DISCRETIZATION IN TIME

Given:

$$\mathbf{u}_{,t} = \mathbf{r}(\mathbf{u})$$

1. Explicit: Runge-Kutta (m-stage)

$$\Delta\mathbf{u}^{n+i} = \alpha_i \; \Delta t \; \mathbf{r}(\mathbf{u}^n + \Delta\mathbf{u}^{n+i-1}) \quad , \quad i = 1, m \quad , \quad \Delta\mathbf{u}^0 = 0$$

α_i: chosen according to desired properties (damping, accuracy)

Examples:

2-Stage: $\alpha_1 = 0.5 \;\; , \;\; \alpha_2 = 1$

3-Stage (Transient): $\alpha_1 = 1/3 \;\; , \;\; \alpha_2 = 1/2 \;\; , \;\; \alpha_3 = 1$

3-Stage (Steady-State): $\alpha_1 = 0.6 \;\; , \;\; \alpha_2 = 0.6 \;\; , \;\; \alpha_3 = 1$

Properties of Explicit Schemes:

- Easy to Code
- Easy to Apply Boundary Cond.
- Easy to Vectorize/Parallelize
- Easy to Maintain/Upgrade
- Δt Limited by Stability Constraint

2. Implicit:

$$\Delta \mathbf{u} = \mathbf{u}^{n+1} - \mathbf{u}^{n} = \Delta t \; \mathbf{r}(\mathbf{u}^{n+\Theta})$$

Approximation

$$\mathbf{r}^{n+\Theta} = \mathbf{r}^{n} + \left.\frac{\partial \mathbf{r}}{\partial \mathbf{u}}\right|^{n} \cdot \Theta \Delta \mathbf{u} = \mathbf{r}^{n} + \mathcal{A}^{n} \cdot \Theta \Delta \mathbf{u}$$

$\Rightarrow$

$$[1 - \Delta t \; \Theta \mathcal{A}^{n}] \cdot \Delta \mathbf{u} = \Delta t \; \mathbf{r}^{n}$$

Examples:

$\Theta = 1.0$: Backward Euler (1st order accurate)

$\Theta = 0.5$: Crank-Nicholson (2nd order accurate)

Properties of Implicit Schemes:

- ‘Arbitrary’ Δt
- Grid Independent Δt
- Large Overhead in Solving System of Equations
- Maximum Accuracy: 2nd Order

SITUATIONS WHERE IMPLICIT SCHEMES PAY OFF

- Physical Stiffness

$$\Delta t|_{phys.relevant} \gg \Delta t_{CFL}$$

- Mesh Stiffness
    - Small Elements
    - Distorted Elements

WORD OF CAUTION

If Δt Too Large, and System Nonlinear $\Rightarrow$

- Chaotic Solutions Possible (see Yee, Sweby...)
  (Steady $\rightarrow$ Unsteady)
- False Steady Solutions Possible
  (Unsteady $\rightarrow$ Steady)

$\Rightarrow$ Conduct Convergence Study !

Possible Way Out: Adaptivity/Sensing in Time