Diplomarbeit

Pathwise Differentiability for Stochastic Differential Equations with Reflection

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1 Introduction

The main purpose of this diploma thesis is to show that the solutions of a system of stochastic differential equations (SDE) with reflection term, also called SKOROHOD SDEs, are pathwise differentiable with respect to the initial value. For a finite set of indices I, we consider the following system of such autonomous SKOROHOD SDEs:

$$
\begin{aligned}
&X_t^i(x) = x^i + \int_0^t b^i(X_r(x))\,dr + l_t^i(x) + \int_0^t \sigma^i(X_r^i(x))\,dw_r^i, \quad t \geq 0,\, i \in I,\\
&X_t^i(x) \geq 0, \quad dl_t^i(x) \geq 0, \quad \int_0^\infty X_t^i(x)\,dl_t^i(x) = 0, \qquad i \in I,
\end{aligned}
\tag{1.1}
$$

for all $x \in \mathbb{R}_+^I$, where the coefficient functions $b^i : \mathbb{R}_+^I \to \mathbb{R}$ and $\sigma^i : \mathbb{R}_+ \to \mathbb{R}$, $i \in I$, are continuously differentiable and LIPSCHITZ continuous, and $(w^i)_{i\in I}$ is a family of independent BROWNian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
For any $i \in I$, the local time $l^i(x)$ is nondecreasing and, since $X_t^i(x) \geq 0$, the condition $\int_0^\infty X_t^i(x)\,dl_t^i(x) = 0$ means that $l^i(x)$ only increases when $X_t^i(x) = 0$, i.e. $l^i(x)$ is there to "push $X^i(x)$ upward", so that it remains nonnegative, and this pushing is minimal in the sense that there is no pushing when $X_t^i(x) > 0$.
Since the coefficient functions are assumed to be LIPSCHITZ continuous and $\mathbb{R}_+^I$ is a convex region, the main result in [15] guarantees the existence and uniqueness of the solution of (1.1).
It is aimed to prove, that the first partial derivatives of $X_t(x)$, $t \geq 0$, w.r.t. the initial value x exist a.s., and to find a probabilistic representation for them. Notice that the differentiability is not trivial because of the presence of the non-smooth reflection term $l^i(x)$.
This problem was discussed for the case $\sigma^i \equiv 1$, $i \in I$, by DEUSCHEL and ZAMBOTTI in [3]. There it is shown that the derivatives evolve according to an ordinary differential equation, when the process is away from the boundary, and that they have a discontinuity and become zero, when the process is at the boundary. This evolution is quite non-trivial, caused by the rather complicated structure of the set at times where the process hits the boundary, which is known to be a set with zero LEBESGUE measure and without isolated points. Nevertheless, the derivatives admit a simple representation in terms of an auxiliary random walk ξ, taking values in the set I of indices.
In this thesis we try to generalize this result to the system in (1.1) under some stronger assumptions on the coefficient functions. We shall apply LAMPERTI's method to transform the system in (1.1) into a system with constant diffusion coefficients and then proceed as in [3].
As a further remarkable addition to the results of [3], we investigate the pathwise differentiability of a BROWNian motion in a wedge with oblique reflection, established by VARADHAN and WILLIAMS in [16]. We shall show, that this process, which can be represented by a two-dimensional system of SKOROHOD SDEs, is pathwise differentiable w.r.t. the initial value up to the time when the process reaches the corner of the wedge. The obtained derivatives are constant on every time interval, when the process is in the interior of the wedge, and they have a discontinuity, when the process hits one side of the wedge and the last hit of the boundary was at the other side. Moreover, the derivatives depend on the number of crossings through the wedge from one side to the other one.
The thesis is organized as follows: In Section 2 we recall at first a differentiability result for SDEs without reflection. Afterwards we summarize the results of [3] and provide some further preparations. In Section 3 we prove that the solutions of (1.1) depend continuously on the initial value x, even when the diffusion coefficients σ^i, $i \in I$, depend on all components of X. Then, in Section 4 we investigate the derivatives of X.

Section 6 contains an alternative approach to the proof of the random walk representation in [3]. So we consider the case $\sigma^i \equiv 1$, $i \in I$, and moreover we assume that the drift coefficients have nonnegative derivatives. We shall use the penalization method to approximate the solution of the SDE with reflection and then the FEYNMAN-KAC formula to obtain a random walk representation of the derivatives for the penalized SDE. Since the auxiliary random walk ξ is a MARKOV process with finite state space I and time-dependent infinitesimal generator, we need to develop FEYNMAN-KAC representations for such MARKOV processes. This is done in Section 5.
In Section 7 we compute the derivatives of the transition semigroup of X:

$$P_t f(x) := \mathbb{E}[f(X_t(x))], \qquad t \geq 0, x \in \mathbb{R}_+^I,$$

for all $f : \mathbb{R}_+^I \to \mathbb{R}$ bounded and continuously differentiable, while we use GIRSANOV's Theorem to decouple the the components of X.
Finally, Section 8 deals with the pathwise differentiability of a BROWNian motion in a wedge.

2 Preliminaries

We summarize some known facts about the pathwise differentiability of solutions for SDEs w.r.t. the initial value. First we consider the "classical" type of SDEs without reflection. Afterwards we state the main results of [3], which deals with SDEs with reflection term. Some basic notation and definitions are given, too. Moreover, we provide some general preparations.

2.1 Pathwise Differentiability for SDEs without Reflection

Let I be a finite set of indices and $(w^i)_{i\in I}$ a family of independent BROWNian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. It is assumed that the filtration $(\mathcal{F}_t)$ satisfies the usual conditions, i.e. it is complete and right-continuous, and that w is adapted to $(\mathcal{F}_t)$. We consider the system

$$X_t^i(x) = x^i + \int_0^t b^i(X_r(x))\, dr + \sum_{k\in I} \int_0^t \sigma^{ik}(X_r(x))\, dw_r^k, \qquad t \geq 0,\ i \in I, \tag{2.1}$$

for all $x \in \mathbb{R}^I$, where the coefficients b^i, $\sigma^{ik} : \mathbb{R}^I \to \mathbb{R}$, $i, k \in I$, are supposed to be continuously differentiable and LIPSCHITZ continuous, such that the pathwise existence and uniqueness of the solution of the system (2.1) is ensured by the PICARD-LINDELÖF Theorem.

Theorem 2.1. *For all $t \geq 0$ the mapping $x \mapsto X_t(x)$ is a.s. continuously differentiable and the partial derivatives $\phi_t^{ij} := \frac{\partial X_t^i(x)}{\partial x^j}$, $i, j \in I$, are the unique pathwise strong solutions of the SDE system*

$$\phi_t^{ij} = \delta_{ij} + \sum_{l\in I} \int_0^t \frac{\partial b^i}{\partial x^l}(X_r(x))\, \phi_r^{lj}\, dr + \sum_{k,l\in I} \int_0^t \frac{\partial \sigma^{ik}}{\partial x^l}(X_r(x))\, \phi_r^{lj}\, dw_r^k.$$

Proof. See Theorem 4.6.5, pp. 173-174 in [8]. □

2.2 Pathwise Differentiability for SDEs with Reflection and Constant Diffusion Coefficients

This subsection contains the main results of [3] and some basic definitions and notation.
Let I and w be as above in Section 2.1. We consider the following system of SKOROHOD SDEs:

$$\begin{aligned} &X_t^i(x) = x^i + \int_0^t b^i(X_r(x))\,dr + l_t^i(x) + w_t^i, \quad t \geq 0,\, i \in I, \\ &X_t^i(x) \geq 0, \quad dl_t^i(x) \geq 0, \quad \int_0^\infty X_t^i(x)\,dl_t^i(x) = 0, \qquad i \in I, \end{aligned} \tag{2.2}$$

for all $x \in \mathbb{R}_+^I := [0,\infty)^I$, where $b^i : \mathbb{R}_+^I \to \mathbb{R}$ is continuously differentiable and LIPSCHITZ continuous. Then, the result in [15] guarantees the existence and uniqueness of a solution of (2.2).
Before we can state the main result of [3], which deals with the derivatives of the solution of the system (2.2), we need to introduce some necessary notation:

$$C^i := \{s \geq 0 : X_s^i(x) = 0\} \text{ and } r^i(t) := \sup\{s \leq t : X_s^i(x) = 0\}, \qquad i \in I,$$

with the convention $\sup \emptyset := 0$.
Let $E := D([0,\infty), I)$ be the space of I-valued càdlàg functions, where I is endowed with the discrete topology, and $\xi_t : E \to I$, $t \in [0,\infty)$, the coordinate process. For any continuous function $c : [0,\infty) \times I \times I \to \mathbb{R}$ and for all $s \in [0,\infty)$ and $i \in I$, $P_{s,i}^c$ denotes the probability measure on E, under which

- $\xi_t = i$ for all $t \in [0,s]$,
- $(\xi_t)_{t\in[s,\infty)}$ has the law of a time continuous MARKOV chain with values in I, starting at $t = s$ from i and with time-dependent generator $(L_t^c)_{t\geq 0}$:

$$L_t^c : \mathbb{R}^I \to \mathbb{R}^I, \qquad L_t^c f(i) := \sum_{k\in I} |c_t(i,k)|\,(f(k) - f(i))\,.$$

MARKOV processes with finite state space and time-dependent generator are investigated in detail later in Section 5.
We denote by $(\eta_l)_l$ the sequence of jump moments of $\xi \in E$:

$$\eta_l : E \to [0,\infty), \qquad \eta_0 := 0, \qquad \eta_{l+1} := \inf\{s > \eta_l : \xi_s \neq \xi_{\eta_l}\}.$$

For all $s,t \in [0,\infty)$ such that $s \leq t$, we define the real bounded measurable function $\rho_{s,t}^c$ on E by

$$\rho_{s,t}^c := \exp\left(\int_s^t \sum_{k\neq\xi_r} |c_r(\xi_r,k)|\,dr + \int_s^t c_r(\xi_r,\xi_r)\,dr\right) \prod_{s<\eta_k\leq t} \operatorname{sign}(c_{\eta_k}(\xi_{\eta_{k-1}},\xi_{\eta_k})),$$

where the sign function is defined by

$$\operatorname{sign}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Finally we define the stopping time τ on $\Omega \times E$ by

$$\tau := \inf\{s > 0 : \, X_s^{\xi_s}(x) = 0\} = \inf_{k \in I} \inf\{s > 0 : \, X_s^k(x) = 0, \, \xi_s = k\}$$

with the convention $\inf \emptyset := +\infty$. In the following we shall often work with the derivatives of functions defined on $\mathbb{R}_+^I = [0, \infty)^I$. Therefore we give the precise definition: A function $f : \mathbb{R}_+^I \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}_+^I$, if there exists a vector $(\partial_i f(x), \, i \in I) \in \mathbb{R}^I$ such that

$$f(x+h) = f(x) + \sum_{i \in I} \partial_i f(x) \, h^i + o(\|h\|), \qquad \forall h : \, x + h \in \mathbb{R}_+^I.$$

Note that when x lies in the boundary of $\mathbb{R}_+^I$, the requirement $x+h \in \mathbb{R}_+^I$ becomes essential. The differentiability for functions on a wedge, which occur in Section 8, can be defined analogously. The main result of [3] is the following

Theorem 2.2. *Let*

$$c_t(i,j) := \frac{\partial b^j}{\partial x^i}(X_t(x)), \qquad t \geq 0, \, i, j \in I.$$

Then, for all $t \geq 0$ and all $x \in \mathbb{R}_+^I$, a.s. the map $x \mapsto X_t(x)$ is continuously differentiable in $\mathbb{R}_+^I$ and the partial derivatives $\eta_t^{ij} := \frac{\partial X_t^i(x)}{\partial x^j}$, $i, j \in I$, a.s. admit the random walk representation

$$\eta_t^{ij} = E_{0,j}^c \left[1\!\!1_{\{\xi_t = i\}} 1\!\!1_{\{\tau > t\}} \rho_{0,t}^c \right]. \tag{2.3}$$

Moreover, the right hand side of (2.3) *defines a right-continuous modification of η such that we have a.s. for all $t \geq 0$:*

$$\eta_t^{ij} = \delta_{ij} + \sum_{k \in I} \int_0^t \frac{\partial b^i}{\partial x^k}(X_r(x)) \, \eta_r^{kj} \, dr, \qquad t \in [0, \inf C^i),$$

$$\eta_t^{ij} = \sum_{k \in I} \int_{r^i(t)}^t \frac{\partial b^i}{\partial x^k}(X_r(x)) \, \eta_r^{kj} \, dr, \qquad t \in [\inf C^i, \infty).$$

Proof. See Theorem 1 in [3]. □

2.3 Skorohod's Lemma about Local Times

In the sequel we shall often use the following lemma to compute local times.

Lemma 2.3 (Skorohod)**.** *Let y be a real-valued continuous function on $[0, \infty)$ such that $y(0) \geq 0$. There exists a unique pair (z, a) of functions on $[0, \infty)$ such that*

i) $z = y + a$,

ii) z *is positive,*

iii) a *is increasing, continuous, vanishing at zero and the corresponding measure da_s is carried by $\{s : z(s) = 0\}$.*

The function a is moreover given by

$$a(t) = \sup_{s \leq t}(-y(s) \vee 0).$$

Proof. See Lemma VI.2.1 in [12]. □

2.4 Minima of perturbed Brownian Motion

For later use we recall the technical lemma the proof of Theorem 2.2 in [3] is based on:

Lemma 2.4. *Let $(w_t)_{t\geq 0}$ be a* BROWN*ian motion on $(\Omega, \mathbb{P})$. For all $T > 0$, let $\theta : \Omega \to [0, T]$ be the random variable such that a.s.*

$$w_\theta < w_s, \qquad \forall s \in [0,T] \backslash \{\theta\}.$$

There exists a random variable $\gamma > 0$, such that for any continuous process $f : [0,T] \to \mathbb{R}$ with

$$f(0) = 0, \qquad |f(t) - f(s)| \leq \gamma\, |t - s|, \qquad t, s \in [0,T],$$

a.s. θ is the only time when $w + f$ attains its minimum over $[0,T]$:

$$w_\theta + f_\theta < w_s + f_s, \qquad \forall s \in [0,T] \backslash \{\theta\}.$$

Proof. See Lemma 1 in [3]. □

2.5 The Burkholder Inequality

Theorem 2.5 (BURKHOLDER inequality)**.** *For every $p > 1$, there exists a constant C_p such that for any continuous local martingale M vanishing at zero,*

$$\mathbb{E}\left[\sup_{t\geq 0} |M_t|^p\right] \leq C_p\, \mathbb{E}\left[\langle M\rangle_\infty^{p/2}\right].$$

Moreover, the constant C_p is given by

$$C_p = \left(6\sqrt{2e}\, q^{3/2}/\sqrt{q-1}\right)^p, \quad \text{where } q := p/(p-1).$$

If we consider the stopped martingale M^T for some stopping time T, it follows immediately that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|^p\right] \leq C_p\, \mathbb{E}\left[\langle M\rangle_T^{p/2}\right].$$

Proof. See, for instance, Theorem IV.4.1 in [12]. The value of the constant C_p has been proved in Theorem 6.3.6 in [4] for discrete time martingales. One can easily generalize this to the continuous time case: For each $T \in (0, \infty)$ and $N \in \mathbb{N}$ apply the discrete version to the discrete martingale $M(\frac{nT}{2^N})$, $0 \leq n \leq 2^N$, observe that

$$\sup\left\{|M(\tfrac{nT}{2^N})|^p : 0 \leq n \leq 2^N\right\} \nearrow \sup_{t\in[0,T]} |M_t|^p \qquad \text{as } N \nearrow \infty$$

and

$$\left(\sum_{n=1}^{2^N} \left(M(\tfrac{nT}{2^N}) - M(\tfrac{(n-1)T}{2^N})\right)^2\right)^{p/2} \nearrow \langle M\rangle_T^{p/2} \qquad \text{as } N \nearrow \infty.$$

Finally apply the monotone convergence theorem.

□

3 Continuous Dependence on the Initial Value

In this section we shall prove that the solutions of SKOROHOD SDEs with diagonal noise depend continuously on the initial value. This result will be useful in the next section, where the differentiability of such processes w.r.t. the initial value is investigated under some stronger assumptions. The basic idea of the proof is the same as in Step 1 in the proof of Theorem 1 in [3] combined with an application of the KOLMOGOROV-CHENTSOV Theorem.

Theorem 3.1 (KOLMOGOROV-CHENTSOV). *If $(X_t)_{t\in\mathbb{R}^n}, n \in \mathbb{N}$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a complete separable metric space (S, ρ), and if there exist positive constants α, C, ε such that for all $s, t \in \mathbb{R}^n$*

$$\mathbb{E}\left[\rho(X_s, X_t)^\alpha\right] \le C\, \|s - t\|^{n+\varepsilon},$$

then, there exists a continuous modification of X, which is HÖLDER *continuous of order θ for each $\theta < \varepsilon/\alpha$.*

Proof. See, for instance, Theorem I.25.2 in [13]. □

Clearly, the theorem is also applicable, if the parameter set of the process is restricted to $[0, \infty)^n$. We consider for a finite set of indices I the following system of stochastic differential equations of the SKOROHOD type:

$$\begin{aligned} &X_t^i(x) = x^i + \int_0^t b^i(X_r(x))\, dr + l_t^i(x) + \int_0^t \sigma^i(X_r(x))\, dw_r^i, \quad t \ge 0,\, i \in I, \\ &X_t^i(x) \ge 0, \quad dl_t^i(x) \ge 0, \quad \int_0^\infty X_t^i(x)\, dl_t^i(x) = 0, \qquad i \in I, \end{aligned} \tag{3.1}$$

for all $x \in \mathbb{R}_+^I$, where $(w^i)_{i\in I}$ are independent BROWNian motions on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ as before. Furthermore, for every $i \in I$ the functions b^i and σ^i: $\mathbb{R}_+^I \to \mathbb{R}$ are assumed to be LIPSCHITZ continuous with constants K_{b^i} and K_{σ^i}. Then, the pathwise existence and uniqueness of solutions of equation (3.1) has been proved in [15] (note that $\mathbb{R}_+^I$ is convex). Obviously, the solutions have continuous sample paths.

Theorem 3.2. *For arbitrary but fixed $T > 0$, let $(X_t(x))$ and $(X_t(y))$, $0 \le t \le T$, be solutions of* (3.1) *for some $x, y \in \mathbb{R}_+^I$. Then,*

$$\mathbb{E}\left[\sup_{0\le t\le T} \|X_t(x) - X_t(y)\|^p\right] \le C\, \|x - y\|^p, \tag{3.2}$$

for any fixed $p \ge 4$, where C is a constant that depends only on T and p. Moreover, there exists a continuous modification of the mapping $x \mapsto (X_t(x))_{t\in[0,T]}$ (continuous with respect to the sup-norm topology).

Proof. Fix $T > 0$. Since $X_t^i\, dl_t^i = 0$ for all $t \ge 0$ and $i \in I$, we have

$$\left(X_t^i(x) - X_t^i(y)\right)\left(dl_t^i(x) - dl_t^i(y)\right) = -X_t^i(x)\, dl_t^i(y) - X_t^i(y)\, dl_t^i(x) \le 0. \tag{3.3}$$

Using the CAUCHY-SCHWARZ inequality for the Euclidian norm in $\mathbb{R}^I$ we obtain

$$
\begin{aligned}
&\sum_{i\in I}\left(X_t^i(x)-X_t^i(y)\right)\left(b^i(X_t(x))-b^i(X_t(y))\right)\\
\leq&\|X_t(x)-X_t(y)\|\left(\sum_{i\in I}\left(b^i(X_t(x))-b^i(X_t(y))\right)^2\right)^{1/2}\\
\leq&\|X_t(x)-X_t(y)\|\left(\sum_{i\in I}K_{b^i}^2\,\|X_t(x)-X_t(y)\|^2\right)^{1/2}\\
=&K_b\,\|X_t(x)-X_t(y)\|^2,
\end{aligned}
\tag{3.4}
$$

where $K_b := \left(\sum_{i\in I} K_{b^i}^2\right)^{1/2}$ denotes the LIPSCHITZ norm of b.
Since for all $i \in I$ and $x \in \mathbb{R}_+^I$ the mapping $t \mapsto l_t^i(x)$ is non-decreasing and thus of bounded variation, we have

$$
d\langle X^i(x)-X^i(y)\rangle_t = \left(\sigma^i(X_t(x))-\sigma^i(X_t(y))\right)^2\,dt. \tag{3.5}
$$

Using ITÔ's integration by parts formula, (3.5), (3.3), (3.4) and the LIPSCHITZ continuity of the functions σ^i, $i \in I$, we obtain

$$
\begin{aligned}
d\|X_t(x)-X_t(y)\|^2 &= \sum_{i\in I} d\left(X_t^i(x)-X_t^i(y)\right)^2\\
&=\sum_{i\in I}\left[2\left(X_t^i(x)-X_t^i(y)\right)\,d\left(X_t^i(x)-X_t^i(y)\right)+d\langle X^i(x)-X^i(y)\rangle_t\right]\\
&=2\sum_{i\in I}\left(X_t^i(x)-X_t^i(y)\right)\left(b^i(X_t(x))-b^i(X_t(y))\right)\,dt\\
&\quad+2\sum_{i\in I}\left(X_t^i(x)-X_t^i(y)\right)\left(\sigma^i(X_t(x))-\sigma^i(X_t(y))\right)\,dw_t^i\\
&\quad+2\sum_{i\in I}\left(X_t^i(x)-X_t^i(y)\right)\left(dl_t^i(x)-dl_t^i(y)\right)\\
&\quad+\sum_{i\in I}\left(\sigma^i(X_t(x))-\sigma^i(X_t(y))\right)^2\,dt\\
&\leq (2K_b+K_\sigma^2)\,\|X_t(x)-X_t(y)\|^2\,dt\\
&\quad+2\sum_{i\in I}\left(X_t^i(x)-X_t^i(y)\right)\left(\sigma^i(X_t(x))-\sigma^i(X_t(y))\right)\,dw_t^i,
\end{aligned}
$$

where $K_\sigma := \left(\sum_{i\in I} K_{\sigma^i}^2\right)^{1/2}$. We write this as

$$
\begin{aligned}
\|X_t(x)-X_t(y)\|^2 \leq \|x-y\|^2 &+ (2K_b+K_\sigma^2)\int_0^t \|X_r(x)-X_r(y)\|^2\,dr\\
&+2\sum_{i\in I}\int_0^t \left(X_r^i(x)-X_r^i(y)\right)\left(\sigma^i(X_r(x))-\sigma^i(X_r(y))\right)\,dw_r^i.
\end{aligned}
\tag{3.6}
$$

In the following C_i, $i = 1, \ldots, 8$, denote positive constants only depending on T and p.

