

BOUNDARY CONDITIONS: FAR FIELD (4)

c) <u>Subsonic Outflow</u>:

- Incoming Characteristics: λ_3
- Outgoing Characteristic : $\lambda_{1,2,4}$

$\Rightarrow$

- W_3: From "∞"
- $W_{1,2,4}$: From Predicted State

$\Rightarrow$

$$\rho - p/\overline{c}^2 = \rho_* - p_*/\overline{c}^2$$

$$v_t = v_{t*}$$

$$v_n + p/(\overline{\rho c}) = \quad v_{n\infty} + p_\infty/(\overline{\rho c})$$

$$-v_n + p/(\overline{\rho c}) = -v_{n*} + p_*/(\overline{\rho c})$$

BOUNDARY CONDITIONS: FAR FIELD (5)

<u>Possibilities</u>:

c1) Prescribed Pressure p_∞:

$$v_t = v_{t*} \quad ; \quad p = p_\infty$$

$$\rightarrow \rho = \rho_* + (p_\infty - p_*)/\overline{c}^2 \quad ; \quad v_n = v_{n*} + (p_\infty - p_*)/(\overline{\rho c})$$

c2) Prescribed Mach-Number m_∞:

$$v_t = v_{t*} \quad ; \quad v_n = -c_* m_\infty$$

$$\rightarrow p = p_* + \rho_* c_* (v_n - v_{n*}) \quad ; \quad \rightarrow \rho = \rho_* + (p - p_*)/\overline{c}^2$$

c3) Prescribed Mass-Flux $(\rho v_n)_\infty$:

$$v_t = v_{t*} \quad ; \quad p = p_* + c_*((\rho v_n)_\infty - \rho_* v_{n*})$$

$$\rightarrow \rho = \rho_* + (p - p_*)/\overline{c}^2 \quad ; \quad \rightarrow v_n = (\rho v_n)_\infty / \rho$$

d) <u>Supersonic Outflow</u>:

- Incoming Characteristics: None
- Outgoing Characteristic : λ_{1-4}

$\Rightarrow$

- W_{1-4}: From Predicted State

BOUNDARY CONDITIONS: WALLS (1)

<u>Flux Through A-B</u>:

$$\int_{\Gamma_{AB}} (\mathbf{F}n_x + \mathbf{G}n_y)ds = \int_{\Gamma_{AB}} \mathbf{F}dy - \mathbf{G}dx$$

a) <u>Node-Centered Schemes</u>:

$$(\mathbf{F}\Delta y - \mathbf{G}\Delta x)_{AB} = \frac{1}{2}\,(p_A + p_B)\begin{pmatrix} 0 \\ \Delta y \\ -\Delta x \\ 0 \end{pmatrix}$$

BOUNDARY CONDITIONS: WALLS (2)

b) <u>Cell-Centered Schemes</u>:

b1) Reflection/Dummy Cell

$$\rho_0 = \rho_1 \quad , \quad p_0 = p_1$$

$$(\mathbf{v} \cdot \mathbf{n})_0 = -(\mathbf{v} \cdot \mathbf{n})_1 \quad , \quad (\mathbf{v} \cdot \mathbf{t})_0 = (\mathbf{v} \cdot \mathbf{t})_1$$

- Exact for 'Symmetry Boundary'
- $p_w = p_1 + (\rho v_n^2)_1$

b2) Simple Extrapolation

$$p_w = p_1$$

- First Order (Error $O(\Delta n)$)

b3) Second Order Extrapolation

$$p_w = p_1 - \frac{\partial p}{\partial n}\Delta n \quad , \quad \frac{\partial p}{\partial n} = \frac{\rho \mathbf{v}^2}{R}$$

- Second Order
- Need to Take Into Account Sign of R !