We fix any $p \geq 2$ and apply the HÖLDER inequality twice to the right hand side of (3.6) to obtain for $t \in [0,T]$:

$$\begin{aligned}
\|X_t(x) - X_t(y)\|^{2p} \leq\ & C_1 \|x-y\|^{2p} + C_2 \left(\int_0^t \|X_r(x) - X_r(y)\|^2 \, dr \right)^p \\
& + C_3 \left(\sum_{i \in I} \int_0^t \left(X_r^i(x) - X_r^i(y) \right) \left(\sigma^i(X_r(x)) - \sigma^i(X_r(y)) \right) dw_r^i \right)^p \\
\leq\ & C_1 \|x-y\|^{2p} + C_4 \int_0^t \|X_r(x) - X_r(y)\|^{2p} \, dr \\
& + C_3 \left(\sum_{i \in I} \int_0^t \left(X_r^i(x) - X_r^i(y) \right) \left(\sigma^i(X_r(x)) - \sigma^i(X_r(y)) \right) dw_r^i \right)^p . \qquad (3.7)
\end{aligned}$$

Note that the last term of the right hand side of (3.6) is a continuous local martingale vanishing at zero. Its quadratic variation can be estimated as follows:

$$\begin{aligned}
& \left\langle 2 \sum_{i \in I} \int_0^\cdot \left(X_r^i(x) - X_r^i(y) \right) \left(\sigma^i(X_r(x)) - \sigma^i(X_r(y)) \right) dw_r^i \right\rangle_t \\
=& 4 \sum_{i,j \in I} \int_0^t \left(X_r^i(x) - X_r^i(y) \right) \left(\sigma^i(X_r(x)) - \sigma^i(X_r(y)) \right) \\
& \qquad \times \left(X_r^j(x) - X_r^j(y) \right) \left(\sigma^j(X_r(x)) - \sigma^j(X_r(y)) \right) d\langle w^i, w^j \rangle_r \qquad (3.8) \\
=& 4 \sum_{i \in I} \int_0^t \left(X_r^i(x) - X_r^i(y) \right)^2 \left(\sigma^i(X_r(x)) - \sigma^i(X_r(y)) \right)^2 dr \\
\leq& C_5 \int_0^t \|X_r(x) - X_r(y)\|^4 \, dr,
\end{aligned}$$

where we used again the LIPSCHITZ continuity of the coefficients σ^i, $i \in I$. From (3.7), the BURKHOLDER inequality (see Theorem 2.5) and (3.8) we obtain

$$\begin{aligned}
& \mathbb{E} \left[\sup_{0 \leq t \leq T} \|X_t(x) - X_t(y)\|^{2p} \right] \\
\leq & C_1 \|x-y\|^{2p} + C_4 \, \mathbb{E} \left[\sup_{0 \leq t \leq T} \int_0^t \|X_r(x) - X_r(y)\|^{2p} \, dr \right] \\
& + C_3 \, \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\sum_{i \in I} \int_0^t \left(X_r^i(x) - X_r^i(y) \right) \left(\sigma^i(X_r(x)) - \sigma^i(X_r(y)) \right) dw_r^i \right)^p \right] \\
\leq & C_1 \|x-y\|^{2p} + C_4 \, \mathbb{E} \left[\int_0^T \|X_r(x) - X_r(y)\|^{2p} \, dr \right] + C_6 \, \mathbb{E} \left[\left(\int_0^T \|X_r(x) - X_r(y)\|^4 \, dr \right)^{p/2} \right].
\end{aligned}$$

Using the HÖLDER inequality and FUBINI's Theorem, this involves

$$\begin{aligned}
\mathbb{E} \left[\sup_{0 \leq t \leq T} \|X_t(x) - X_t(y)\|^{2p} \right] & \leq C_1 \|x-y\|^{2p} + C_7 \, \mathbb{E} \left[\int_0^T \|X_r(x) - X_r(y)\|^{2p} \, dr \right] \\
& \leq C_1 \|x-y\|^{2p} + C_7 \int_0^T \mathbb{E} \left[\sup_{0 \leq s \leq r} \|X_s(x) - X_s(y)\|^{2p} \right] dr. \qquad (3.9)
\end{aligned}$$

Now we can apply GRONWALL's Lemma to (3.9) to obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\|X_t(x)-X_t(y)\|^{2p}\right]\le C_8\,\|x-y\|^{2p},\qquad p\ge 2,$$

and this is equivalent to the desired estimate (3.2). By the KOLMOGOROV-CHENTSOV Theorem (see Theorem 3.1), choosing the Polish space $C([0,T])$ equipped with the supremum-norm and $p>\max(|I|,4)$, we conclude that $x\mapsto (X_t(x))_{t\in[0,T]}$ has a continuous modification. □

Note that the result of Theorem 3.2 is still valid, if we replace the term $l_t^i(x)$ in SDE (3.1) by $c\,l_t^i(x)$ for any constant $c>0$.

4 The Derivatives for Skorohod SDEs with Diagonal Noise

In this section we study the differentiability of solutions of SKOROHOD SDEs with diagonal noise. Compared to the last section we need some stronger assumptions on the coefficients b and σ. For instance their components are required to be differentiable, which is also necessary in the case without reflection (cf. Theorem 2.1); moreover, for technical reasons the diffusion coefficients are supposed to be strictly positive and decoupled. In the proof we shall proceed as follows: By LAMPERTI's method (cf. Section 3.4 in [9]) we transform the system with diagonal noise into a system very similar to that in [3]. Then we use nearly the same arguments as in the proof of Theorem 2.2 in [3] to show, that the solution of the transformed system is continuously differentiable, and that the derivatives satisfy pathwise a system of ordinary differential equations like that in Theorem 2.2. Finally we reconvert the received derivatives into derivatives for the original system.
We consider for a finite set of indices I a system of SKOROHOD SDEs with decoupled diffusion term:

$$\begin{aligned}
&X_t^i(x)=x^i+\int_0^t b^i(X_r(x))\,dr+l_t^i(x)+\int_0^t\sigma^i(X_r^i(x))\,dw_r^i,\quad t\ge 0,\ i\in I,\\
&X_t^i(x)\ge 0,\quad dl_t^i(x)\ge 0,\quad \int_0^\infty X_t^i(x)\,dl_t^i(x)=0,\qquad i\in I,
\end{aligned}\tag{4.1}$$

for all $x\in\mathbb{R}_+^I:=[0,\infty)^I$. As in the previous sections $(w^i)_{i\in I}$ is a family of independent BROWNian motions on a probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$. For every $i\in I$ we assume that the coefficients $b^i:\mathbb{R}_+^I\to\mathbb{R}$ and $\sigma^i:\mathbb{R}_+\to\mathbb{R}$ satisfy the following conditions:

i) $b^i\in C^1(\mathbb{R}_+^I)$, $\sigma^i\in C^2(\mathbb{R}_+)$,

ii) $\exists c>0$ such that $|b^i(x)-b^i(y)|\le c\,\|x-y\|,\qquad \forall x,y\in\mathbb{R}_+^I$,

iii) $\exists c>0$ such that $|\sigma^i(x)-\sigma^i(y)|\le c\,|x-y|,\qquad \forall x,y\in\mathbb{R}_+$,

iv) $\sigma^i(x)>0,\qquad \forall x\in\mathbb{R}_+$,

v) the function $x\mapsto \frac{b^i(x)}{\sigma^i(x^i)}-\frac12(\sigma^i)'(x^i)$, $x\in\mathbb{R}_+^I$, is LIPSCHITZ continuous.

Then, existence and uniqueness of the solution are ensured as before.
In the following we shall use again the notations c, ξ, $P_{s,i}^c$, $\rho_{s,t}^c$ and τ introduced in Section 2. We recall that

$$C^i:=\{s\ge0:X_s^i(x)=0\}\text{ and }r^i(t):=\sup\{s\le t:X_s^i(x)=0\},\qquad t\ge0,\ i\in I,\tag{4.2}$$

with the convention $\sup\emptyset:=0$.

Theorem 4.1. *For all $t \geq 0$ and $x \in \mathbb{R}_+^I$ a.s. the map $x \mapsto X_t(x)$ is continuously differentiable. Setting $\eta_t^{ij} := \frac{\partial X_t^i(x)}{\partial x^j}$, $i, j \in I$, and*

$$c_t(i,j) := \begin{cases} \frac{\sigma^i(X_t^i(x))}{\sigma^j(X_t^j(x))} \frac{\partial b^j}{\partial x^i}(X_t(x)) & \text{if } i \neq j, \\ \frac{\partial b^i}{\partial x^i}(X_t(x)) - \sigma^i(X_t^i(x))\, g^i(X_t(x)) & \text{if } i = j, \end{cases}$$

where

$$g^i(x) := \frac{(\sigma^i)'(x^i)}{\sigma^i(x^i)^2} b^i(x) + \tfrac{1}{2}(\sigma^i)''(x^i), \qquad i \in I,$$

the derivatives a.s. admit the representation

$$\eta_t^{ij} = \frac{\sigma^i(X_t^i(x))}{\sigma^j(x^j)} E_{0,j}^c \left[\mathbb{1}_{\{\xi_t = i\}} \mathbb{1}_{\{\tau > t\}} \rho_{0,t}^c \right], \tag{4.3}$$

which defines a right-continuous modification of η such that we have a.s. for all $t \geq 0$:

$$\eta_t^{ij} = \sigma^i(X_t^i(x)) \left\{ \frac{\delta_{ij}}{\sigma^j(x^j)} + \int_0^t \left(\left[\frac{1}{\sigma^i(X_r^i(x))} \sum_{k \in I} \frac{\partial b^i}{\partial x^k}(X_r(x))\, \eta_r^{kj} \right] - g^i(X_r(x))\, \eta_r^{ij} \right) dr \right\} \tag{4.4}$$

if $t \in [0, \inf C^i)$ and

$$\eta_t^{ij} = \sigma^i(X_t^i(x)) \left\{ \int_{r^i(t)}^t \left(\left[\frac{1}{\sigma^i(X_r^i(x))} \sum_{k \in I} \frac{\partial b^i}{\partial x^k}(X_r(x))\, \eta_r^{kj} \right] - g^i(X_r(x))\, \eta_r^{ij} \right) dr \right\} \tag{4.5}$$

if $t \in [\inf C^i, \infty)$.

Note that, if we set $\sigma^i \equiv 1$ for all $i \in I$, we obtain the result of Theorem 2.2.

Proof. We divide the proof into several parts.

Lamperti-Transformation

For all $i \in I$ we define the functions a^i on $\mathbb{R}_+$ by

$$a^i := \frac{1}{\sigma^i} \text{ with the first derivative } (a^i)' := -\frac{(\sigma^i)'}{(\sigma^i)^2}. \tag{4.6}$$

Note that a^i, $i \in I$, is well-defined on $\mathbb{R}_+$, since σ^i is required to be strictly positive in condition iv). We set

$$A^i(x) := \int_0^x a^i(z)\, dz, \qquad x \in \mathbb{R}_+, i \in I,$$

so that A^i, $i \in I$, is that function on $\mathbb{R}_+$ satisfying $(A^i)' = a^i$ and $A^i(0) = 0$. Hence we have $(A^i)' = \frac{1}{\sigma^i} > 0$, i.e. A^i is strictly increasing and there exists the inverse function $(A^i)^{-1} : [0, A^i(\infty)) \to \mathbb{R}_+$.

Note that, since $t \mapsto l_t^i$, $i \in I$, is a process of bounded variation, we have

$$d\langle X_t^i(x) \rangle_t = \sigma^i(X_t^i(x))^2\, dt, \qquad i \in I.$$

Then, by ITÔ's formula and (4.6) we obtain for $A^i(X^i_t(x))$:

$$\begin{aligned} dA^i(X^i_t(x)) &= a^i(X^i_t(x))\,dX^i_t(x) + \tfrac{1}{2}(a^i)'(X^i_t(x))\,d\langle X^i(x)\rangle_t \\ &= a^i(X^i_t(x))\left\{b^i(X_t(x))\,dt + dl^i_t(x) + \sigma^i(X^i_t(x))\,dw^i_t\right\} + \tfrac{1}{2}(a^i)'(X^i_t(x))\,\sigma^i(X^i_t(x))^2\,dt \\ &= \left[a^i(X^i_t(x))\,b^i(X_t(x)) - \tfrac{1}{2}(\sigma^i)'(X^i_t(x))\right]\,dt + a^i(X^i_t(x))\,dl^i_t(x) + dw^i_t. \end{aligned}$$

Since $dl^i_t(x) > 0$ only if $X^i_t(x) = 0$ for every $i \in I$, we have

$$a^i(X^i_t(x))\,dl^i_t(x) = a^i_0\,dl^i_t(x), \quad t \geq 0, \quad \text{with } a^i_0 := a^i(0) > 0.$$

Now we define the change of variables:

$$y = A(x) := (A^i(x^i))_{i\in I} \in M \subseteq \mathbb{R}^I_+,$$

where we denote by M the cartesian product of the sets $[0, A^i(\infty))$, $i \in I$, i.e. M is the domain of the inverse function $x = A^{-1}(y) := ((A^i)^{-1}(y^i))_{i\in I}$. In the following we consider the transformed process

$$Y^i_t(y) := A^i(X^i_t(x)), \qquad t \geq 0,\, i \in I.$$

Furthermore, we define the functions

$$\hat{b}^i : M \to \mathbb{R} \qquad y \mapsto a^i\big((A^i)^{-1}(y^i)\big)\; b^i\big(A^{-1}(y)\big) - \tfrac{1}{2}\,(\sigma^i)'\big((A^i)^{-1}(y^i)\big)\,, \qquad i \in I.$$

Note that by our assumptions on the coefficients, in particular by v), $\hat{b}^i$ is LIPSCHITZ continuous for every $i \in I$.
By our choice of A^i, $i \in I$, in particular since A^i is strictly increasing, we have $Y^i_t(y) = 0$ if and only if $X^i_t(x) = 0$, i.e. both processes have the same local time in zero. Thus we obtain the transformed system

$$\begin{aligned} &Y^i_t(y) = y^i + \int_0^t \hat{b}^i(Y_r(y))\,dr + a^i_0\,l^i_t(y) + w^i_t, \qquad t \geq 0,\, i \in I, \\ &Y^i_t(y) \geq 0, \quad dl^i_t(y) \geq 0, \quad \int_0^\infty Y^i_t(y)\,dl^i_t(y) = 0, \qquad i \in I. \end{aligned}$$

The Derivative of Y

Fix any time $T > 0$. We set $C^i := \{s \in [0,T] : X^i_s(x) = 0\} = \{s \in [0,T] : Y^i_s(y) = 0\}$ and $r^i(t) := \sup(C^i \cap [0,t])$, $t \in [0,T]$, $i \in I$. Recall that C^i, $i \in I$, is known to be a.s. a closed set with zero LEBESGUE measure without isolated points and that a.s. C^i is equal to the support of the measure $dl^i_t(x)$ on $[0,T]$ (see Proposition VI.2.5 in [12]). Let $C := \bigcup_{i\in I} C^i$. Then, the sets C^i, $i \in I$, satisfy the conditions of Proposition 1 in [3].
We set

$$\hat{W}^i_t(y) := \int_0^t \hat{b}^i(Y_r(y))\,dr + w^i_t, \qquad t \in [0,T],\, i \in I.$$

By GIRSANOV's Theorem there exists a probability measure $\tilde{\mathbb{P}}(y)$, which is equivalent to $\mathbb{P}$ and under which $(\hat{W}^i_t(y))_{i\in I}$ is a BROWNian motion in $\mathbb{R}^I$ (cf. Section 7.2 for details). Hence:

$$Y^i_t(y) = y^i + \hat{W}^i_t(y) + a^i_0\,l^i_t(y), \qquad t \in [0,T],\, i \in I. \tag{4.7}$$

Let $(A_n)_n$ be the countable collection of the connected components of the set $[0,T]\backslash C$ and $a_n := \inf A_n$. Since A_n is open, there exists a $q_n \in A_n \cap \mathbb{Q}$. We denote by A_n^i the connected component of $[0,T]\backslash C^i$ that contains q_n. Then, $A_n \subseteq A_n^i$.
In the following we use the abbreviation $y_\varepsilon = y + \varepsilon\, e^j$ for all $\varepsilon > 0$, where $(e^i)_{i\in I}$ is the canonical basis of $\mathbb{R}^I$, i.e. $e^j(i) = \delta_{ij}$. Fix now $i \in I$. First we consider the case $t < \inf C^i$, i.e. in particular $l_t^i(y) = 0$. From Theorem 3.2 we know that the map $y \mapsto (Y_t(y))_{t\in[0,T]}$ has a continuous modification w.r.t. the sup-norm topology. Working with this modification, we can find a random $\tilde{\Delta}^i > 0$ such that a.s.

$$\sup_{0\le s\le t} |Y_s^i(y_\varepsilon) - Y_s^i(y)| < \tfrac{1}{2} \inf_{0\le s\le t} Y_s^i(y), \qquad \forall \varepsilon \in (0, \tilde{\Delta}^i)$$

(the right hand side is strictly positive a.s. since $t < \inf C^i$ and $Y^i(y)$ has continuous sample paths). For such ε we have a.s. $Y_s^i(y_\varepsilon) > 0$ for all $s \in [0,t]$ and thus $l_t^i(y_\varepsilon) = 0$ a.s.
Hence we obtain a.s.

$$Y_t^i(y_\varepsilon) - Y_t^i(y) = y_\varepsilon^i - y^i + \int_0^t \left(\hat{b}^i(Y_r(y_\varepsilon)) - \hat{b}^i(Y_r(y))\right) dr, \qquad t < \inf C^i,\ i \in I.$$

Now we consider the case $t > \inf C^i$: Let $n \in \mathbb{N}$ such that $t \in A_n^i$, i.e. $l_t^i(y) = l_{q_n}^i(y) = l_{r^i(q_n)}^i(y)$, since by construction $Y^i(y)$ is strictly positive on A_n^i. By Skorohod's Lemma (see Lemma 2.3) we have for all $t \in [0,T]$:

$$a_0^i\, l_t^i(y) = \sup_{s\le t} \left[y^i + \hat{W}_s^i(y)\right]^- = \left[-y^i - \inf_{s\le t} \hat{W}_s^i(y)\right]^+.$$

Since $Y_{r^i(q_n)}^i(y) = 0$, $a_0^i > 0$ and $s \mapsto l_s^i(y)$ is non-decreasing, we see that

$$\hat{W}_{r^i(q_n)}^i(y) = -y^i - a_0^i\, l_{r^i(q_n)}^i(y) \le -y^i - a_0^i\, l_s^i(y) = -Y_s^i(y) + \hat{W}_s^i(y) \le \hat{W}_s^i, \qquad \forall s \in [0, r^i(q_n)].$$

Hence, using $l_{q_n}^i(y) = l_{r^i(q_n)}^i(y)$, we conclude that $\inf_{s\le q_n} \hat{W}_s^i(y) = \inf_{s\le r^i(q_n)} \hat{W}_s^i(y) = \hat{W}_{r^i(q_n)}^i(y)$ and

$$a_0^i\, l_t^i(y) = a_0^i\, l_{r^i(q_n)}^i(y) = \left[-y^i - \inf_{s\le r^i(q_n)} \hat{W}_s^i(y)\right]^+ = \left[-y^i - \hat{W}_{r^i(q_n)}^i(y)\right]^+, \qquad t \in A_n^i.$$

On the other hand, we have by Skorohod's Lemma for all $\varepsilon > 0$:

$$\begin{aligned} a_0^i\, l_t^i(y_\varepsilon) &= \sup_{s\le t} \left[y_\varepsilon^i + \hat{W}_s^i(y_\varepsilon)\right]^- = \sup_{s\le t} \left[y_\varepsilon^i + \hat{W}_s^i(y) + \left(\hat{W}_s^i(y_\varepsilon) - \hat{W}_s^i(y)\right)\right]^- \\ &= \left[-y_\varepsilon^i - \inf_{s\le t} \left(\hat{W}_s^i(y) + \left(\hat{W}_s^i(y_\varepsilon) - \hat{W}_s^i(y)\right)\right)\right]^+, \qquad t \in [0,T],\ i \in I. \end{aligned}$$

Since $\hat{W}^i(y)$ is a Brownian motion under $\tilde{\mathbb{P}}(y)$, i.e. the law of $\hat{W}^i(y)$ is absolutely continuous w.r.t. the law of a Brownian motion, applying Lemma 2.4 a.s. for every $q \in [0,T] \cap \mathbb{Q}$, there exists a random variable θ_q^i such that $\hat{W}^i(y)$ attains its minimum over $[0,q]$ only at θ_q^i. Moreover, we can find a random variable $\gamma_q^i > 0$ such that every γ_q^i-Lipschitz perturbation of $\hat{W}^i(y)$ attains its minimum over $[0,q]$ only at θ_q^i. Fix now q_n. From Theorem 3.2 we know that $y \mapsto (Y_r(y))_{r\in[0,q_n]}$ is continuous w.r.t. the sup-norm topology. We choose $\tilde{\Delta}_n^i > 0$ such that a.s.

$$\sup_{r\le q_n} \|Y_r(y_\varepsilon) - Y_r(y)\| \le \frac{1}{K_{\hat{b}^i}} \gamma_{q_n}^i, \qquad \forall \varepsilon \in (0, \tilde{\Delta}_n^i),$$

where $K_{\hat{b}^i}$ denotes the LIPSCHITZ norm of $\hat{b}^i$. For such ε we have

$$\sup_{r\le q_n}\left|\hat{b}^i(Y_r(y_\varepsilon))-\hat{b}^i(Y_r(y))\right|\le K_{\hat{b}^i}\sup_{r\le q_n}\|Y_r(y_\varepsilon)-Y_r(y)\|\le\gamma^i_{q_n},$$

and we obtain for $f(s):=\hat{W}^i_s(y_\varepsilon)-\hat{W}^i_s(y)$

$$\begin{aligned}|f(t)-f(s)|&=\left|\int_s^t\left(\hat{b}^i(Y_r(y_\varepsilon))-\hat{b}^i(Y_r(y))\right)dr\right|\le\sup_{r\le q_n}\left|\hat{b}^i(Y_r(y_\varepsilon))-\hat{b}^i(Y_r(y))\right||t-s|\\&\le\gamma^i_{q_n}|t-s|\end{aligned}$$

for all $s,t\in[0,q_n]$, i.e. f has LIPSCHITZ continuous sample paths if $\varepsilon\in(0,\tilde{\Delta}^i_n)$. By Lemma 2.4, since $\hat{W}^i(y)$ attains its minimum over $[0,q_n]$ only at $\theta^i_{q_n}=r^i(q_n)$, we get for such ε:

$$a^i_0\,l^i_{q_n}(y_\varepsilon)=\left[-y^i_\varepsilon-\hat{W}^i_{r^i(q_n)}(y)-\int_0^{r^i(q_n)}\left(\hat{b}^i(Y_r(y_\varepsilon))-\hat{b}^i(Y_r(y))\right)dr\right]^+.$$

Then, for $\varepsilon\in(0,\Delta^i_n)$ with $\Delta^i_n:=\min(\tilde{\Delta}^i,\tilde{\Delta}^i_n)$, we have $l^i_{q_n}(y_\varepsilon)=l^i_{q_n}(y)=0$ if $q_n<\inf C^i$. In the other case $q_n>\inf C^i$ we have $a^i_0\,l^i_{q_n}(y)=-y^i-\hat{W}^i_{r^i(q_n)}(y)$ and, possibly after choosing a smaller Δ^i_n, $y^i_\varepsilon+\hat{W}^i_{r^i(q_n)}(y_\varepsilon)<0$ resp. $a^i_0\,l^i_{q_n}(y_\varepsilon)>0$, so that

$$a^i_0\,l^i_{q_n}(y_\varepsilon)=-y^i_\varepsilon-\hat{W}^i_{r^i(q_n)}(y)-\int_0^{r^i(q_n)}\left(\hat{b}^i(Y_r(y_\varepsilon))-\hat{b}^i(Y_r(y))\right)dr.$$

Obviously we have for all $t\in[0,T]$:

$$Y^i_t(y_\varepsilon)-Y^i_t(y)=\varepsilon\,\delta_{ij}+\int_0^t\left(\hat{b}^i(Y^i_r(y_\varepsilon))-\hat{b}^i(Y^i_r(y))\right)dr+a^i_0\,l^i_t(y_\varepsilon)-a^i_0\,l^i_t(y).$$

We set for all $i\in I$, $t\in[0,T]$ and $\varepsilon>0$

$$\begin{aligned}\hat{\eta}^i_t(\varepsilon)&:=\frac{Y^i_t(y_\varepsilon)-Y^i_t(y)}{\varepsilon},\qquad\Delta_n:=\min_{i\in I}\Delta^i_n\\Y^{\alpha,\varepsilon}_r&:=\alpha\,Y_r(y_\varepsilon)+(1-\alpha)\,Y_r(y),\qquad\alpha\in[0,1].\end{aligned}$$

Since by chain rule

$$\begin{aligned}\hat{b}^i(Y^i_r(y_\varepsilon))-\hat{b}^i(Y^i_r(y))&=\hat{b}^i(Y^{\alpha,\varepsilon}_r)\Big|^1_{\alpha=0}=\int_0^1\frac{d}{d\alpha}\hat{b}^i(Y^{\alpha,\varepsilon}_r)\,d\alpha\\&=\int_0^1\frac{d}{d\alpha}\hat{b}^i(\alpha Y_r(y_\varepsilon)+(1-\alpha)Y_r(y))\,d\alpha\\&=\int_0^1\sum_{k\in I}\frac{\partial\hat{b}^i}{\partial y^k}(Y^{\alpha,\varepsilon}_r)\left(Y^k_r(y_\varepsilon)-Y^k_r(y)\right)d\alpha,\end{aligned}$$

we obtain a.s. for all $n\in\mathbb{N}$, $\varepsilon\in(0,\Delta^i_n)$ and $t\in A^i_n$:

$$\begin{aligned}\hat{\eta}^i_t(\varepsilon)&=\delta_{ij}+\frac{1}{\varepsilon}\int_0^t\left(\hat{b}^i(Y^i_r(y_\varepsilon))-\hat{b}^i(Y^i_r(y))\right)dr\\&=\delta_{ij}+\int_0^t\sum_{k\in I}\left[\int_0^1\frac{\partial\hat{b}^i}{\partial y^k}(Y^{\alpha,\varepsilon}_r)\,d\alpha\right]\hat{\eta}^k_r(\varepsilon)\,dr,\qquad t\in[0,\inf C^i)\end{aligned}$$

and

$$\hat{\eta}_t^i(\varepsilon) = \frac{1}{\varepsilon} \int_{r^i(t)}^t \left(\hat{b}^i(Y_r^i(y_\varepsilon)) - \hat{b}^i(Y_r^i(y)) \right) dr$$
$$= \int_{r^i(t)}^t \sum_{k\in I} \left[\int_0^1 \frac{\partial \hat{b}^i}{\partial y^k}(Y_r^{\alpha,\varepsilon})\, d\alpha \right] \hat{\eta}_r^k(\varepsilon)\, dr, \qquad t \in (\inf C^i, T].$$

Thus we have a.s. for all $\varepsilon \in (0, \Delta_n)$, $i \in I$ and $t \in A_n$:

$$\hat{\eta}_t^i(\varepsilon) = \hat{\eta}_{a_n}^i(\varepsilon) + \int_{a_n}^t \sum_{k\in I} \left[\int_0^1 \frac{\partial \hat{b}^i}{\partial y^k}(Y_r^{\alpha,\varepsilon})\, d\alpha \right] \hat{\eta}_r^k(\varepsilon)\, dr.$$

Now we are exactly in the same situation as before Step 5 in the proof of Theorem 1 in [3]. Therefore, by the same reasoning as in Step 5 and 6 of that proof we can conclude that $y \mapsto Y_t(y)$ is continuously differentiable a.s. for all $t \geq 0$ and, setting

$$c_t(i,j) := \frac{\partial \hat{b}^j}{\partial y^i}(Y_t(y)), \qquad t \geq 0,\ i,j \in I, \tag{4.8}$$

we obtain that the derivatives of $Y_t(y)$ a.s. admit the following random walk representation:

$$\frac{\partial Y_t^i(y)}{\partial y^j} = E_{0,j}^c \left[\mathbb{1}_{\{\xi_t = i\}} \mathbb{1}_{\{\tau > t\}} \rho_{0,t}^c \right]. \tag{4.9}$$

Moreover, the partial derivatives satisfy a.s.:

$$\frac{\partial Y_t^i(y)}{\partial y^j} = \delta_{ij} + \int_0^t \left(\sum_{k\in I} \frac{\partial \hat{b}^i}{\partial y^k}(Y_r(y))\, \frac{\partial Y_r^k(y)}{\partial y^j} \right) dr, \qquad \text{if } t \in [0, \inf C^i), \tag{4.10}$$

$$\frac{\partial Y_t^i(y)}{\partial y^j} = \int_{r^i(t)}^t \left(\sum_{k\in I} \frac{\partial \hat{b}^i}{\partial y^k}(Y_r(y))\, \frac{\partial Y_r^k(y)}{\partial y^j} \right) dr, \qquad \text{if } t \in [\inf C^i, \infty), \tag{4.11}$$

where C^i and r^i, $i \in I$, are defined as in (4.2).
Notice that the derivatives do not depend on the constants a_0^i, $i \in I$.

Reconversion

From the definition of A and σ it is obvious that for every $k \in I$ the k-th component of $A(x)$, $x \in \mathbb{R}_+^I$, does only depend on the k-th component of x, i.e. $A^k(x) = A^k(x^k)$, so that

$$\frac{\partial A^k(x)}{\partial x^j} = 0 \quad \text{if } j \neq k. \tag{4.12}$$

The inverse function A^{-1} has this property, too. Using the formula for the derivative of inverse functions we obtain

$$((A^i)^{-1})'(y^i) = \frac{1}{(A^i)'((A^i)^{-1}(y^i))} = \frac{1}{(A^i)'(x^i)} = \sigma^i(x^i), \qquad i \in I. \tag{4.13}$$

By construction we have

$$X_t^i(x) = (A^i)^{-1}(Y_t^i(A(x))), \qquad t \geq 0,\ i \in I,$$

and by chain rule, (4.12) and (4.13) we get

$$\begin{aligned}\frac{\partial X_t^i(x)}{\partial x^j} &= ((A^i)^{-1})'(Y_t^i(A(x))\sum_{k\in I}\frac{\partial Y_t^i(y)}{\partial y^k}\,\frac{\partial A^k(x)}{\partial x^j}\\ &= \frac{\sigma^i(X_t^i(x))}{\sigma^j(x^j)}\,\frac{\partial Y_t^i(y)}{\partial y^j}, \qquad t\geq 0,\ i\in I. \end{aligned}\tag{4.14}$$

If we replace the partial derivative of Y in (4.14) by the right hand side of (4.10) resp. (4.11), we still have to reconvert the integrands: First we obtain by chain rule and (4.13)

$$\begin{aligned}\frac{\partial Y_t^k(y)}{\partial y^j} &= \frac{\partial}{\partial y^j}\left[A^k(X_t^k(A^{-1}(y)))\right] = (A^k)'(X_t^k(A^{-1}(y)))\sum_{l\in I}\frac{\partial X_t^k(x)}{\partial x^l}\,\frac{\partial (A^{-1})^l(y)}{\partial y^j}\\ &= \frac{1}{\sigma^k(X_t^k(x))}\,\frac{\partial X_t^k(x)}{\partial x^j}\,\frac{\partial (A^{-1})^j(y)}{\partial y^j}\\ &= \frac{\sigma^j(x^j)}{\sigma^k(X_t^k(x))}\,\frac{\partial X_t^k(x)}{\partial x^j}, \qquad t\geq 0,\ k,j\in I.\end{aligned}\tag{4.15}$$

We recall that

$$\hat b^i(y) = a^i\big((A^i)^{-1}(y^i)\big)\; b^i\big(A^{-1}(y)\big) - \tfrac12(\sigma^i)'\big((A^i)^{-1}(y^i)\big)\,, \qquad i\in I.$$

Using chain rule, (4.6) and (4.13), we obtain for the partial derivatives of $\hat b^i$, $i\in I$:

$$\frac{\partial \hat b^i(y)}{\partial y^k} = a^i((A^i)^{-1}(y^i))\sum_{l\in I}\frac{\partial b^i}{\partial y^l}(A^{-1}(y))\,\frac{\partial (A^l)^{-1}}{\partial y^k}(y) = \frac{1}{\sigma^i(x^i)}\,\frac{\partial b^i}{\partial x^k}(x)\,\sigma^k(x^k), \qquad \text{if } k\neq i,$$

and differentiating w.r.t. the i-th variable yields by product rule, chain rule, (4.6) and (4.13):

$$\begin{aligned}\frac{\partial \hat b^i(y)}{\partial y^i} &= (a^i)'((A^i)^{-1}(y^i))\,((A^i)^{-1})'(y^i)\,b^i(A^{-1}(y))\\ &\quad + \frac{1}{\sigma^i(x^i)}\,\frac{\partial b^i}{\partial x^i}(x)\,\sigma^i(x^i) - \tfrac12(\sigma^i)''(x^i)\,((A^i)^{-1})'(y^i)\\ &= \sigma^i(x^i)\left(-\frac{(\sigma^i)'(x^i)}{(\sigma^i(x^i))^2}b^i(x) + \frac{1}{\sigma^i(x^i)}\,\frac{\partial b^i}{\partial x^i}(x) - \tfrac12(\sigma^i)''(x^i)\right)\\ &= \frac{\partial b^i}{\partial x^i}(x) - \sigma^i(x^i)\,g^i(x),\end{aligned}$$

where

$$g^i(x) := \frac{(\sigma^i)'(x^i)}{\sigma^i(x^i)^2}\,b^i(x) + \frac12(\sigma^i)''(x^i), \qquad i\in I.$$

Thus, we have for all $t\geq 0$ and $i\in I$

$$\frac{\partial \hat b^i}{\partial y^k}(Y_t(y)) = \frac{\sigma^k(X_t^k(x))}{\sigma^i(X_t^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_t(x)), \qquad \text{if } k\neq i,\tag{4.16}$$

and

$$\frac{\partial \hat b^i}{\partial y^i}(Y_t(y)) = \frac{\partial b^i}{\partial x^i}(X_t(x)) - \sigma^i(X_t^i(x))\,g^i(X_t(x)).\tag{4.17}$$

Hence, we obtain by (4.15), (4.16) and (4.17) for all $t \geq 0$ and $i \in I$:

$$\begin{aligned}
&\sum_{k\in I} \frac{\partial \hat{b}^i}{\partial y^k}(Y_t(y))\,\frac{\partial Y_t^k(y)}{\partial y^j} \\
&= \sum_{k\in I}\left[\frac{\sigma^k(X_t^k(x))}{\sigma^i(X_t^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_t(x))\,\frac{\sigma^j(x^j)}{\sigma^k(X_t^k(x))}\,\frac{\partial X_t^k(x)}{\partial x^j}\right] - \sigma^i(X_t^i(x))\,g^i(X_t(x))\,\frac{\sigma^j(x^j)}{\sigma^i(X_t^i(x))}\,\frac{\partial X_t^i(x)}{\partial x^j} \\
&= \sum_{k\in I}\left[\frac{\sigma^j(x^j)}{\sigma^i(X_t^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_t(x))\,\frac{\partial X_t^k(x)}{\partial x^j}\right] - \sigma^j(x^j)\,g^i(X_t(x))\,\frac{\partial X_t^i(x)}{\partial x^j}. \qquad (4.18)
\end{aligned}$$

Setting $\eta_t^{ij} := \frac{\partial X_t^i(x)}{\partial x^j}$, $t \geq 0$, $i,j \in I$, we use (4.14), (4.10), (4.11) and (4.18) to obtain a.s. for $t \in [0, \inf C^i)$

$$\begin{aligned}
\eta_t^{ij} &= \frac{\sigma^i(X_t^i(x))}{\sigma^j(x^j)}\,\frac{\partial Y_t^i(y)}{\partial y^j} \\
&= \frac{\sigma^i(X_t^i(x))}{\sigma^j(x^j)}\left\{\delta_{ij} + \sigma^j(x^j)\int_0^t\left(\sum_{k\in I}\left[\frac{1}{\sigma^i(X_r^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_r(x))\,\eta_r^{kj}\right] - g^i(X_r(x))\,\eta_r^{ij}\right)dr\right\} \\
&= \sigma^i(X_t^i(x))\left\{\frac{1}{\sigma^j(x^j)}\delta_{ij} + \int_0^t\left(\sum_{k\in I}\left[\frac{1}{\sigma^i(X_r^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_r(x))\,\eta_r^{kj}\right] - g^i(X_r(x))\,\eta_r^{ij}\right)dr\right\},
\end{aligned}$$

and for $t \in [\inf C^i, \infty)$

$$\eta_t^{ij} = \sigma^i(X_t^i(x))\int_{r^i(t)}^t\left(\sum_{k\in I}\left[\frac{1}{\sigma^i(X_r^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_r(x))\,\eta_r^{kj}\right] - g^i(X_r(x))\,\eta_r^{ij}\right)dr,$$

i.e. we obtain (4.4) and (4.5). Finally, inserting (4.9) into (4.14) leads to

$$\eta_t^{ij} = \frac{\sigma^i(X_t^i(x))}{\sigma^j(x^j)}\,E_{0,j}^c\left[\mathbb{1}_{\{\xi_t=i\}}\mathbb{1}_{\{\tau>t\}}\rho_{0,t}^c\right] \qquad \text{a.s.},$$

where c, defined in (4.8), is equal to that in the statement by (4.16) and (4.17), which proves the representation in (4.3). □

5 Excursus: Feynman-Kac Formula for inhomogeneous Markov Processes

This excursus deals with time-inhomogeneous MARKOV processes. At first we define the transition functions for such processes and the corresponding time-dependent infinitesimal generators. Then we prove the analogues to the forward and backward equation in the time-inhomogeneous case, where we often use the relationship between generators and solutions of martingale problems. To keep things simple we assume the state space of the process to be finite. The main purpose of this section is to deduce FEYNMAN-KAC formulae for solutions of CAUCHY problems, whose PDEs contain time-dependent generators. We investigate two types of CAUCHY problems differing in the time variable, whose derivative appears in the PDE. Thus two FEYNMAN-KAC representation are given, called forward and backward FEYNMAN-KAC formula. The backward FEYNMAN-KAC formula is a generalization of the backward equation like in the time-homogeneous case. Unfortunately, the forward version cannot be transfered directly from the

homogeneous case, which is certainly due to the time-dependence of the generator. Therefore, we have to change the underlying MARKOV process by a time-transformation in the generator. In this section we denote by X the considered MARKOV process, because this is the most conventional notation, and not the solution of a SKOROHOD SDE as in the other sections. Since this section doesn't deal with SDEs at all, there should be no danger of confusion.
Some of the results in the first subsection can be transfered directly from the homogeneous case. Therefore, the proofs of the homogeneous analogues are simply adapted to the time-inhomogeneous case (cf. the references).

5.1 Basic Definitions and first Properties

We consider a time-inhomogeneous MARKOV process $(X_t)_{t\geq 0}$ with finite state space I: Let $\Omega := D([0,\infty), I)$ be the space of I-valued càdlàg functions, where I is endowed with the discrete topology. For $\omega \in \Omega$, set $X_t(\omega) = \omega(t)$, i.e. X is the coordinate process, and define $\mathcal{F}_t^0 := \sigma(X_s;\ s \leq t)$ and $\mathcal{F}^0 := \sigma(X_s;\ s < \infty)$. Furthermore, for each $s \geq 0$ and $i \in I$ we denote by $P_{s,i}$ that probability measure on $(\Omega, \mathcal{F}^0)$, under which $X_t = i$ for all $t \in [0, s]$ and $(X_t)_{t\geq s}$ has the law of a time continuous MARKOV process, that starts in i at time $t = s$ and is associated with a given transition function $P(s, t, ., .)$ on $(I, \mathcal{P}(I))$ in the sense of

Definition 5.1. *A transition function on $(I, \mathcal{P}(I))$ is a family $P(s, t, ., .)$, $0 \leq s \leq t$, of functions $P(s, t, ., .) : I \times \mathcal{P}(I) \to [0, 1]$ such that*

i) *$P(s, t, ., .)$ is a stochastic kernel, i.e. $P(s, t, i, .)$ is a probability measure on $(I, \mathcal{P}(I))$ for each $i \in I$ and $P(s, t, ., A)$ is $\mathcal{P}(I)$ measurable for each $A \in \mathcal{P}(I)$,*

ii) $P(s, s, i, \{j\}) = \delta_{ij}, \qquad i, j \in I,\ s \geq 0,$

iii) *the* CHAPMAN-KOLMOGOROV *equation is satisfied:*

$$\sum_{k\in I} P(s, t, i, \{k\})\, P(t, u, k, \{j\}) = P(s, u, i, \{j\}), \qquad 0 \leq s \leq t \leq u, \quad i, j \in I. \tag{5.1}$$

Of course, $(P(s, t, ., .))_{t\geq s\geq 0}$ can be interpreted as a family of stochastic matrices, since the state space I is finite. To simplify the notation, we set

$$p(s, t, i, j) := P(s, t, i, \{j\}), \qquad i, j \in I,\ 0 \leq s \leq t.$$

Moreover, we shall assume that X is a regular jump MARKOV process, i.e.

- for arbitrary $(s, i, j) \in [0,\infty) \times I \times I$ the limit

$$\lim_{t\downarrow s} \frac{p(s, t, i, j) - \delta_{ij}}{t - s} =: q_{ij}(s) \tag{5.2}$$

  exists and

- the convergence in (5.2) is uniform in $(s, i, j) \in [0, t] \times I \times I$ and the function $q_{ij}(s)$ for fixed $i, j \in I$ is continuous in $s \in [0, t]$ where $t \in [0,\infty)$ is arbitrary.

These conditions are necessary to ensure the differentiability of the transition probabilities:

Remark 5.2. The transition probabilities $p(s, t, i, j)$ of a regular jump process are differentiable w.r.t. t for $t > s$ and they are differentiable w.r.t. s, $s < t$.

Proof. See Theorem 2 and 3 in Chapter 1 in [6]. □

The differentiability w.r.t. t will also be pointed out later in Proposition 5.4.
Now we consider the family of bounded operators $(P_{s,t})$, $0 \leq s \leq t$, on the real-valued functions on I, generated by the transition functions:

$$P_{s,t}f(i) := E_{s,i}[f(X_t)] = \sum_{j \in I} p(s,t,i,j)\, f(j). \tag{5.3}$$

Then the CHAPMAN-KOLMOGOROV equation implies the following analogue to the semigroup property:

$$P_{s,t}(P_{t,u}f) = P_{s,u}f, \qquad 0 \leq s \leq t \leq u, \tag{5.4}$$

which can easily be checked as follows: For $0 \leq s \leq t \leq u$ and $i \in I$ we have

$$\begin{aligned} P_{s,t}(P_{t,u}f)(i) &= \sum_{k \in I} p(s,t,i,k)\, P_{t,u}f(k) = \sum_{k \in I} p(s,t,i,k) \sum_{j \in I} p(t,u,k,j)\, f(j) \\ &= \sum_{j \in I} \left(\sum_{k \in I} p(s,t,i,k)\, p(t,u,k,j) \right) f(j) = \sum_{j \in I} p(s,u,i,j)\, f(j) \\ &= P_{s,u}f(i). \end{aligned}$$

Definition 5.3. *The time-dependent operator $(L_t)_{t \geq 0}$ on the real-valued functions on I, defined by*

$$L_t f := \lim_{h \downarrow 0} \frac{1}{h}(P_{t,t+h}f - f),$$

is called (time-dependent) infinitesimal generator of the process X.

Note that by our assumptions, in particular by (5.2), $L_t f$ is well-defined for all real-valued functions f on I and all $t \geq 0$.
From now on, f always denotes a real-valued function on I. Since we have assumed the state space I to be finite, in this context the generator can be interpreted as time-dependent Q-matrix $Q(s) := \{q_{ij}(s);\, i,j \in I\}$, $s \geq 0$, where $q_{ij}(s)$ is defined as above in (5.2), i.e. $q_{ij}(s)$ is the right-hand derivative of $t \mapsto p(s,t,i,j)$, $t \in [s,\infty)$, at $t = s$. The following properties of the Q-matrix are obvious by definition:

$$q_{ij}(s) \geq 0 \quad \text{if } i \neq j, \qquad \sum_k q_{ik}(s) = 0, \qquad i,j \in I,\ s \geq 0.$$

Finally, the generator is given by

$$L_t f(i) = (Q(t)f)(i) = \sum_{k \in I} q_{ik}(t) f(k) = \sum_{k \in I} q_{ik}(t)\, (f(k) - f(i)), \qquad t \geq 0,\ i \in I,$$

since for all $i \in I$ and $t \geq 0$

$$\begin{aligned} \frac{1}{h}\,(P_{t,t+h}f(i) - f(i)) &= \frac{1}{h}\,(P_{t,t+h}f(i) - P_{t,t}f(i)) = \sum_{k \in I} \frac{1}{h}\,(p(t,t+h,i,k) - p(t,t,i,k))\, f(k) \\ &\xrightarrow[h \downarrow 0]{} \sum_{k \in I} q_{ik}(t)\, f(k). \end{aligned}$$

Next we state the analogue to the forward equation (cf. Proposition VII.1.2 in [12]).

Proposition 5.4. *For any fixed $s \geq 0$,*

i) the function $t \mapsto P_{s,t}f$ on $[s,\infty)$ is differentiable with derivative $\frac{\partial}{\partial t}P_{s,t}f = P_{s,t}(L_t f)$,

ii) $P_{s,t}f - f = \int_s^t P_{s,r}(L_r f)\,dr$.

Proof. Fix $s \geq 0$. From the definition of $P_{s,t}$ for $t \geq s$ in (5.3) it is obvious that this operator is linear and continuous in f. Thus we obtain using (5.4)

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\left[P_{s,t+h}f - P_{s,t}f\right] &= \lim_{h\downarrow 0}\frac{1}{h}\left[P_{s,t}(P_{t,t+h}f) - P_{s,t}f\right] = \lim_{h\downarrow 0}P_{s,t}\left(\frac{P_{t,t+h}f - f}{h}\right)\\ &= P_{s,t}(L_t f),\end{aligned}$$

i.e. the right-hand derivative of $t \mapsto P_{s,t}f$ exists and is equal to $P_{s,t}(L_t f)$. On the other hand, the function $t \mapsto \int_s^t P_{s,r}(L_r f)\,dr$, $t \geq s$, is differentiable and its derivative is equal to $P_{s,t}(L_t f)$. Since two functions, which have the same right-hand derivative, only differ by a constant, it follows that

$$P_{s,t}f = \int_s^t P_{s,r}(L_r f)\,dr + c$$

for some c not depending on t. This proves i) and, choosing $t = s$, it follows that $c = f$, which proves ii).

□

The following proposition describes the probabilistic significance of generators. It points out the relationship between generators and solutions of martingale problems (cf. Proposition VII.1.6 in [12]).

Proposition 5.5. *For any fixed $s \geq 0$ and $i \in I$, the process $(M_t^{s,f})_{t\geq s}$, defined by*

$$M_t^{s,f} := f(X_t) - f(X_s) - \int_s^t L_r f(X_r)\,dr,$$

is a $((\mathcal{F}_t^0)_{t\geq s}, P_{s,i})$-martingale.

Proof. Fix any $s \geq 0$ and $i \in I$. Obviously the process $M^{s,f}$ is adapted to the natural filtration of X and $M_t^{s,f}$ is integrable for each $t \geq s$, since f and $L_t f$ are bounded for all $t \geq s$. Hence, it is sufficient to show that the martingale property holds: For $s \leq u \leq t$ we obtain by the Markov property:

$$\begin{aligned}E_{s,i}\left[M_t^{s,f}\middle|\,\mathcal{F}_u^0\right] &= M_u^{s,f} + E_{s,i}\left[f(X_t) - f(X_u) - \int_u^t L_r f(X_r)\,dr\middle|\,\mathcal{F}_u^0\right]\\ &= M_u^{s,f} + E_{u,X_u}\left[f(X_t) - f(X_u) - \int_u^t L_r f(X_r)\,dr\right].\end{aligned}$$

But for any $k \in I$ we have by Fubini's Theorem and Proposition 5.4

$$E_{u,k}\left[f(X_t) - f(X_u) - \int_u^t L_r f(X_r)\,dr\right] = P_{u,t}f(k) - f(k) - \int_u^t P_{u,r}(L_r f)(k)\,dr = 0,$$

which completes the proof. □

We extend the last result to the time-space process (t, X_t) (cf. Lemma IV.20.12 in [14]).

Lemma 5.6. *For any fixed $s \geq 0$ and $i \in I$, let g be a real-valued function on $[s, \infty) \times I$: $(t, j) \mapsto g(t, j)$ such that $\frac{\partial}{\partial t} g(t, j)$ exists and is continuous for every $j \in I$. Then, the process $(\hat{M}_t^{s,g})_{t \geq s}$ defined by*

$$\hat{M}_t^{s,g} := g(t, X_t) - g(s, X_s) - \int_s^t \left(\frac{\partial}{\partial r} g(r, X_r) + L_r g(r, .)(X_r) \right) dr$$

is a local $((\mathcal{F}_t^0)_{t \geq s}, P_{s,i})$-martingale.

Proof. Fix any $s \geq 0$ and $i \in I$. Since

$$g(t, X_t) = \sum_{j \in I} g(t, j) \, \mathbb{1}_{\{X_t = j\}}$$

and the martingale property is preserved under linear transformations, it is sufficient to prove the lemma when g has the form

$$g(t, j) = h(t) f(j),$$

where h is a continuously differentiable function on $[s, \infty)$. From the definition of the process $M^{s,f}$ in the statement of Proposition 5.5 we have

$$df(X_t) = dM_t^{s,f} + L_t f(X_t) \, dt.$$

Using the integration by parts formula for finite variation processes (see e.g. Section IV.18, p. 27 in [14]), we obtain

$$\begin{aligned} dg(t, X_t) = h'(t) f(X_t) \, dt + h(t) \, df(X_t) &= \left[h'(t) f(X_t) + h(t) L_t f(X_t) \right] dt + h(t) \, dM_t^{s,f} \\ &= \left[\frac{\partial}{\partial t} g(t, X_t) + L_t g(t, .)(X_t) \right] dt + h(t) \, dM_t^{s,f}. \end{aligned}$$

Since $M^{s,f}$ is a martingale by Proposition 5.5, $\left(\int_s^t h(r) \, dM_r^{s,f} \right)_{t \geq s}$ is a local martingale and the result follows.

□

5.2 The Backward Equation and a Backward Feynman-Kac Representation

Now we are going to prove the analogue to the backward equation in the time-inhomogeneous case. For that purpose we need the following

Lemma 5.7. *For any fixed $t > 0$ and $i \in I$, the process $(P_{s,t} f(X_s))_{0 \leq s \leq t}$ is a $((\mathcal{F}_s^0)_{0 \leq s \leq t}, P_{0,i})$-martingale.*

Proof. Since the process is obviously adapted and integrable, we have only to check the martingale property: Using the Markov property and the projectivity of conditional expectations, we obtain for $0 \leq u \leq s \leq t$:

$$\begin{aligned} E_{0,i} \left[P_{s,t} f(X_s) \,|\, \mathcal{F}_u^0 \right] = E_{0,i} \left[E_{s,X_s}[f(X_t)] \,|\, \mathcal{F}_u^0 \right] &= E_{0,i} \left[E_{0,i}[f(X_t) \,|\, \mathcal{F}_s^0] \,|\, \mathcal{F}_u^0 \right] = E_{0,i} \left[f(X_t) \,|\, \mathcal{F}_u^0 \right] \\ &= E_{u,X_u}[f(X_t)] = P_{u,t} f(X_u). \end{aligned}$$

□

For the next result we need the additional assumption that the transition probabilities $p(s,t,i,j)$ are even continuously differentiable w.r.t. s, $s<t$ (cf. Remark 5.2).

Proposition 5.8 (Backward Equation)**.** *For any fixed $t>0$ we define the function u on $[0,t]\times I$ by $(s,i)\mapsto u(s,i):=P_{s,t}f(i)$. Then, $u(.,i)$ is continuously differentiable for every $i\in I$, and satisfies the* CAUCHY *problem:*

$$\begin{aligned}\frac{\partial}{\partial s}u(s,i) &= -L_s u(s,.)(i), \qquad \forall s\in[0,t],\, i\in I,\\ u(t,.) &= f.\end{aligned} \tag{5.5}$$

Proof. Clearly, the function u defined as in the statement satisfies $u(t,.)=f$ and u is continuously differentiable in s by the regularity assumptions to the transition probabilities. We recall that by Lemma 5.6

$$\begin{aligned}\hat{M}_s^{0,u} &= u(s,X_s)-u(0,X_0)-\int_0^s\left[\frac{\partial}{\partial r}u(r,X_r)+L_r u(r,.)(X_r)\right]dr\\ &=: u(s,X_s)-u(0,X_0)-A_s, \qquad\qquad s\in[0,t],\end{aligned}$$

is a local martingale relative to $P_{0,i}$ for any $i\in I$. Since by Lemma 5.7 $(u(s,X_s))_{0\le s\le t}$ is a $P_{0,i}$-martingale, we conclude that $(A_s)_{0\le s\le t}$ is a continuous local martingale of bounded variation. We recall that such martingales are constant a.s. (cf. Proposition IV.1.12 in [12]), i.e. we obtain $A\equiv 0$ a.s. and thus

$$\frac{\partial}{\partial s}u(s,X_s)+L_s u(s,.)(X_s)=0 \qquad ds\otimes dP_{0,i}\text{ a.e.}$$

Since $\frac{\partial}{\partial s}u(s,j)+L_s u(s,.)(j)$ is continuous in s for every $j\in I$, it follows that u solves (5.5). □

Next we shall show that u, defined as in Proposition 5.8, is the unique solution of the CAUCHY problem in (5.5). To this aim we shall transfer the uniqueness theorem of TYCHONOV (cf. Section 4.3.A in [7]) into our setting. This requires some preparations:
For fixed $s\ge 0$ and $t>s$, we denote by $\tilde{P}_{0,i}^{s,t}$ the probability measure on $D([0,s],I)$, under which the coordinate process $(X_r)_{0\le r\le s}$ is a MARKOV process with state space I, that starts at time $r=0$ in i and is associated with the time-dependent generator $(L_{t-s+r})_{0\le r\le s}$.

Remark 5.9. Analogously to Lemma 5.6 one can show that for arbitrary $t>s\ge 0$, $i\in I$ and g as in Lemma 5.6 the process $(\tilde{M}_r^{s,t,g})_{0\le r\le s}$ defined by

$$\begin{aligned}\tilde{M}_r^{s,t,g} :=& g(t-s+r,X_r)-g(t-s,X_0)\\ &-\int_0^r\left(\frac{\partial}{\partial r'}g(t-s+r',X_{r'})+L_{t-s+r'}g(t-s+r',.)(X_{r'})\right)dr'\end{aligned}$$

is a local martingale relative to $\tilde{P}_{0,i}^{s,t}$.

Theorem 5.10 (TYCHONOV)**.** *For an arbitrary but fixed $t>0$, let u be a function on $[0,t]\times I$ such that $u(.,i)$ is continuously differentiable for all $i\in I$, and suppose that u satisfies*

$$\begin{aligned}\frac{\partial}{\partial s}u(s,i) &= -L_s u(s,.)(i), \qquad \forall i\in I,\, s\in[0,t],\\ u(t,.) &= 0.\end{aligned} \tag{5.6}$$

Then $u=0$ on $[0,t]\times I$.

Proof. Since u satisfies the boundary condition, it suffices to show $u(s,.) = 0$ for all $s \in [0,t)$. Let $s \in [0,t)$ be arbitrary. By Remark 5.9 we know that

$$\begin{aligned}\tilde{M}_r^{s,t,u} &= u(t-s+r, X_r) - u(t-s, X_0)\\ &\quad - \int_0^r \left(\frac{\partial}{\partial r'} u(t-s+r', X_{r'}) + L_{t-s+r'} u(t-s+r', .)(X_{r'})\right) dr', \qquad r \in [0,s],\end{aligned} \tag{5.7}$$

is a local martingale relative to $\tilde{P}_{0,i}^{s,t}$ for some $i \in I$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence. Then the stopped process is a martingale with zero mean. Since u satisfies the Cauchy problem in (5.6), the integral in the right hand side of (5.7) is equal to zero. Thus, taking expectations in (5.7) w.r.t. $\tilde{P}_{0,i}^{s,t}$ leads to

$$u(t-s, i) = \tilde{E}_{0,i}^{s,t}\left[u(t-s+(r\wedge\tau_n), X_{r\wedge\tau_n})\right], \qquad n \in \mathbb{N},\ r \in [0,s].$$

We let n tend to infinity and obtain by the dominated convergence theorem

$$u(t-s, i) = \tilde{E}_{0,i}^{s,t}\left[u(t-s+r, X_r)\right], \qquad r \in [0,s].$$

In particular, choosing $r = s$:

$$u(t-s, i) = \tilde{E}_{0,i}^{s,t}\left[u(t, X_s)\right] = 0,$$

since $u(t,.) = 0$ from (5.6), and the proof is complete. □

Corollary 5.11. *In the situation of Proposition 5.8, $u(s,i) := P_{s,t}f(i)$ is the unique solution of* (5.5)*.*

Proof. Let u_1 and u_2 be two solutions of the Cauchy problem in (5.5). Then $u_1 - u_2$ satisfies (5.6) and from Theorem 5.10 it follows that $u_1 - u_2 = 0$.

□

Now we are able to state a backward Feynman-Kac representation for time-inhomogeneous Markov chains, which is a generalization of the backward equation, since we only modify the Cauchy problem by adding a linear inhomogenity κ:

Theorem 5.12. *For an arbitrary but fixed $T > 0$, let v be a real-valued function on $[0,T] \times I$ such that $v(.,i)$ is continuously differentiable for every $i \in I$ and suppose that v satisfies the* Cauchy *problem*

$$\begin{aligned}-\frac{\partial}{\partial s} v(s,i) &= L_s v(s,.)(i) - \kappa(s,i)\, v(s,i), \qquad \forall s \in [0,T],\ i \in I,\\ v(T,.) &= f,\end{aligned} \tag{5.8}$$

where κ denotes a real-valued function on $[0,T] \times I$ such that $\kappa(.,i)$ is continuous for every $i \in I$. Then, v admits the stochastic representation on $[0,T] \times I$:

$$v(s,i) = E_{s,i}\left[f(X_T) \exp\left(-\int_s^T \kappa(r, X_r)\, dr\right)\right]. \tag{5.9}$$

Proof. In the case $s = T$ the representation in (5.9) is obvious. For any $s \in [0, T)$ and $i \in I$, the process

$$\hat{M}_t^{s,v} = v(t, X_t) - v(s, X_s) - \int_s^t \left(\frac{\partial}{\partial r} v(r, X_r) + L_r v(r, .)(X_r) \right) dr, \qquad t \in [s, T],$$

is a local martingale relative to $P_{s,i}$ by Lemma 5.6. We write this as

$$dv(t, X_t) = \left(\frac{\partial}{\partial t} v(t, X_t) + L_t v(t, .)(X_t) \right) dt + d\hat{M}_t^{s,v}.$$

Using the integration by parts formula and the fact that v is a solution of (5.8) we obtain

$$\begin{aligned} & d\left[v(t, X_t) \, \exp\left(- \int_s^t \kappa(r, X_r) \, dr \right) \right] \\ &= \exp\left(- \int_s^t \kappa(r, X_r) \, dr \right) [dv(t, X_t) - \kappa(t, X_t) \, v(t, X_t) \, dt] \\ &= \exp\left(- \int_s^t \kappa(r, X_r) \, dr \right) \left[d\hat{M}_t^{s,v} + \left(\frac{\partial}{\partial t} v(t, X_t) + L_t v(t, .)(X_t) - \kappa(t, X_t) \, v(t, X_t) \right) dt \right] \\ &= \exp\left(- \int_s^t \kappa(r, X_r) \, dr \right) d\hat{M}_t^{s,v}, \end{aligned}$$

so that

$$v(t, X_t) \exp\left(- \int_s^t \kappa(r, X_r) \, dr \right) = v(s, X_s) + \int_s^t \exp\left(- \int_s^u \kappa(r, X_r) \, dr \right) d\hat{M}_u^{s,v}, \qquad t \in [s, T]. \tag{5.10}$$

Notice that the integral in the right hand side of (5.10) is a local martingale relative to $P_{s,i}$. Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence. Since the stopped process is a martingale with zero mean, taking expectation w.r.t. $P_{s,i}$ in (5.10) yields:

$$v(s, i) = E_{s,i} \left[v(t \wedge \tau_n, X_{t \wedge \tau_n}) \, \exp\left(- \int_s^{t \wedge \tau_n} \kappa(r, X_r) \, dr \right) \right], \qquad n \in \mathbb{N}.$$

We let n tend to infinity and obtain by the dominated convergence theorem:

$$v(s, i) = E_{s,i} \left[v(t, X_t) \, \exp\left(- \int_s^t \kappa(r, X_r) \, dr \right) \right], \qquad t \in [s, T]. \tag{5.11}$$

In particular, choosing $t = T$ in (5.11), we obtain

$$v(s, i) = E_{s,i} \left[f(X_T) \, \exp\left(- \int_s^T \kappa(r, X_r) \, dr \right) \right],$$

which completes the proof. □

5.3 A Forward Feynman-Kac Representation

For the forward version of the FEYNMAN-KAC formula it is required to consider a transformation of the underlying MARKOV process in the following manner: For arbitrary but fixed $s \geq 0$ and $T \geq s$, we define the time-reversion

$$\mu(t) := s + T - t, \qquad t \in [s, T],$$

to simplify the notation in the sequel. Furthermore, for $i \in I$ let $\hat{P}^T_{s,i}$ be the probability measure on $D([0,T], I)$, under which the coordinate process $(X_r)_{0\le r\le T}$ is constant equal to i up to time $r = s$ and $(X_r)_{s\le r\le T}$ is a MARKOV process with state space I, that starts at time $r = s$ in i and is associated with the time-dependent generator $(L_{\mu(r)})_{s\le r\le T}$. Note that under $\hat{P}^s_{s,i}$ we have simply $(X_r)_{0\le r\le s} \equiv i$.

Remark 5.13. Analogously to Lemma 5.6 one can show that for arbitrary $s \ge 0$, $i \in I$ and $T > s$ and g as in Lemma 5.6 the process $(\bar{M}^{s,T,g}_t)_{s\le t\le T}$ defined by

$$\bar{M}^{s,T,g}_t := g(\mu(t), X_t) - g(\mu(s), X_s) - \int_s^t \left(-\frac{\partial}{\partial r} g(\mu(r), X_r) + L_{\mu(r)} g(\mu(r), .)(X_r) \right) dr$$

is a local martingale relative to $\hat{P}^T_{s,i}$.

This can be generalized as follows:

Lemma 5.14. *For arbitrary $s \ge 0$, $i \in I$ and $T > s$, we consider the process $(H_t)_{s\le t\le T}$ defined by*

$$H_t := (\mu(t), X_t, Z_t) \quad \text{with } Z_t := \int_s^t \kappa(\mu(r), X_r)\, dr,$$

where $\kappa : [0,\infty) \times I \to \mathbb{R}$ is a function such that $\kappa(., j)$ is continuous for every $j \in I$. Moreover, let $\phi : [0,\infty) \times I \times \mathbb{R} \to \mathbb{R}$ be a function such that $\frac{\partial}{\partial t}\phi(t, j, z)$ and $\frac{\partial}{\partial z}\phi(t, j, z)$ exist and are continuous for every $j \in I$. Then, the process $(\check{M}^{s,T,\phi})_{s\le t\le T}$ defined by

$$\begin{aligned}
\check{M}^{s,T,\phi}_t &:= \phi(H_t) - \phi(H_s) \\
&\quad - \int_s^t \left(-\frac{\partial}{\partial r}\phi(H_r) + L_{\mu(r)}\phi(\mu(r), ., Z_r)(X_r) + \kappa(\mu(r), X_r)\frac{\partial}{\partial z}\phi(H_r) \right) dr \\
&=: \phi(H_t) - \phi(H_s) - \int_s^t L_H\phi(H_r)\, dr \qquad (5.12)
\end{aligned}$$

is a local martingale relative to $\hat{P}^T_{s,i}$.

Proof. It is sufficient to prove the lemma when ϕ has the form

$$\phi(t, j, z) = \tilde{h}(z)\, g(t, j)$$

with $\tilde{h} \in C^1(\mathbb{R})$ and g as above in Lemma 5.6 (for general ϕ we can use a monotone-class argument). By definition of $\bar{M}^{s,T,g}$ in Remark 5.13 we have

$$dg(\mu(t), X_t) = d\bar{M}^{s,T,g}_t + \left(-\frac{\partial}{\partial t} g(\mu(t), X_t) + L_{\mu(t)} g(\mu(t), .)(X_t) \right) dt.$$

Applying the integration by parts formula yields:

$$\begin{aligned}
d\phi(H_t) &= g(\mu(t), X_t)\, d\tilde{h}(Z_t) + \tilde{h}(Z_t)\, dg(\mu(t), X_t) \\
&= \kappa(\mu(t), X_t)\, \tilde{h}'(Z_t)\, g(\mu(t), X_t)\, dt \\
&\quad + \tilde{h}(Z_t) \left[\left(-\frac{\partial}{\partial t} g(\mu(t), X_t) + L_{\mu(t)} g(\mu(t), .)(X_t) \right) dt + d\bar{M}^{s,T,g}_t \right] \\
&= \left[\kappa(\mu(t), X_t)\, \frac{\partial}{\partial z}\phi(H_t) + L_{\mu(t)}\phi(\mu(t), ., Z_t)(X_t) - \frac{\partial}{\partial t}\phi(H_t) \right] dt + \tilde{h}(Z_t)\, d\bar{M}^{s,T,g}_t.
\end{aligned}$$

By Remark 5.13 the last term is a local martingale and the result follows.

□

Now we are able to prove a forward version of the Feynman-Kac formula, following the idea of Theorem 8.2.1 in [11]:

Theorem 5.15. *For an arbitrary but fixed $s \in [0,\infty)$, let $v = v(s,.,.)$ be a real-valued function on $[s,\infty) \times I$ such that $v(s,.,i)$ is continuously differentiable for every $i \in I$ and suppose that v satisfies the* Cauchy *problem*

$$\begin{aligned} \frac{\partial}{\partial t} v(s,t,i) &= L_t v(s,t,.)(i) - \kappa(t,i)\, v(s,t,i), \qquad \forall t \in [s,\infty),\ i \in I, \\ v(s,s,.) &= f, \end{aligned} \tag{5.13}$$

where κ denotes a real-valued function on $[0,\infty) \times I$ such that $\kappa(.,i)$ is continuous for every $i \in I$. Then, v admits the stochastic representation on $[s,\infty) \times I$:

$$v(s,t,i) = \hat{E}^t_{s,i}\left[f(X_t) \exp\left(- \int_s^t \kappa(s+t-r, X_r)\, dr \right)\right]. \tag{5.14}$$

Proof. Clearly, the representation in (5.14) holds for $t = s$. We use the same notation as in Lemma 5.14. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of $\check{M}^{s,T,\phi}$, i.e. for every $n \in \mathbb{N}$ the stopped process $(\check{M}^{s,T,\phi})^{\tau_n}$ is a martingale relative to $\hat{P}^T_{s,i}$ with zero mean. Thus, taking expectations in (5.12) yields:

$$\hat{E}^T_{s,i}\left[\phi(H_{t\wedge\tau_n})\right] = \phi(T,i,0) + \hat{E}^T_{s,i}\left[\int_s^{t\wedge\tau_n} L_H \phi(H_r)\, dr\right], \qquad t \in [s,T].$$

Choosing a special ϕ by

$$\phi(t,j,z) = \exp(-z)\, v(s,t,j),$$

we obtain for all $t \in [s,T]$:

$$\begin{aligned} L_H\phi(H_t) &= \exp(-Z_t)\left(-\frac{\partial}{\partial t} v(s,\mu(t),X_t) + L_{\mu(t)} v(s,\mu(t),.)(X_t) - \kappa(\mu(t),X_t)\, v(s,\mu(t),X_t)\right) \\ &= 0, \end{aligned}$$

since v satisfies (5.13). Thus we have for all $t \in [s,T]$ by our choice of ϕ, using the dominated convergence theorem

$$\begin{aligned} v(s,T,i) = \phi(T,i,0) &= \hat{E}^T_{s,i}\left[\phi(H_{t\wedge\tau_n})\right] \\ &= \hat{E}^T_{s,i}\left[\exp\left(-\int_s^{t\wedge\tau_n} \kappa(\mu(r),X_r)\, dr\right) v(s,\mu(t\wedge\tau_n),X_{t\wedge\tau_n})\right] \\ &\longrightarrow \hat{E}^T_{s,i}\left[\exp\left(-\int_s^{t} \kappa(\mu(r),X_r)\, dr\right) v(s,\mu(t),X_t)\right], \qquad \text{as } n \uparrow \infty. \end{aligned}$$

In particular, choosing $t = T$, we obtain

$$v(s,T,i) = \hat{E}^T_{s,i}\left[\exp\left(-\int_s^T \kappa(s+T-r,X_r)\, dr\right) f(X_T)\right].$$

Since $T > s$ is arbitrary, the result follows.

□

6 Random Walk Reprensentation

In this section we discuss an alternative approach to prove the random walk representation given in Theorem 2.2, while we assume that the drift coefficients have nonnegative derivatives. The idea is based on the penalization method and an application of the FEYNMAN-KAC formula, established in the preceding section. This approach seems to be the most convenient method to prove random walk representations, but, unfortunately, it does not lead to a complete proof here.

We consider the following system of SKOROHOD SDEs for a finite set of indices I, introduced in Section 2.2:

$$\begin{aligned} &X_t^i(x) = x^i + \int_0^t b^i(X_r(x))\,dr + l_t^i(x) + w_t^i, \qquad t \geq 0,\, i \in I, \\ &X_t^i(x) \geq 0, \quad dl_t^i(x) \geq 0, \quad \int_0^\infty X_t^i(x)\,dl_t^i(x) = 0, \qquad i \in I, \end{aligned} \tag{6.1}$$

for $x \in \mathbb{R}_+^I$ and a BROWNian motion w as before. In this section we number the elements of I, i.e. without loss of generality we set $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. Recall that the coefficients b^i, $i \in I$, are supposed to be continuously differentiable and LIPSCHITZ continuous. For all $i \in I$ we extend the domain of b^i to $\mathbb{R}^I$ by setting $b^i(x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^N) = b^i(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^N)$ for $x^k \leq 0$, $k = 1, \ldots, N$. Moreover, we assume the partial derivatives of b^i to be nonnegative:

$$\frac{\partial b^i}{\partial x^j}(x) \geq 0, \qquad \forall x \in \mathbb{R}^I, \quad i, j \in I. \tag{6.2}$$

First we shall apply the penalization method to approximate the solution of (6.1). To this aim we shall use some of the techniques in [5] and [10].

6.1 Penalized SDEs

We replace the local times $dl_t^i(x)$, $i \in I$, appearing in the SDE (6.1) by the penalization term $\frac{1}{\varepsilon}\left(X_t^i(x)\right)^- dt$, $\varepsilon > 0$, i.e. the penalized SDE is given by

$$X_t^{\varepsilon,i}(x) = x^i + \int_0^t \left(b^i(X_r^\varepsilon(x)) + \frac{1}{\varepsilon}\left(X_r^{\varepsilon,i}(x)\right)^-\right) dr + w_t^i, \qquad t \geq 0,\, i \in I. \tag{6.3}$$

Notice that the solution of the penalized SDE can take negative values (for this reason the domain of the coefficients b^i, $i \in I$, was extended to $\mathbb{R}^I$). Setting

$$h_\varepsilon^i(x) := b^i(x) + \frac{1}{\varepsilon}\left(x^i\right)^-, \qquad i \in I,\, \varepsilon > 0, \tag{6.4}$$

the SDE (6.3) reads

$$X_t^{\varepsilon,i}(x) = x^i + \int_0^t h_\varepsilon^i(X_r^\varepsilon(x))\,dr + w_t^i, \qquad t \geq 0,\, i \in I. \tag{6.5}$$

We observe that for all $i \in I$ the LIPSCHITZ continuity of the functions b^i and $x \mapsto (x)^-$, $x \in \mathbb{R}$, implies that h_ε^i is also LIPSCHITZ continuous for all $\varepsilon > 0$. Thus, existence and uniqueness of solutions of the penalized SDE are ensured for every $\varepsilon > 0$. As we have mentioned above, we shall prove that the solutions of the SDE (6.1) can be approximated by those of the penalized SDE in (6.3). First we state a comparison theorem for the penalized SDEs.

In the sequel the argument x of the process X resp. X^ε is suppressed.

Proposition 6.1. *Assume that* $0 < \varepsilon < \bar{\varepsilon} < 1$. *Then,* $X_t^{\bar{\varepsilon},i} \leq X_t^{\varepsilon,i}$ *holds a.s. for all* $t \geq 0$ *and* $i \in I$.

Observe that for $0 < \varepsilon < \bar{\varepsilon} < 1$

$$h_{\bar{\varepsilon}}^i(x) \leq h_{\varepsilon}^i(x), \qquad \forall x \in \mathbb{R}^I,\ i \in I. \tag{6.6}$$

Unfortunately, we cannot apply the comparison theorem in Proposition 5.2.18 in [7] to prove Proposition 6.1, because this is only applicable in the one-dimensional case. So we prove Proposition 6.1 in a similar manner to Proposition 4.2 in [5]:

Proof. We have to show that for fixed $0 < \varepsilon < \bar{\varepsilon} < 1$

$$\Delta_t^i := X_t^{\bar{\varepsilon},i} - X_t^{\varepsilon,i} \leq 0, \qquad \forall t \geq 0,\ i \in I.$$

Recall that we have set $I = \{1, \ldots, N\}$. We introduce the following notation:

$$\begin{aligned} X_t^{\varepsilon,\bar{\varepsilon},k} &:= \left(X_t^{\bar{\varepsilon},1}, \ldots, X_t^{\bar{\varepsilon},k}, X_t^{\varepsilon,k+1}, \ldots, X_t^{\varepsilon,N} \right), \quad k = 1, \ldots, N, \\ X_t^{\varepsilon,\bar{\varepsilon},0} &:= X_t^{\varepsilon} \end{aligned}$$

for all $t \geq 0$. Note that $X_t^{\varepsilon,\bar{\varepsilon},k}$ and $X_t^{\varepsilon,\bar{\varepsilon},k-1}$, $k = 1, \ldots, N$, differ only in the k-th component and the k-th component of their difference is equal to Δ_t^k.
We apply the mean value theorem of differential calculus to obtain for all $i \in I$ and $t \geq 0$:

$$\begin{aligned} b^i(X_t^{\bar{\varepsilon}}) - b^i(X_t^{\varepsilon}) =& \sum_{k=1}^N b^i(X_t^{\varepsilon,\bar{\varepsilon},k}) - b^i(X_t^{\varepsilon,\bar{\varepsilon},k-1}) = \sum_{k=1}^N \frac{\partial b^i(\xi^k)}{\partial x^k} \Delta_t^k \\ \text{with} \quad \xi^k :=& X_t^{\varepsilon,k} + \vartheta_k \left(X_t^{\bar{\varepsilon},k} - X_t^{\varepsilon,k} \right) \qquad \text{for some } \vartheta_k \in (0,1),\ k = 1, \ldots, N. \end{aligned} \tag{6.7}$$

By our assumptions we have

$$\frac{\partial b^i}{\partial x^k}(x) \in [0, K_{b^i}], \qquad \forall x \in \mathbb{R}^I,\ i, k \in I,$$

where K_{b^i} denotes again the Lipschitz constant of b^i, $i \in I$. Hence, we can estimate (6.7) as follows:

$$b^i(X_t^{\bar{\varepsilon}}) - b^i(X_t^{\varepsilon}) \leq \sum_{k=1}^N K_{b^i} \left(\Delta_t^k \right)^+ = K_{b^i} \sum_{k=1}^N \left(\Delta_t^k \right)^+. \tag{6.8}$$

Note that $\Delta_t^i = \int_0^t \left(h_{\bar{\varepsilon}}^i(X_r^{\bar{\varepsilon}}) - h_{\varepsilon}^i(X_r^{\varepsilon}) \right) dr$, $t \geq 0$, is a continuous process of bounded variation for every $i \in I$. Thus, using the integration by parts formula, (6.5) and (6.6) we obtain:

$$\begin{aligned} \frac{1}{2} d \left\{ \sum_{i \in I} \left[(\Delta_t^i)^+ \right]^2 \right\} &= \frac{1}{2} \sum_{i \in I} d \left[(\Delta_t^i)^+ \right]^2 = \sum_{i \in I} (\Delta_t^i)^+ \, d\Delta_t^i = \sum_{i \in I} (\Delta_t^i)^+ \left(h_{\bar{\varepsilon}}^i(X_t^{\bar{\varepsilon}}) - h_{\varepsilon}^i(X_t^{\varepsilon}) \right) dt \\ &= \sum_{i \in I} (\Delta_t^i)^+ \left[\left(h_{\bar{\varepsilon}}^i(X_t^{\bar{\varepsilon}}) - h_{\bar{\varepsilon}}^i(X_t^{\varepsilon}) \right) + \left(h_{\bar{\varepsilon}}^i(X_t^{\varepsilon}) - h_{\varepsilon}^i(X_t^{\varepsilon}) \right) \right] dt \\ &\leq \sum_{i \in I} (\Delta_t^i)^+ \left(h_{\bar{\varepsilon}}^i(X_t^{\bar{\varepsilon}}) - h_{\bar{\varepsilon}}^i(X_t^{\varepsilon}) \right) dt. \end{aligned}$$

From (6.4), (6.8) and the fact that $x \mapsto (x)^-$ is LIPSCHITZ continuous, we obtain

$$\begin{aligned}
\frac{1}{2}d\left\{\sum_{i\in I}\left[\left(\Delta_t^i\right)^+\right]^2\right\} &\leq \left\{\sum_{i\in I}\left(\Delta_t^i\right)^+\left(b^i(X_t^{\bar\varepsilon})-b^i(X_t^{\varepsilon})\right)+\frac{1}{\bar\varepsilon}\sum_{i\in I}\left(\Delta_t^i\right)^+\left[\left(X_t^{\bar\varepsilon,i}\right)^- - \left(X_t^{\varepsilon,i}\right)^-\right]\right\}dt\\
&\leq \left\{\sum_{i\in I}\left(\Delta_t^i\right)^+ K_{b^i}\sum_{k\in I}\left(\Delta_t^k\right)^+ + C_1\sum_{i\in I}\left(\Delta_t^i\right)^+\left|\Delta_t^i\right|\right\}dt\\
&\leq \left\{C_2\left[\sum_{i\in I}\left(\Delta_t^i\right)^+\right]^2 + C_1\sum_{i\in I}\left[\left(\Delta_t^i\right)^+\right]^2\right\}dt,
\end{aligned}$$

and by the CAUCHY-SCHWARZ inequality we get

$$\frac{1}{2}d\left\{\sum_{i\in I}\left[\left(\Delta_t^i\right)^+\right]^2\right\} \leq C_3\sum_{i\in I}\left[\left(\Delta_t^i\right)^+\right]^2 dt, \tag{6.9}$$

where C_i, $i \in \{1,2,3\}$, are positive constants not depending on t. We integrate both sides of (6.9) in t and, since $\Delta_0^i = 0$ for every $i \in I$, this involves

$$\frac{1}{2}\sum_{i\in I}\left[\left(\Delta_t^i\right)^+\right]^2 \leq C_3\int_0^t\sum_{i\in I}\left[\left(\Delta_r^i\right)^+\right]^2 dr. \tag{6.10}$$

Now we can apply GRONWALL's Lemma to (6.10) to obtain

$$\sum_{i\in I}\left[\left(\Delta_t^i\right)^+\right]^2 = 0, \qquad t \geq 0,\, i \in I,$$

in particular $\left(\Delta_t^i\right)^+ = 0$ for all $i \in I$ and $t \geq 0$, which completes the proof. □

Next we shall prove the existence of $\sup_{0<\varepsilon<1} X_t^{\varepsilon}$. To this aim we need the following proposition, which shows that the solution of the SDE with reflection term in (6.1) is an upper bound of the solutions of the penalized SDE (6.3) (cf. again Proposition 4.2 in [5]).

Proposition 6.2. *For all $0 < \varepsilon < 1$, $X_t^{\varepsilon,i} \leq X_t^i$ holds a.s. for all $t \geq 0$ and $i \in I$.*

Proof. Proceeding as in the proof of Proposition 6.1, we have to show that for any fixed $0 < \varepsilon < 1$

$$\psi_t^i := X_t^{\varepsilon,i} - X_t^i \leq 0, \qquad \forall t \geq 0,\, i \in I.$$

Analogously to (6.7) and (6.8) we can conclude that

$$b^i(X_t^{\varepsilon}) - b^i(X_t) \leq K_{b^i}\sum_{k=1}^N\left(\psi_t^k\right)^+. \tag{6.11}$$

Furthermore, if $\psi_t^i > 0$, we have $X_t^{\varepsilon,i} > 0$ and therefore $\left(X_t^{\varepsilon,i}\right)^- = 0$ for every $i \in I$. Since on the other hand $dl_t^i \geq 0$ for all $t \geq 0$, $i \in I$, it follows that

$$\left(\psi_t^i\right)^+\left[\frac{1}{\varepsilon}\left(X_t^{i,\varepsilon}\right)^- dt - dl_t^i\right] \leq 0, \qquad \forall t \geq 0, i \in I. \tag{6.12}$$

Since ψ^i, $i \in I$, is a continuous process of bounded variation, we obtain by the integration by parts formula, (6.1), (6.3), (6.11) and (6.12)

$$
\begin{aligned}
\frac{1}{2}d\left\{\sum_{i\in I}\left[\left(\psi_t^i\right)^+\right]^2\right\} &= \sum_{i\in I}\left(\psi_t^i\right)^+ d\psi_t^i \\
&= \sum_{i\in I}\left(\psi_t^i\right)^+\left[b^i(X_t^\varepsilon) - b^i(X_t)\right]dt + \sum_{i\in I}\left(\psi_t^i\right)^+\left[\frac{1}{\varepsilon}\left(X_t^{i,\varepsilon}\right)^- dt - dl_t^i\right] \\
&\le \sum_{i\in I}\left(\psi_t^i\right)^+ K_{b^i}\sum_{k\in I}\left(\psi_t^k\right)^+ dt \\
&\le C_1\left[\sum_{i\in I}\left(\psi_t^i\right)^+\right]^2 dt,
\end{aligned}
$$

and finally by the Cauchy-Schwarz inequality

$$
\frac{1}{2}d\left\{\sum_{i\in I}\left[\left(\psi_t^i\right)^+\right]^2\right\} \le C_2\sum_{i\in I}\left[\left(\psi_t^i\right)^+\right]^2 dt, \tag{6.13}
$$

where C_1 and C_2 are positive constants independent of t. Clearly, $\psi_0^i = 0$ for all $i \in I$ and we can apply Gronwall's Lemma to (6.13) to obtain

$$
\sum_{i\in I}\left[\left(\psi_t^i\right)^+\right]^2 = 0, \qquad \forall t \ge 0,\ i \in I,
$$

in particular, $\left(\psi_t^i\right)^+ = 0$ for all $t \ge 0$ and $i \in I$, and the proof is complete.

□

Remark 6.3. From Proposition 6.1 and Proposition 6.2 we can conclude that a.s.

$$
\tilde{X}_t := \lim_{\varepsilon\downarrow 0} X_t^\varepsilon = \sup_{0<\varepsilon<1} X_t^\varepsilon \tag{6.14}
$$

exists for all $t \ge 0$.

Now we are able to show that even $X_t = \sup_{0<\varepsilon<1} X_t^\varepsilon$, respectively $X_t = \lim_{\varepsilon\downarrow 0} X_t^\varepsilon$, a.s. for all $t \ge 0$. We shall proceed in a similar manner to [10] pp. 84-85.

Theorem 6.4. *$X_t = \sup_{0<\varepsilon<1} X_t^\varepsilon$ holds a.s. for all $t \ge 0$.*

Proof. Defining $\tilde{X}_t := \sup_{0<\varepsilon<1} X_t^\varepsilon$ for $t \ge 0$ as above in (6.14), it is sufficient to show that a.s.

i) $\tilde{X}_t^i \ge 0, \qquad \forall t \ge 0,\ i \in I,$

ii) $d\tilde{X}_t^i = b^i(\tilde{X}_t)\,dt + l_t^i + w_t^i$ for every $i \in I$, where $dl_t^i \ge 0$ and $\int_0^\infty \tilde{X}_s^i\,dl_s^i = 0$,

since the solution of that SDE is unique. X^ε satisfies by definition

$$
dX_t^{\varepsilon,i} = \left\{b^i(X_t^\varepsilon) + \frac{1}{\varepsilon}\left(X_t^{\varepsilon,i}\right)^-\right\}dt + dw_t^i, \qquad i \in I, \varepsilon > 0. \tag{6.15}
$$

We multiply both sides of (6.15) by ε and let ε tend to zero. Since Proposition 6.2 ensures the existence of $\lim_{\varepsilon\downarrow 0} X_t^\varepsilon$ a.s. for all $t \geq 0$, this leads to

$$\int_0^t \left(\tilde{X}_s^i\right)^- ds = 0, \qquad t \geq 0,\, i \in I,$$

and therefore $\tilde{X}_t^i \geq 0$ for all $t \geq 0$ and $i \in I$, which proves i). We define for all $\varepsilon \in (0,1)$ and $i \in I$ the random measure $l^{\varepsilon,i}$ on $[0,\infty)$ by

$$dl_t^{\varepsilon,i} := \frac{1}{\varepsilon}\left(X_t^{\varepsilon,i}\right)^- dt.$$

Then, equation (6.15) combined with Remark 6.3 ensures that a.s. for every $i \in I$, $l^{\varepsilon,i}$ converges weakly to some positive measure l^i on $[0,\infty)$ as $\varepsilon \downarrow 0$. By letting ε tend to zero in (6.15), it becomes clear that $\tilde{X}$ satisfies the SDE in ii).
Thus, it only remains to check that the limiting measure l^i is in fact the local time of $\tilde{X}$ in zero: Obviously $t \mapsto l_t^i$ is non-decreasing for all $i \in I$, since $dl_t^{\varepsilon,i} \geq 0$ for all ε. On the other hand, by definition of $l^{\varepsilon,i}$ it is clear that

$$\operatorname{supp} l^{\varepsilon,i} \subseteq \left\{t : X_t^{\varepsilon,i} \leq 0\right\}, \qquad \forall \varepsilon \in (0,1),\, i \in I.$$

Note that the set $\left\{t : X_t^{\varepsilon,i} \leq 0\right\}$ decreases, when ε decreases. Hence, we can conclude that

$$\operatorname{supp} l^{i} \subseteq \left\{t : X_t^{\varepsilon,i} \leq 0\right\}, \qquad \forall \varepsilon \in (0,1),\, i \in I$$

and therefore, since $dl_t^i \geq 0$ for all $i \in I$:

$$\int_0^t X_s^{\varepsilon,i}\, dl_s^i \leq 0, \qquad \forall t \geq 0,\, \varepsilon \in (0,1),\, i \in I.$$

We apply the monotone convergence theorem to obtain

$$\int_0^t \tilde{X}_s^i\, dl_s^i \leq 0, \qquad \forall t \geq 0,\, i \in I,$$

and, since $\tilde{X}_t^i \geq 0$ for all $t \geq 0$ and $i \in I$, it follows that

$$\int_0^t \tilde{X}_s^i\, dl_s^i = 0, \qquad \forall t \geq 0,\, i \in I,$$

and therefore

$$\int_0^\infty \tilde{X}_s^i\, dl_s^i = 0, \qquad i \in I,$$

which completes the proof. □

6.2 Random Walk Representation for Penalized SDEs

In this subsection we shall establish a random walk representation for the derivatives of the solution of the penalized SDE. To this aim we shall use the forward FEYNMAN-KAC formula developed in Section 5. First we need to modify and extend the notation introduced in Section 2: Recall that $E := D([0,\infty), I)$ denotes the space of I-valued càdlàg functions and $\xi_t : E \to I$, $t \in [0,\infty)$, the coordinate process. In contrast to Section 2 the continuous function $c : [0,\infty) \times I \times I \to \mathbb{R}$ is supposed to be nonnegative, so that some other definitions become a bit simpler. Namely, for all $s \in [0,\infty)$ and $i \in I$, $P^c_{s,i}$ denotes the probability measure on E, under which

- $\xi_t = i$ for all $t \in [0,s]$,
- $(\xi_t)_{t\in[s,\infty)}$ has the law of the time continuous MARKOV chain with values in I starting at $t = s$ from i and with time-dependent generator $(L^c_t)_{t\geq 0}$:

$$L^c_t : \mathbb{R}^I \to \mathbb{R}^I, \qquad L^c_t f(i) := \sum_{k\in I} c_t(i,k)\,(f(k) - f(i)),$$

and the function $\rho^c_{s,t}$, $0 \leq s \leq t$, becomes

$$\rho^c_{s,t} := \exp\left(\int_s^t \sum_{k\in I} c_r(\xi_r, k)\, dr\right).$$

Fix now any arbitrary $T > 0$. In the following we shall need a time-reverse argument. We set for all $0 \leq s \leq t$ and $i, j \in I$:

$$\hat{c}_t(i,j) := c_{T-t}(j,i) \quad \text{if } t \leq T, \qquad \hat{c}_t(i,j) := c_0(j,i) \quad \text{if } t \geq T,$$
$$\hat{P}^c_{0,j} := P^{\hat{c}}_{0,j}, \qquad \hat{\rho}^c_{s,t} := \rho^{\hat{c}}_{s,t}.$$

Let $E_T := D([0,T], I)$. If $e : [0,T] \to I$ has right limit at any $t \in [0,T]$, we set $e^*(t) := \lim_{s\downarrow t} e(s)$. Then, for all $e \in E_T$, $[e_{T-.}]^* \in E_T$.

Lemma 6.5. *For all bounded* BOREL *measurable* $\Phi : E_T \to \mathbb{R}$ *and* $i, j \in I$*:*

$$E^c_{0,i}\left[\Phi(\xi)\,\mathbb{1}_{\{\xi_T=j\}}\,\rho^c_{0,T}\right] = \hat{E}^c_{0,j}\left[\Phi([\xi_{T-.}]^*)\,\mathbb{1}_{\{\xi_T=i\}}\,\hat{\rho}^c_{0,T}\right].$$

Proof. See Lemma 4 in [3]. □

Now we consider again for $\varepsilon > 0$ the solution $X^\varepsilon(x)$ of the penalized SDE

$$X^{\varepsilon,i}_t(x) = x^i + \int_0^t \left(b^i(X^\varepsilon_r(x)) + \frac{1}{\varepsilon}\left(X^{\varepsilon,i}_r(x)\right)^-\right) dr + w^i_t, \qquad t \geq 0,\ i \in I, \tag{6.16}$$

and define

$$c^\varepsilon(t,i,j) := \frac{\partial b^j}{\partial x^i}(X^\varepsilon_t(x)).$$

Notice that we have assumed c^ε to be nonnegative in (6.2). Set

$$\tilde{c}^\varepsilon(t,i,j) := c^\varepsilon(t,j,i),$$

in particular:

$$\tilde{c}^\varepsilon(t,i,j) = \hat{c}^\varepsilon(T-t,i,j), \qquad t \leq T. \tag{6.17}$$

Theorem 6.6. *For any $\varepsilon > 0$ the partial derivatives of the mapping $x \mapsto X_t^\varepsilon(x)$ exist a.s. for all $t \geq 0$ and admit a.s. the following random walk representation:*

$$\frac{\partial X_t^{\varepsilon,i}(x)}{\partial x^j} = E_{0,j}^{c^\varepsilon}\left[\mathbb{1}_{\{\xi_t=i\}} \exp\left(-\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{(-\infty,0]}(X_r^{\varepsilon,\xi_r}(x))\,dr\right)\rho_{0,t}^{c^\varepsilon}\right], \qquad i,j \in I.$$

Proof. From Theorem 2.1 we know that for all $t \geq 0$ a.s. the mapping $x \mapsto X_t^\varepsilon(x)$ is continuously differentiable and for all $i,j \in I$ the derivatives satisfy a.s.:

$$\frac{\partial}{\partial x^j} X_t^{\varepsilon,i}(x) = \delta_{ij} + \int_0^t \left\{\sum_{k\in I} \frac{\partial b^i}{\partial x^k}(X_r^\varepsilon(x))\,\frac{\partial X_r^{\varepsilon,k}(x)}{\partial x^j} - \frac{1}{\varepsilon}\,\mathbb{1}_{(-\infty,0]}(X_r^{\varepsilon,i}(x))\,\frac{\partial X_r^{\varepsilon,i}(x)}{\partial x^j}\right\} dr.$$

Therefore, we have for all $i,j \in I$ and $t \geq 0$:

$$\begin{aligned}
\frac{\partial^2}{\partial t\,\partial x^j} X_t^{\varepsilon,i}(x) &= \sum_{k\in I} \frac{\partial b^i}{\partial x^k}(X_t^\varepsilon(x))\,\frac{\partial X_t^{\varepsilon,k}(x)}{\partial x^j} - \frac{1}{\varepsilon}\,\mathbb{1}_{(-\infty,0]}(X_t^{\varepsilon,i}(x))\,\frac{\partial X_t^{\varepsilon,i}(x)}{\partial x^j} \\
&= \sum_{k\in I} \frac{\partial b^i}{\partial x^k}(X_t^\varepsilon(x))\left(\frac{\partial X_t^{\varepsilon,k}(x)}{\partial x^j} - \frac{\partial X_t^{\varepsilon,i}(x)}{\partial x^j}\right) \\
&\quad + \left(\sum_{k\in I} \frac{\partial b^i}{\partial x^k}(X_t^\varepsilon(x)) - \frac{1}{\varepsilon}\,\mathbb{1}_{(-\infty,0]}(X_t^{\varepsilon,i}(x))\right)\frac{\partial X_t^{\varepsilon,i}(x)}{\partial x^j}.
\end{aligned} \tag{6.18}$$

For fixed $j \in I$ we set

$$v^\varepsilon(t,i) := \frac{\partial X_t^{\varepsilon,i}(x)}{\partial x^j}, \quad \kappa^\varepsilon(t,i) := \sum_{k\in I} \frac{\partial b^i}{\partial x^k}(X_t^\varepsilon(x)) - \frac{1}{\varepsilon}\,\mathbb{1}_{(-\infty,0]}(X_t^{\varepsilon,i}(x)), \qquad t \geq 0,\ i \in I,$$

in particular

$$v^\varepsilon(0,i) = \delta_{ij}.$$

It follows from equation (6.18) that v^ε satisfies the Cauchy problem

$$\begin{aligned}
\frac{\partial}{\partial t} v^\varepsilon(t,i) &= L_t^{\tilde{c}^\varepsilon} v(t,.)(i) + \kappa^\varepsilon(t,i)\,v^\varepsilon(t,i), && \text{on } [0,\infty)\times I, \\
v^\varepsilon(0,i) &= \delta_{ij}, && \forall i \in I.
\end{aligned}$$

We apply the forward Feynman-Kac formula, established in Theorem 5.15, for $t = T$ and obtain using (6.17):

$$v^\varepsilon(T,i) = \hat{E}_{0,i}^{c^\varepsilon}\left[\mathbb{1}_{\{\xi_T=j\}} \exp\left(\int_0^T \kappa^\varepsilon(T-r,\xi_r)\,dr\right)\right].$$

In particular, we have:

$$\begin{aligned}
\frac{\partial X_T^{\varepsilon,i}(x)}{\partial x^j} &= \hat{E}_{0,i}^{c^\varepsilon}\left[\mathbb{1}_{\{\xi_T=j\}} \exp\left(-\frac{1}{\varepsilon}\int_0^T \mathbb{1}_{(-\infty,0]}\left(X_{T-r}^{\varepsilon,\xi_r}(x)\right)\,dr\right)\hat{\rho}_{0,T}^{c^\varepsilon}\right] \\
&= \hat{E}_{0,i}^{c^\varepsilon}\left[\mathbb{1}_{\{\xi_T=j\}} \exp\left(-\frac{1}{\varepsilon}\int_0^T \mathbb{1}_{(-\infty,0]}\left(X_r^{\varepsilon,\xi_{T-r}}(x)\right)\,dr\right)\hat{\rho}_{0,T}^{c^\varepsilon}\right].
\end{aligned}$$

Since the set of jump moments of ξ is countable and has therefore zero LEBESGUE measure, this is equivalent to

$$\frac{\partial X_T^{\varepsilon,i}(x)}{\partial x^j} = \hat{E}_{0,i}^{c^\varepsilon}\left[\mathbb{1}_{\{\xi_T=j\}}\exp\left(-\frac{1}{\varepsilon}\int_0^T \mathbb{1}_{(-\infty,0]}\left(X_r^{\varepsilon,[\xi_{T-.}]^*(r)}(x)\right)\,dr\right)\hat{\rho}_{0,T}^{c^\varepsilon}\right].$$

Now we can apply Lemma 6.5 and obtain

$$\frac{\partial X_T^{\varepsilon,i}(x)}{\partial x^j} = E_{0,j}^{c^\varepsilon}\left[\mathbb{1}_{\{\xi_T=i\}}\exp\left(-\frac{1}{\varepsilon}\int_0^T \mathbb{1}_{(-\infty,0]}(X_r^{\varepsilon,\xi_r}(x))\,dr\right)\rho_{0,T}^{c^\varepsilon}\right].$$

Since $T>0$ is arbitrary, the result follows.

□

We have proved so far that the derivatives of the solutions $X^\varepsilon(x)$ admit a.s. the random walk representation established in Theorem 6.6, which seems to be nearly the same as the representation in Theorem 2.2. The problem is to show that the derivatives of $X^\varepsilon(x)$ are convergent if $\varepsilon \downarrow 0$ and that the limit is equal to the random walk representation given in Theorem 2.2; but it seems to be quite difficult even to show that the derivatives are really convergent.

7 The Derivative of the Semigroup by Girsanov Transformation

In this section we study the derivatives of the transition semigroup of X, defined by

$$P_t f(x) := \mathbb{E}[f(X_t(x))], \qquad t \geq 0,\, x \in \mathbb{R}_+^I,$$

for all $f : \mathbb{R}_+^I \to \mathbb{R}$ bounded and continuously differentiable, where X is again the unique solution of a SKOROHOD SDE. At first we prove a preparing technical proposition, before we investigate the case, where X is the solution of a SKOROHOD SDE with constant diffusion coefficients. After that we discuss the SDE type with diagonal noise established in Section 4. First of all, let us recall the GIRSANOV Theorem, which will be crucial in the sequel:

Theorem 7.1 (GIRSANOV). *Let $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space such that $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0}\mathcal{F}_t\right)$ and the filtration $(\mathcal{F}_t)$ satisfies the usual conditions, i.e. it's right-continuous and complete. Suppose that the probability measure Q on $(\Omega, \mathcal{F}_\infty)$ is locally absolutely continuous to P, i.e. for each $t \geq 0$, the restriction of Q to $\mathcal{F}_t$ is absolutely continuous w.r.t. the restriction of P to $\mathcal{F}_t$. Let $(Z_t)_{t\geq 0}$ be the corresponding density process. If M is a continuous local P-martingale, then*

$$\tilde{M}_t = M_t - \int_0^t Z_s^{-1}\,d\langle M, Z\rangle_s, \qquad t \geq 0,$$

is a continuous local Q-martingale.

Proof. See, for instance, Theorem VIII.1.4 in [12]. □

7.1 Interchanging of Differentiation and Stochastic Integration

Proposition 7.2. *Let $(w_t)_{t\geq 0}$ be a* BROWN*ian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $(\mathcal{F}_t)$ satisfies the usual conditions and w is adapted to $(\mathcal{F}_t)$, and let $(z_t(x))_{t\geq 0}$ be a continuous, progressively measurable process depending on $x \in \mathbb{R}$. Suppose that for all $t \geq 0$ a.s. the mapping $x \mapsto z_t(x)$ is continuously differentiable and that $t \mapsto \frac{\partial}{\partial x} z_t(x)$ is right-continuous and locally bounded. Then, setting*

$$Z_t(x) := \int_0^t z_r(x)\, dw_r, \qquad t \geq 0,$$

we have for $p \geq 2$ and all $t \geq 0$

$$\frac{Z_t(x+h) - Z_t(x)}{h} \xrightarrow[h\to 0]{} \int_0^t \frac{\partial}{\partial x} z_r(x)\, dw_r \qquad \text{in } \mathcal{L}^p(\mathbb{P}),$$

i.e.

$$\frac{\partial}{\partial x} \int_0^t z_r(x)\, dw_r = \int_0^t \frac{\partial}{\partial x} z_r(x)\, dw_r \qquad \text{in } \mathcal{L}^p(\mathbb{P}).$$

Proof. We have to show that for $p \geq 2$ and $t \geq 0$

$$\mathbb{E}\left[\left|Z_t(x+h) - Z_t(x) - h\int_0^t \frac{\partial}{\partial x} z_r(x)\, dw_r\right|^p\right] \in o(|h|^p).$$

Using the linearity of stochastic integrals, the BURKHOLDER inequality (see Theorem 2.5) and finally the HÖLDER inequality, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|Z_t(x+h) - Z_t(x) - h\int_0^t \frac{\partial}{\partial x} z_r(x)\, dw_r\right|^p\right]\\
=&\mathbb{E}\left[\left|\int_0^t \left(z_r(x+h) - z_r(x) - h\frac{\partial}{\partial x} z_r(x)\right) dw_r\right|^p\right]\\
\leq& C_1\, \mathbb{E}\left[\left(\int_0^t \left(z_r(x+h) - z_r(x) - h\frac{\partial}{\partial x} z_r(x)\right)^2 dr\right)^{p/2}\right]\\
\leq& C_2\, \mathbb{E}\left[\int_0^t \left|z_r(x+h) - z_r(x) - h\frac{\partial}{\partial x} z_r(x)\right|^p dr\right],
\end{aligned}$$

where C_1 and C_2 are constants only depending on p and t. We apply the mean value theorem of differential calculus to z_r, $r \in [0,t]$, and obtain

$$\begin{aligned}
\mathbb{E}\left[\left(Z_t(x+h) - Z_t(x) - h\int_0^t \frac{\partial}{\partial x} z_r(x)\, dw_r\right)^2\right] &\leq C_2\, \mathbb{E}\left[\int_0^t \left|h\,\frac{\partial}{\partial x} z_r(\tilde{x}_r) - h\,\frac{\partial}{\partial x} z_r(x)\right|^p dr\right]\\
&= |h|^p\, \mathbb{E}\left[\int_0^t \left|\frac{\partial}{\partial x} z(r,\tilde{x}_r) - \frac{\partial}{\partial x} z(r,x)\right|^p dr\right]
\end{aligned}$$

for some $\tilde{x}_r$ between x and $x+h$, $r \in [0,t]$. Since $\frac{\partial}{\partial x} z$ is locally bounded and continuous in x, we obtain by the dominated convergence theorem

$$\mathbb{E}\left[\int_0^t \left|\frac{\partial}{\partial x} z_r(\tilde{x}) - \frac{\partial}{\partial x} z_r(x)\right|^p dr\right] \xrightarrow[h\to 0]{} 0,$$

and the claim follows.

□

7.2 Derivative of the Semigroup for Skorohod SDEs with Constant Diffusion Coefficients

We compute the derivatives of the transition semigroup $P_t f(x)$, $t \geq 0$, $x \in \mathbb{R}_+^I$, for the solution X of the system

$$
\begin{aligned}
&X_t^i(x) = x^i + \int_0^t b^i(X_r(x))\, dr + l_t^i(x) + w_t^i, \qquad t \geq 0,\, i \in I,\\
&X_t^i(x) \geq 0, \quad dl_t^i(x) \geq 0, \quad \int_0^\infty X_t^i(x)\, dl_t^i(x) = 0, \qquad i \in I,
\end{aligned} \tag{7.1}
$$

introduced in Section 2.2. We shall proceed as follows: We decouple the system (7.1) by the Girsanov transformation and obtain a system of reflected Brownian motions. Then, the derivatives of that decoupled processes have a quite simple form by Theorem 2.2.
Let $\tilde{\mathbb{P}}(x)$ be that probability measure, which is locally equivalent to $\mathbb{P}$, with the density process $\tilde{Z}(x)$ given by

$$
\tilde{Z}_t(x) := \left.\frac{d\tilde{\mathbb{P}}(x)}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \exp\left(-\int_0^t b(X_r(x))\, dw_r - \frac{1}{2}\int_0^t \|b(X_r(x))\|^2\, dr\right), \quad t \geq 0,
$$

i.e. $\tilde{Z}(x)$ is the Doleans-Dade exponential of $-\int_0^\cdot b(X_r(x))\, dw_r$, while we use the usual short hand notation for the integrals. Applying Itô's formula we get:

$$
\begin{aligned}
d\tilde{Z}_t(x) &= \tilde{Z}_t(x)\left(-b(X_t(x))\, dw_t - \frac{1}{2}\|b(X_t(x))\|^2\, dt\right) + \frac{1}{2}\tilde{Z}_t(x)\, \|b(X_t(x))\|^2\, dt\\
&= -\tilde{Z}_t(x)\, b(X_t(x))\, dw_t.
\end{aligned}
$$

We use the associativity of stochastic integrals to obtain:

$$
\int_0^t \frac{1}{\tilde{Z}_s(x)}\, d\tilde{Z}_s(x) = -\int_0^t b(X_s(x))\, dw_s, \qquad t \geq 0.
$$

Since w is a $\mathbb{P}$-martingale, from the Girsanov Theorem we know that

$$
\begin{aligned}
w_t - \int_0^t \frac{1}{\tilde{Z}_s(x)} d\langle w, \tilde{Z}(x)\rangle_s &= w_t - \left\langle w, -\int_0^\cdot b(X_s(x))\, dw_s\right\rangle_t\\
&= w_t + \int_0^t b(X_s(x))\, ds\\
&=: W_t(x), \qquad\qquad t \geq 0,
\end{aligned}
$$

is a local martingale relative to $\tilde{\mathbb{P}}(x)$. Clearly, its quadratic variation holds $\langle W^i(x), W^j(x)\rangle_t = \delta_{ij} t$ a.s. for all $t \geq 0$, $i, j \in I$. Thus, we can apply Levy's characterization theorem (cf. Theorem IV.3.6 in [12]) and conclude that $W(x)$ is a $\tilde{\mathbb{P}}(x)$-Brownian motion in $\mathbb{R}^I$, so that under $\tilde{\mathbb{P}}(x)$ the process X has the law of an i.i.d. family of reflected Brownian motions on $[0, \infty)$ starting at the initial value x:

$$
X_t^i(x) = x^i + W_t^i(x) + l_t^i(x), \qquad t \geq 0,\, i \in I.
$$

We can apply Theorem 2.2 and obtain that under $\tilde{\mathbb{P}}(x)$ the first partial derivatives of X w.r.t. the initial value have the following form:

$$
\frac{\partial X_t^i(x)}{\partial x^j} = \delta_{ij} \mathbb{1}_{\{\tau_j > t\}} \qquad \text{with } \tau_j := \inf\{s > 0 : X_s^j(x) = 0\}, \qquad t \geq 0,\, i, j \in I. \tag{7.2}
$$

We set

$$Z_t(x) := \left.\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}(x)}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t b(X_r(x))\,dw_r + \frac{1}{2}\int_0^t \|b(X_r(x))\|^2\,dr\right) =: \exp(\Lambda_t(x)), \qquad t \geq 0.$$

In the following we shall use again the notation $x_\varepsilon := x + \varepsilon\, e^j$, $j \in I$, where $(e^j)_{j\in I}$ is the canonical basis of $\mathbb{R}^I$ and $\varepsilon \in \mathbb{R}$ such that $x_\varepsilon \in \mathbb{R}^I_+$.

Proposition 7.3. *Let $j \in I$ be arbitrary but fixed and $x_\varepsilon := x + \varepsilon\, e^j$. Then, for any $t \geq 0$*

$$\frac{1}{\varepsilon}(Z_t(x_\varepsilon) - Z_t(x)) \xrightarrow[\varepsilon\to 0]{\mathcal{L}^1(\tilde{\mathbb{P}}(x))} Z_t(x)\left(\int_0^t \frac{\partial}{\partial x^j}[b(X_r(x))]\,dw_r + \frac{1}{2}\int_0^t \frac{\partial}{\partial x^j}\|b(X_r(x))\|^2\,dr\right).$$

Proof. At first, we observe that

$$\begin{aligned} Z_t(x_\varepsilon) - Z_t(x) &= \exp(\Lambda_t(x_\varepsilon)) - \exp(\Lambda_t(x)) = \exp(\Lambda_t(x))\,[\exp(\Lambda_t(x_\varepsilon) - \Lambda_t(x)) - 1] \\ &= Z_t(x)\left[\Lambda_t(x_\varepsilon) - \Lambda_t(x) + \sum_{p=2}^{\infty}\frac{(\Lambda_t(x_\varepsilon) - \Lambda_t(x))^p}{p!}\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\left|\frac{1}{\varepsilon}(Z_t(x_\varepsilon) - Z_t(x)) - Z_t(x)\left(\int_0^t \frac{\partial}{\partial x^j}[b(X_r(x))]\,dw_r + \frac{1}{2}\int_0^t \frac{\partial}{\partial x^j}\|b(X_r(x))\|^2\,dr\right)\right|\right] \\ \leq&\tilde{\mathbb{E}}\left[Z_t(x)\left|\frac{1}{\varepsilon}(\Lambda_t(x_\varepsilon) - \Lambda_t(x)) + \frac{1}{\varepsilon}\sum_{p=2}^{\infty}\frac{(\Lambda_t(x_\varepsilon) - \Lambda_t(x))^p}{p!} - \int_0^t \frac{\partial}{\partial x^j}[b(X_r(x))]\,dw_r\right.\right. \\ &\quad\left.\left. - \frac{1}{2}\int_0^t \frac{\partial}{\partial x^j}\|b(X_r(x))\|^2\,dr\right|\right] \\ \leq&\mathbb{E}\left[\left|\frac{1}{\varepsilon}\int_0^t (b(X_r(x_\varepsilon)) - b(X_r(x)))\,dw_r - \int_0^t \frac{\partial}{\partial x^j}[b(X_r(x))]\,dw_r\right|\right] \\ &+ \frac{1}{2}\,\mathbb{E}\left[\left|\frac{1}{\varepsilon}\int_0^t \left(\|b(X_r(x_\varepsilon))\|^2 - \|b(X_r(x))\|^2\right)\,dr - \int_0^t \frac{\partial}{\partial x^j}\|b(X_r(x))\|^2\,dr\right|\right] \\ &+ \mathbb{E}\left[\left|\frac{1}{\varepsilon}\sum_{p=2}^{\infty}\frac{(\Lambda_t(x_\varepsilon) - \Lambda_t(x))^p}{p!}\right|\right]. \end{aligned} \tag{7.3}$$

For every $i \in I$ and all $r \in [0,t]$ a.s. $b^i(X_r(x))$ is continuously differentiable w.r.t. x^j, there exists a right-continuous modification of $r \mapsto \frac{\partial}{\partial x^j}[b^i(X_r(x))]$ and the derivatives are uniformly bounded in $r \in [0,t]$ (see Theorem 2.2 and recall that b^i is LIPSCHITZ continuous). Thus, we can apply Proposition 7.2 and obtain that the first term in (7.3) tends to zero as $\varepsilon \to 0$. Moreover, we get that a.s.

$$\frac{\partial}{\partial x^j}\int_0^t \|b(X_r(x))\|^2\,dr = \int_0^t \frac{\partial}{\partial x^j}\|b(X_r(x))\|^2\,dr$$

(dominated convergence resp. see e.g. Korollar 16.3 in [1]), so we can conclude that also the second term in (7.3) tends to zero. Thus, it remains to show that the last term tends to zero, too. By the BEPPO LEVI Theorem and the HÖLDER inequality, we get

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{\varepsilon}\sum_{p=2}^{\infty}\frac{(\Lambda_t(x_\varepsilon)-\Lambda_t(x))^p}{p!}\right|\right] &\le \sum_{p=2}^{\infty}\frac{1}{p!}\,\mathbb{E}\left[\left|\frac{1}{\varepsilon}\left(\Lambda_t(x_\varepsilon)-\Lambda_t(x)\right)^p\right|\right]\\ &\le \sum_{p=2}^{\infty}\frac{C_1^p}{p!}\sum_{i\in I}\left\{\frac{1}{|\varepsilon|}\,\mathbb{E}\left[\left|\int_0^t (b^i(X_r(x_\varepsilon))-b^i(X_r(x)))\,dw_r^i\right|^p\right]\right.\\ &\qquad\left.+\mathbb{E}\left[\frac{1}{|\varepsilon|}\left|\int_0^t \left(b^i(X_r(x_\varepsilon))^2-b^i(X_r(x))^2\right)\,dr\right|^p\right]\right\},\end{aligned}$$

where C_1 (and C_2 below) are positive constants independent of p. We apply the BURKHOLDER inequality (see Theorem 2.5) and again the HÖLDER inequality to obtain:

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{\varepsilon}\sum_{p=2}^{\infty}\frac{(\Lambda_t(x_\varepsilon)-\Lambda_t(x))^p}{p!}\right|\right] &\le \sum_{p=2}^{\infty}\frac{C_2^p}{p!}\sum_{i\in I}\left\{C_p\,\frac{1}{|\varepsilon|}\,\mathbb{E}\left[\int_0^t \left|b^i(X_r(x_\varepsilon))-b^i(X_r(x))\right|^p\,dr\right]\right.\\ &\qquad\left.+\mathbb{E}\left[\frac{1}{|\varepsilon|}\int_0^t \left|b^i(X_r(x_\varepsilon))^2-b^i(X_r(x))^2\right|^p\,dr\right]\right\},\end{aligned}$$

where the BURKHOLDER constant is given by $C_p = \left(6\sqrt{2e}\,q^{3/2}/\sqrt{q-1}\right)^p$ with $q := p/(p-1)$, i.e.

$$C_p = \left(6\sqrt{2e}\right)^p\left(1+\frac{1}{p-1}\right)^{3p/2}(p-1)^{p/2}\,. \tag{7.4}$$

Since for every $i \in I$, $b^i(X_r(x))$ resp. $b^i(X_r(x))^2$, $r \in [0,t]$, are differentiable a.s. w.r.t. x^j and their derivatives are bounded, we get for $p \ge 2$ by the dominated convergence theorem:

$$\mathbb{E}\left[\frac{1}{|\varepsilon|}\int_0^t \left|b^i(X_r(x_\varepsilon))-b^i(X_r(x))\right|^p\,dr\right]\xrightarrow[\varepsilon\to 0]{} 0$$

resp.

$$\mathbb{E}\left[\frac{1}{|\varepsilon|}\int_0^t \left|b^i(X_r(x_\varepsilon))^2-b^i(X_r(x))^2\right|^p\,dr\right]\xrightarrow[\varepsilon\to 0]{} 0.$$

Thus, the proof is complete by dominated convergence, if we can show that

$$\sum_{p=2}^{\infty}\frac{M^p}{p!}\,C_p < \infty \tag{7.5}$$

for some positive constant M independent of p and with C_p as in (7.4). Setting $a_p := \frac{M^p}{p!}\,C_p$, $p \ge 2$, we obtain:

$$\left|\frac{a_{p+1}}{a_p}\right| = 6\sqrt{2e}\,M\left(1+\frac{1}{p}\right)^{3(p+1)/2}\left(1+\frac{1}{p-1}\right)^{-3p/2}\left(1+\frac{1}{p-1}\right)^{p/2}\frac{\sqrt{p}}{p+1}\xrightarrow[p\to\infty]{} 0,$$

since

$$\left(1+\frac{1}{p}\right)^{3(p+1)/2}\to e^{3/2},\quad \left(1+\frac{1}{p-1}\right)^{-3p/2}\to e^{-3/2},\quad \left(1+\frac{1}{p-1}\right)^{p/2}\to\sqrt{e},\quad \frac{\sqrt{p}}{p+1}\to 0$$

as $p \to \infty$. Hence, (7.5) holds by the quotient criterion.

□

Theorem 7.4. *For all $f : \mathbb{R}_+^I \to \mathbb{R}$ bounded and continuously differentiable, $t \geq 0$, $x \in \mathbb{R}_+^I$ and any $j \in I$:*

$$\frac{\partial}{\partial x^j} P_t f(x) = \mathbb{E}\left[\frac{\partial f}{\partial x^j}(X_t(x))\, \mathbb{1}_{\{\tau_j > t\}} + f(X_t(x)) \left(\int_0^t \frac{\partial b}{\partial x^j}(X_r(x))\, \mathbb{1}_{\{\tau_j > r\}}(dr + dw_r) \right) \right],$$

where $\tau_j := \inf\{s > 0 : X_s^j(x) = 0\}$.

Proof. By a density argument we may assume that f has bounded derivatives. For any $j \in I$, we set again $x_\varepsilon = x + \varepsilon\, e^j$, so that

$$\begin{aligned}
&\frac{\partial}{\partial x^j} \mathbb{E}[f(X_t(x))] \\
=&\frac{\partial}{\partial x^j} \tilde{\mathbb{E}}[f(X_t(x))\, Z_t(x)] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, \tilde{\mathbb{E}}[f(X_t(x_\varepsilon)\, Z_t(x_\varepsilon) - f(X_t(x))\, Z_t(x)] \\
=&\lim_{\varepsilon \to 0} \left\{ \tilde{\mathbb{E}}\left[\frac{1}{\varepsilon}\, (f(X_t(x_\varepsilon)) - f(X_t(x)))\, Z_t(x) \right] + \tilde{\mathbb{E}}\left[\frac{1}{\varepsilon}\, (Z_t(x_\varepsilon) - Z_t(x))\, f(X_t(x_\varepsilon)) \right] \right\}.
\end{aligned}$$

Since f and $X_t(x)$ are continuous, bounded and have bounded derivatives a.s., we can apply the dominated convergence theorem to the first term, and Proposition 7.3 to the second term to obtain

$$\begin{aligned}
\frac{\partial}{\partial x^j} \mathbb{E}[f(X_t(x))] =&\tilde{\mathbb{E}}\left[\frac{\partial}{\partial x^j}\, [f(X_t(x))]\ Z_t(x) \right] \\
&+ \tilde{\mathbb{E}}\left[f(X_t(x))\, Z_t(x)\ \left(\int_0^t \frac{\partial}{\partial x^j} [b(X_r(x))]\, dw_r + \frac{1}{2} \int_0^t \frac{\partial}{\partial x^j} \|b(X_r(x))\|^2\, dr \right) \right].
\end{aligned}$$

Using chain rule we get

$$\begin{aligned}
\frac{\partial}{\partial x^j} \mathbb{E}[f(X_t(x))] &= \tilde{\mathbb{E}}\left[Z_t(x) \sum_{k \in I} \frac{\partial f}{\partial x^k}(X_t(x))\, \frac{\partial X_t^k(x)}{\partial x^j} \right] \\
&\quad + \tilde{\mathbb{E}}\left[f(X_t(x))\, Z_t(x) \sum_{i \in I} \int_0^t \frac{\partial}{\partial x^j}\, \left[b^i(X_r(x)) \right]\ dr \right] \\
&\quad + \tilde{\mathbb{E}}\left[f(X_t(x))\, Z_t(x) \sum_{i \in I} \int_0^t \frac{\partial}{\partial x^j}\, \left[b^i(X_r(x)) \right]\ dw_r^i \right] \\
&= \tilde{\mathbb{E}}\left[Z_t(x) \sum_{k \in I} \frac{\partial f}{\partial x^k}(X_t(x))\, \frac{\partial X_t^k(x)}{\partial x^j} \right] \\
&\quad + \tilde{\mathbb{E}}\left[f(X_t(x))\, Z_t(x) \sum_{i \in I} \int_0^t \sum_{k \in I} \frac{\partial b^i}{\partial x^k}(X_r(x))\, \frac{\partial X_r^k(x)}{\partial x^j}\, dr \right] \\
&\quad + \tilde{\mathbb{E}}\left[f(X_t(x))\, Z_t(x) \sum_{i \in I} \int_0^t \sum_{k \in I} \frac{\partial b^i}{\partial x^k}(X_r(x))\, \frac{\partial X_r^k(x)}{\partial x^j}\, dw_r^i \right].
\end{aligned}$$

We insert the representation of the derivatives of X under $\tilde{\mathbb{P}}(x)$ in (7.2) and get

$$\begin{aligned}\frac{\partial}{\partial x^j}\mathbb{E}[f(X_t(x))] = \tilde{\mathbb{E}}\left[\frac{\partial f}{\partial x^j}(X_t(x))\,\mathbb{1}_{\{\tau_j>t\}}\,Z_t(x)\right]\\ +\tilde{\mathbb{E}}\left[f(X_t(x))\,Z_t(x)\sum_{i\in I}\int_0^t\frac{\partial b^i}{\partial x^j}(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,dr\right]\\ +\tilde{\mathbb{E}}\left[f(X_t(x))\,Z_t(x)\sum_{i\in I}\int_0^t\frac{\partial b^i}{\partial x^j}(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,dw_r^i\right].\end{aligned}$$

Finally, using short hand notation, we change again the measure to obtain:

$$\begin{aligned}\frac{\partial}{\partial x^j}\mathbb{E}[f(X_t(x))] = \mathbb{E}\left[\frac{\partial f}{\partial x^j}(X_t(x))\,\mathbb{1}_{\{\tau_j>t\}}\right]+\mathbb{E}\left[f(X_t(x))\int_0^t\frac{\partial b}{\partial x^j}(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,dr\right]\\ +\mathbb{E}\left[f(X_t(x))\int_0^t\frac{\partial b}{\partial x^j}(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,dw_r\right],\end{aligned}$$

and the result follows.

□

7.3 Derivative of the Semigroup for Skorohod SDEs with Diagonal Noise

We shall generalize the preceding result: Now we investigate the transition semigroup of X, where X is the solution of the system

$$\begin{aligned}&X_t^i(x) = x^i+\int_0^t b^i(X_r(x))\,dr + l_t^i(x)+\int_0^t\sigma^i(X_r^i(x))\,dw_r^i, \quad t\ge 0,\, i\in I\\ &X_t^i(x)\ge 0, \quad dl_t^i(x)\ge 0, \quad \int_0^\infty X_t^i(x)\,dl_t^i(x)=0, \qquad i\in I,\end{aligned}$$

established in Section 4. The coefficients b^i and σ^i, $i\in I$, are supposed to satisfy the conditions i)-v) stated at the beginning of Section 4. We shall compute the derivatives of the semigroup by combining the techniques of the last subsection with the LAMPERTI transformation introduced in the proof of Theorem 4.1.

Theorem 7.5. *For all $f:\mathbb{R}_+^I\to\mathbb{R}$ bounded and continuously differentiable, $t\ge 0$, $x\in\mathbb{R}_+^I$ and any $j\in I$:*

$$\begin{aligned}\frac{\partial}{\partial x^j}P_tf(x) =\mathbb{E}\Bigg[&\frac{\sigma^j(X_t^j(x))}{\sigma^j(x^j)}\frac{\partial f}{\partial x^j}(X_t(x))\mathbb{1}_{\{\tau_j>t\}}\\ &+f(X_t(x))\,\frac{1}{\sigma^j(x^j)}\Bigg\{\sum_{i\in I}\int_0^t\frac{\sigma^j(X_r^j(x))}{\sigma^i(X_r^i(x))}\frac{\partial b^i}{\partial x^j}(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,(dr+dw_r^i)\\ &-\int_0^t\sigma^j(X_r^j(x))\,g^j(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}(dr+dw_r^j)\Bigg\}\Bigg],\end{aligned}$$

where

$$g^j(x) := \frac{(\sigma^j)'(x^j)}{\sigma^j(x^j)^2}b^j(x)+\tfrac{1}{2}(\sigma^j)''(x^j) \text{ and } \tau_j:=\inf\{s>0:\,X_s^j(x)=0\}, \qquad j\in I.$$

Proof. We shall use the same notation as in the proof of Theorem 4.1. Recall that Y, which denotes the LAMPERTI transform of X, satisfies

$$Y_t^i(y) = y^i + \int_0^t \hat{b}^i(Y_r(y))\,dr + a_0^i\, l_t^i(y) + w_t^i, \qquad t \geq 0,\, i \in I,$$

where $\hat{b}^i$, $i \in I$, is defined by

$$\hat{b}^i(y) := a^i\big((A^i)^{-1}(y^i)\big)\; b^i\big(A^{-1}(y)\big) - \tfrac{1}{2}\,(\sigma^i)'\big((A^i)^{-1}(y^i)\big)\,.$$

Furthermore, we recall that

$$\frac{\partial \hat{b}^i}{\partial y^k}(Y_r(y)) = \frac{\sigma^k(X_r^k(x))}{\sigma^i(X_r^i(x))}\,\frac{\partial b^i}{\partial x^k}(X_r(x)), \qquad \text{if } k \neq i, \tag{7.6}$$

and

$$\frac{\partial \hat{b}^i}{\partial y^i}(Y_r(y)) = \frac{\partial b^i}{\partial x^i}(X_r(x)) - \sigma^i(X_r^i(x))\, g^i(X_r(x)) \tag{7.7}$$

with g^i, $i \in I$, defined as in the statement (cf. equation (4.16) and (4.17)). We define the probability measure $\hat{\mathbb{P}}(y)$, which is locally equivalent to $\mathbb{P}$, by the density process

$$\hat{Z}_t(y) := \left.\frac{d\mathbb{P}}{d\hat{\mathbb{P}}(y)}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \hat{b}(Y_r(y))\,dw_r + \frac{1}{2}\int_0^t \|\hat{b}(Y_r(y))\|^2\,dr\right), \qquad t \geq 0.$$

Analogously to the preceding subsection, $(\hat{W}_t^i(y))_{i\in I}$, defined by

$$\hat{W}_t^i(y) := \int_0^t \hat{b}^i(Y_r(y))\,dr + w_t^i, \qquad t \in [0,T],\, i \in I,$$

is a BROWNian motion in $\mathbb{R}^I$ under $\hat{\mathbb{P}}(y)$ by GIRSANOV's Theorem. In particular, under $\hat{\mathbb{P}}(y)$

$$Y_t^i(y) = y^i + \hat{W}_t^i(y) + a_0^i\, l_t^i(y), \qquad t \geq 0,\, i \in I,$$

is an i.i.d. family of reflected BROWNian motions. From the equation for the derivatives of Y in (4.10) and (4.11) we obtain that under $\hat{\mathbb{P}}(y)$ a.s.

$$\frac{\partial Y_t^i(y)}{\partial y^j} = \delta_{ij}\,\mathbb{1}_{\{\tau_j > t\}}, \qquad \text{with } \tau_j := \inf\{s > 0:\, Y_s^j(y) = 0\}, \qquad t \geq 0,\, i,j \in I. \tag{7.8}$$

Notice that by construction of Y

$$\tau_j = \inf\{s > 0:\, X_s^j(x) = 0\}, \qquad j \in I.$$

Finally, we recall that

$$\frac{\partial X_t^i(x)}{\partial x^j} = \frac{\sigma^i(X_t^i(x))}{\sigma^j(x^j)}\,\frac{\partial Y_t^i(y)}{\partial y^j}, \qquad t \geq 0,\, i,j \in I \tag{7.9}$$

(cf. equation (4.14)).

Now we compute the partial derivatives of the semigroup of X. At first, proceeding analogously to the proof of Theorem 7.4, we have for any $j \in I$ and $t \geq 0$:

$$\begin{aligned}
&\frac{\partial}{\partial x^j}\mathbb{E}[f(X_t(x))] \\
=&\frac{\partial}{\partial x^j}\hat{\mathbb{E}}[f(X_t(x))\,\hat{Z}_t(y)] = \frac{\partial}{\partial x^j}\hat{\mathbb{E}}[f(X_t(x))\,\hat{Z}_t(A(x))] \\
=&\hat{\mathbb{E}}\left[\frac{\partial}{\partial x^j}\,[f(X_t(x))]\;\hat{Z}_t(y)\right] \\
&+\hat{\mathbb{E}}\left[f(X_t(x))\,\hat{Z}_t(y)\left(\int_0^t \frac{\partial}{\partial x^j}[\hat{b}(Y_r(A(x)))]\,dw_r + \frac{1}{2}\int_0^t \frac{\partial}{\partial x^j}\|\hat{b}(Y_r(A(x)))\|^2\,dr\right)\right] \\
=&:I + II.
\end{aligned} \tag{7.10}$$

Using chain rule, (7.9) and (7.8) we obtain

$$\begin{aligned}
I &= \hat{\mathbb{E}}\left[\hat{Z}_t(y)\sum_{k\in I}\frac{\partial f}{\partial x^k}(X_t(x))\,\frac{\partial X_t^k(x)}{\partial x^j}\right] = \hat{\mathbb{E}}\left[\hat{Z}_t(y)\sum_{k\in I}\frac{\partial f}{\partial x^k}(X_t(x))\,\frac{\sigma^k(X_t^k(x))}{\sigma^j(x^j)}\,\frac{\partial Y_t^k(y)}{\partial y^j}\right] \\
&= \hat{\mathbb{E}}\left[\hat{Z}_t(y)\frac{\partial f}{\partial x^j}(X_t(x))\,\frac{\sigma^j(X_t^j(x))}{\sigma^j(x^j)}\,\mathbb{1}_{\{\tau_j>t\}}\right] \\
&= \mathbb{E}\left[\frac{\partial f}{\partial x^j}(X_t(x))\,\frac{\sigma^j(X_t^j(x))}{\sigma^j(x^j)}\,\mathbb{1}_{\{\tau_j>t\}}\right].
\end{aligned} \tag{7.11}$$

On the other hand, by construction of the change of variables, chain rule and (7.8), we have a.s. under $\hat{\mathbb{P}}(y)$ for all $r \geq 0$ and $i \in I$:

$$\begin{aligned}
\frac{\partial}{\partial x^j}\left[\hat{b}^i(Y_r(A(x)))\right] &= \sum_{k\in I}\frac{\partial}{\partial y^k}\left[\hat{b}^i(Y_r(y))\right]\,\frac{\partial A^k(x)}{\partial x^j} = \frac{1}{\sigma^j(x^j)}\,\frac{\partial}{\partial y^j}\left[\hat{b}^i(Y_r(y))\right] \\
&= \frac{1}{\sigma^j(x^j)}\sum_{k\in I}\frac{\partial \hat{b}^i}{\partial y^k}(Y_r(y))\,\frac{\partial Y_r^k(y)}{\partial y^j} = \frac{1}{\sigma^j(x^j)}\,\frac{\partial \hat{b}^i}{\partial y^j}(Y_r(y))\,\mathbb{1}_{\{\tau_j>r\}},
\end{aligned}$$

so that a.s. under $\hat{\mathbb{P}}(y)$

$$\begin{aligned}
&\int_0^t \frac{\partial}{\partial x^j}[\hat{b}(Y_r(A(x)))]\,dw_r + \frac{1}{2}\int_0^t \frac{\partial}{\partial x^j}\|\hat{b}(Y_r(A(x)))\|^2\,dr \\
=&\sum_{i\in I}\int_0^t \frac{\partial}{\partial x^j}[\hat{b}^i(Y_r(A(x)))]\,(dw_r^i + dr) = \frac{1}{\sigma^j(x^j)}\sum_{i\in I}\int_0^t \frac{\partial \hat{b}^i}{\partial y^j}(Y_r(y))\,\mathbb{1}_{\{\tau_j>r\}}\,(dw_r^i + dr).
\end{aligned}$$

Hence, we obtain using (7.6) and (7.7):

$$\begin{aligned}
II &= \hat{\mathbb{E}}\left[f(X_t(x))\,\hat{Z}_t(y)\,\frac{1}{\sigma^j(x^j)}\sum_{i\in I}\int_0^t \frac{\partial \hat{b}^i}{\partial y^j}(Y_r(y))\,\mathbb{1}_{\{\tau_j>r\}}\,(dw_r^i + dr)\right] \\
&= \hat{\mathbb{E}}\Bigg[f(X_t(x))\,\hat{Z}_t(y)\,\frac{1}{\sigma^j(x^j)}\Bigg\{\sum_{i\in I}\int_0^t \frac{\sigma^j(X_r^j(x))}{\sigma^i(X_r^i(x))}\,\frac{\partial b^i}{\partial x^j}(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,(dw_r^i + dr) \\
&\qquad - \int_0^t \sigma^j(X_r^j(x))\,g^j(X_r(x))\,\mathbb{1}_{\{\tau_j>r\}}\,(dw_r^j + dr)\Bigg\}\Bigg].
\end{aligned} \tag{7.12}$$

Finally, we insert (7.11) and (7.12) into (7.10) and this yields the claim.

□

8 Reflected Brownian Motion in a Wedge

In this section we investigate the pathwise differentiability of a BROWNian motion in a wedge with oblique reflection, i.e. we consider a MARKOV process that has continuous sample paths and the following three properties:

a) The state space S is an infinite two-dimensional wedge, and the process behaves in the interior like a BROWNian motion.

b) The process reflects instantaneously at the boundary of the wedge, the direction of reflection being constant along each side.

c) The amount of time that the process spends at the corner of the wedge has zero LEBESGUE measure.

Without loss of generality, we may suppose that the corner of the wedge is at the origin, and one side is along the x^1-axis. Let $\xi \in (0,\pi)$ be the angle of the wedge. The two sides of the wedge will be denoted by ∂S_1 and ∂S_2, and the direction of reflection on these sides will be denoted by constant vectors v_1 and v_2. Associated with these directions of reflection are angles θ_1 and θ_2, taking values in $(-\frac{\pi}{2},\frac{\pi}{2})$, where θ_j, $j=1,2$, is defined as the angle between v_j and the inward normal n_j to the side ∂S_j. The sign convention for the angles is that they are positive if the associated direction of reflection points towards the corner. See Figure 8.1 for an example of an acute wedge, where θ_1 and θ_2 shown there are both positive. Define $\alpha := (\theta_1+\theta_2)/\xi$.
In [16] the process was characterized in law as the solution of a submartingale problem. There it was proved that the solution of the associated submartingale problem exists and is unique if $\alpha < 2$, i.e. in this case there is a unique continuous strong MARKOV process satisfying a)-c). If $\alpha \geq 2$, there is no solution for the submartingale problem. Nevertheless, in this case there exists a unique continuous strong MARKOV process satisfying a) and b), which almost surely reaches the corner and remains there. As a further result it was shown in [16] that the process starting away from the corner does not reach the corner with probability one if $\alpha \leq 0$, and that it does reach the corner almost surely if $\alpha > 0$.
Unlike [16], we shall describe the process pathwise by a system of SKOROHOD SDEs and apply some of the techniques used in [3] resp. in Section 4. The aim is to compute the pathwise derivatives w.r.t. the starting point up to time τ_0, when the process hits the corner of the wedge for the first time (if $\alpha \leq 0$ we might have $\tau_0 = \infty$ a.s.).

8.1 Model and Notation

Let $\xi \in (0,\pi)\backslash\{\frac{\pi}{2}\}$, $m := \tan\xi$ (the case $\xi = \frac{\pi}{2}$ will be discussed later in Section 8.6) and

$$S := \left\{(x^1,x^2)\in\mathbb{R}^2 : x^2 \geq 0,\ \mathrm{sign}(m)\,(mx^1 - x^2) \geq 0\right\}.$$

The boundary ∂S of the wedge S consists of the two sides

$$\partial S_1 := \left\{(x^1,x^2)\in S : x^2 = 0\right\} \quad \text{and} \quad \partial S_2 := \left\{(x^1,x^2)\in S : \mathrm{sign}(m)\,(mx^1 - x^2) = 0\right\}.$$

If $\xi \in (0,\frac{\pi}{2})$, it can easily be verified, that the cartesian coordinates of v_1 and v_2 are given by

$$v_1 = (-\tan\theta_1, 1), \qquad v_2 = \begin{cases} (1, -\tan(\frac{\pi}{2}-\xi+\theta_2)) & \text{if } \theta_2 \in (-\frac{\pi}{2},\xi), \\ (0,-1) & \text{if } \theta_2 = \xi, \\ (-1, \tan(\frac{\pi}{2}-\xi+\theta_2)) & \text{if } \theta_2 \in (\xi,\frac{\pi}{2}), \end{cases}$$

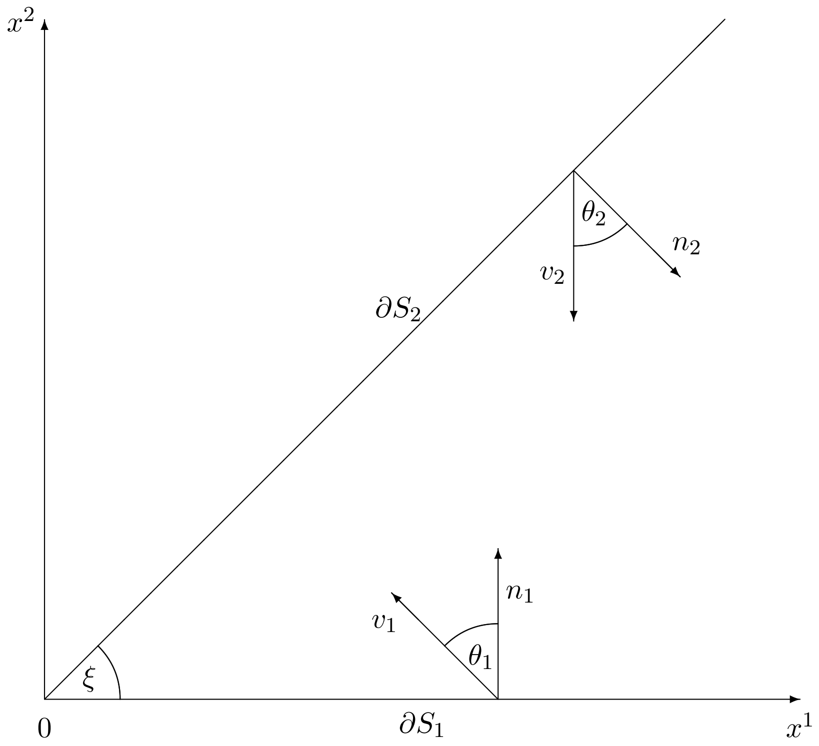

Figure 8.1: Acute Wegde with $\theta_1 > 0$ and $\theta_2 > 0$

resp. if $\xi \in (\frac{\pi}{2}, \pi)$:

$$v_1 = (-\tan\theta_1, 1), \qquad v_2 = \begin{cases} (-1, \tan(\frac{\pi}{2} - \xi + \theta_2)) & \text{if } \theta_2 \in (-\frac{\pi}{2}, -(\pi - \xi)), \\ (0, 1) & \text{if } \theta_2 = -(\pi - \xi), \\ (1, -\tan(\frac{\pi}{2} - \xi + \theta_2)) & \text{if } \theta_2 \in (-(\pi - \xi), \frac{\pi}{2}). \end{cases}$$

Let us assume that

$$\theta_2 \in \begin{cases} (-\frac{\pi}{2}, \xi) & \text{if } \xi \in (0, \frac{\pi}{2}), \\ (-(\pi - \xi), \frac{\pi}{2}) & \text{if } \xi \in (\frac{\pi}{2}, \pi), \end{cases} \quad \text{and} \quad \alpha := \frac{\theta_1 + \theta_2}{\xi} < 2. \tag{8.1}$$

Then, we have

$$v_1 = (-a, 1) \quad \text{and} \quad v_2 = (1, -b) \quad \text{with} \quad a := \tan\theta_1, \quad b := \tan(\tfrac{\pi}{2} - \xi + \theta_2).$$

Moreover, for later use we set

$$c := \frac{b\,(ma + 1)}{m + b}.$$

Then, it can easily be verified that

$$c = (\cos\xi + \sin\xi\tan\theta_1)(\cos\xi + \sin\xi\tan\theta_2) = \frac{\cos(\xi - \theta_1)}{\cos\theta_1}\,\frac{\cos(\xi - \theta_2)}{\cos\theta_2}. \tag{8.2}$$

Since $\alpha < 2$ by assumption, the existence and uniqueness in law is ensured by the main result in [16]. Now, for any starting point $x \in S\backslash\{0\}$, the process is equivalent in law to a continuous process X obtained by the following pathwise construction in terms of a SKOROHOD SDE:

$$X_t(x) = x + w_t + \left(v_1^T, v_2^T\right) \begin{pmatrix} l_t^1(x) \\ l_t^2(x) \end{pmatrix}, \qquad t \geq 0,$$

where $w = (w^1, w^2)$ is a two-dimensional BROWNian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $l^i(x)$, $i \in \{1,2\}$, denotes the local time of $X(x)$ in ∂S_i (not the local time of $X^i(x)$ in zero in contrast to the previous sections) (cf. example in Section 8.4, p. 170 in [2]). The matrix $\left(v_1^T, v_2^T\right)$ is called the reflection matrix. Setting

$$Z_t(x) := \operatorname{sign}(m)\left(mX_t^1(x) - X_t^2(x)\right), \qquad t \geq 0,$$

this leads to the following system of SKOROHOD SDEs: For $x \in S\backslash\{0\}$,

i) $$X_t^1(x) = x^1 + w_t^1 - a\, l_t^1(x) + l_t^2(x), \tag{8.3}$$
$$X_t^2(x) = x^2 + w_t^2 + l_t^1(x) - b\, l_t^2(x), \qquad t \geq 0, \tag{8.4}$$

ii) $$X_t^2(x) \geq 0,\ Z_t(x) \geq 0, \qquad t \geq 0, \tag{8.5}$$

iii) $dl_t^i(x) \geq 0$ for $t \geq 0$ and $i \in \{1,2\}$, $\displaystyle\int_0^\infty X_t^2(x)\, dl_t^1(x) = 0$, $\displaystyle\int_0^\infty Z_t(x)\, dl_t^2(x) = 0$.

In the sequel we shall often use the abbreviation $\tilde{x} := m\, x^1 - x^2$ and $\tilde{w}_t := m\, w_t^1 - w_t^2$, $t \geq 0$, so that $Z(x)$ becomes

$$Z_t(x) = \operatorname{sign}(m)\left(\tilde{x} + \tilde{w}_t - (ma+1)\, l_t^1(x) + (m+b)\, l_t^2(x)\right), \qquad t \geq 0.$$

Note that $\tilde{w}$ is again a BROWNian motion rescaled with the constant factor $\sqrt{m^2+1}$.
We define the stopping time τ_0 by

$$\tau_0 := \inf\{t \geq 0 :\ X_t(x) = 0\}, \qquad x \in S\backslash\{0\}, \tag{8.6}$$

to be the first hitting time of the corner. From Theorem 2.2 in [16] we know that

$$\mathbb{P}[\tau_0 < \infty] = \begin{cases} 0 & \text{if } \alpha \leq 0, \\ 1 & \text{if } 0 < \alpha < 2. \end{cases} \tag{8.7}$$

Moreover, τ_0 has infinite expectation (see Corollary 2.3 in [16]).
Now we introduce some notation corresponding to that in Section 4:

$$C^1 := \{s \geq 0 :\ X_s(x) \in \partial S_1\} = \{s \geq 0 :\ X_s^2(x) = 0\}, \qquad r_1(t) := \sup(C^1 \cap [0,t]),$$
$$C^2 := \{s \geq 0 :\ X_s(x) \in \partial S_2\} = \{s \geq 0 :\ Z_s(x) = 0\}, \qquad r_2(t) := \sup(C^2 \cap [0,t]),$$

with $\sup \emptyset := 0$, and

$$C := C^1 \cup C^2, \qquad r(t) := \max(r_1(t), r_2(t)).$$

Furthermore, for $t \in [0, \tau_0)\backslash C$ we set

$$s(t) := \begin{cases} 0 & \text{if } t < \inf C, \\ 1 & \text{if } r(t) = r_1(t), \\ 2 & \text{if } r(t) = r_2(t), \end{cases}$$

i.e. $s(t) = i$, if the last hit of the boundary before time t was in ∂S_i, $i \in \{1, 2\}$, and $s(t) = 0$, if at time t the process hasn't hit the boundary yet.
Since the process stays away from the corner on $[0, \tau_0)$, the number of reflections before any time $t \in [0, \tau_0)$ is finite a.s. Let $(A_n)_n$ be the family of connected components of $[0, \tau_0)\backslash C$. A_n is open, so that there exists $q_n \in A_n \cap \mathbb{Q}$, $n \in \mathbb{N}$. Then, we may assume that the q_n are arranged such that $q_1 < q_2 < \ldots$, i.e. $(q_n)_n$ is strictly increasing. Notice that on every interval $[q_{n-1}, q_n]$ the process hits only one side of the wedge, since otherwise it would hit the corner.
We extract a subsequence (q_{n_k}) defined by

$$\begin{aligned} q_{n_1} &:= \min\{q_n :\ s(q_n) \neq 0,\ s(q_n) \neq s(q_{n+1})\}, \\ q_{n_k} &:= \min\{q_n :\ q_n > q_{n_{k-1}},\ s(q_n) \neq s(q_{n+1})\}, \qquad k \geq 2, \end{aligned} \tag{8.8}$$

i.e. in every time interval A_{n_k}, $k \geq 1$, the process crosses the wedge S from one side to the other one. Finally we set

$$\begin{aligned} N_{12}(q_n) &:= \left|\{q_k \leq q_n :\ s(q_{k-1}) = 1,\ s(q_k) = 2\}\right|, \\ N_{21}(q_n) &:= \left|\{q_k \leq q_n :\ s(q_{k-1}) = 2,\ s(q_k) = 1\}\right| \end{aligned}$$

and

$$N_{12}(t) := N_{12}(q_n), \quad N_{21}(t) := N_{21}(q_n) \quad \text{for } t \in A_n,$$

i.e. $N_{12}(t)$ denotes the number of crossings from ∂S_1 to ∂S_2 before time t, and $N_{21}(t)$ the number of crossings from ∂S_2 to ∂S_1. For later use we observe that

$$N_{21}(t) - N_{12}(t) = \begin{cases} 0 & \text{if } \inf C^1 < \inf C^2, \\ 1 & \text{if } \inf C^2 < \inf C^1, \end{cases} \qquad \text{for } t \text{ satisfying } s(t) = 1, \tag{8.9}$$

and

$$N_{12}(t) - N_{21}(t) = \begin{cases} 0 & \text{if } \inf C^2 < \inf C^1, \\ 1 & \text{if } \inf C^1 < \inf C^2, \end{cases} \qquad \text{for } t \text{ satisfying } s(t) = 2. \tag{8.10}$$

Now we are able to state the main result of this section:

Theorem 8.1. *The mapping $x \mapsto X_t(x)$, $x \in S\backslash\{0\}$, is differentiable a.s. for all $t \in [0, \tau_0)\backslash C$, and the derivatives are given by:*

$$\frac{\partial X_t^1(x)}{\partial x^1} = \begin{cases} 1 & \textit{if } s(t) = 0, \\ c^{N_{21}(t)} & \textit{if } s(t) = 1, \\ \frac{b}{m+b}\, c^{N_{21}(t)} & \textit{if } s(t) = 2, \end{cases}$$

$$\frac{\partial X_t^1(x)}{\partial x^2} = \begin{cases} 0 & \textit{if } s(t) = 0, \\ \frac{1}{m}\left((ma+1)\, c^{N_{12}(t)} - c^{N_{21}(t)}\right) & \textit{if } s(t) = 1, \\ \frac{1}{m}\left(c^{N_{12}(t)} - \frac{b}{m+b}\, c^{N_{21}(t)}\right) & \textit{if } s(t) = 2, \end{cases}$$

and

$$\frac{\partial X_t^2(x)}{\partial x^1} = \begin{cases} 0 & \textit{if } s(t) = 0, \\ 0 & \textit{if } s(t) = 1, \\ \frac{mb}{m+b}\, c^{N_{21}(t)} & \textit{if } s(t) = 2, \end{cases} \qquad \frac{\partial X_t^2(x)}{\partial x^2} = \begin{cases} 1 & \textit{if } s(t) = 0, \\ 0 & \textit{if } s(t) = 1, \\ c^{N_{12}(t)} - \frac{b}{m+b}\, c^{N_{21}(t)} & \textit{if } s(t) = 2. \end{cases}$$

Remark 8.2. We can easily drop the assumption $\alpha < 2$: If $\alpha \geq 2$, we can consider the unique continuous strong Markov process satisfying a) and b) with absorption at the corner. Of course, for that process we obtain the same pathwise derivatives as in Theorem 8.1 up to the hitting time of the corner.

As before, the first step to prove Theorem 8.1 is to check the continuity w.r.t. the sup-norm topology.

8.2 Continuity in x

Lemma 8.3. *Let $x \in S\backslash\{0\}$. For all q_n, $n \in \mathbb{N}$, we have that for all random $h > 0$ there exists a random $\Delta_n > 0$ such that a.s.*

$$\sup_{s\leq q_n} |X^i_s(x) - X^i_s(y)| < h, \quad \sup_{s\leq q_n} |l^i_s(x) - l^i_s(y)| < h, \quad \sup_{s\leq q_n} |Z_s(x) - Z_s(y)| < h, \tag{8.11}$$

$i \in \{1, 2\}$, for all $y \in S\backslash\{0\}$ satisfying $\|x - y\| < \Delta_n$.

Proof. We prove the lemma by induction over n. For $n = 1$ note first that by construction either $q_1 < \inf C^1$ or $q_1 < \inf C^2$. Let us assume that $q_1 < \inf C^2$ (the other case can be treated analogously), so that $X_s(x) \notin \partial S_2$, i.e. $Z_s(x) > 0$ and $l^2_s(x) = 0$, for all $s \in [0, q_1]$. Let $d := \frac{1}{2}\inf_{s\leq q_1} Z_s(x) > 0$.
We claim that there exists a $\Delta'_1 > 0$ such that

$$\sup_{s\leq q_1} |Z_s(x) - Z_s(y)| < d, \quad \text{i.e. } Z_s(y) > 0 \text{ and } l^2_s(y) = 0 \text{ on } [0, q_1], \text{ for all } \|x - y\| < \Delta'_1. \tag{8.12}$$

For $x \in S\backslash\{0\}$ we define the process $\hat{X}(x)$ by

$$\hat{X}^1_t(x) = x^1 + w^1_t - a\,\hat{l}^1_t(x), \tag{8.13}$$

$$\hat{X}^2_t(x) = x^2 + w^2_t + \hat{l}^1_t(x), \tag{8.14}$$

$$\hat{X}^2_t(x) \geq 0,\ d\hat{l}^1_t(x) \geq 0, \quad \int_0^\infty \hat{X}^2_t(x)\, d\hat{l}^1_t(x) = 0, \qquad t \geq 0. \tag{8.15}$$

and set $\hat{Z}_t(x) := \operatorname{sign}(m)\left(m\hat{X}^1_t(x) - \hat{X}^2_t(x)\right)$, $t \geq 0$. Note that $X_s(x)$ (resp. $Z_s(x)$) coincides with $\hat{X}_s(x)$ (resp. with $\hat{Z}_s(x)$) for all $s \in [0, q_1]$. Moreover, if (8.12) holds, then this is also true for any starting point y in place of x with $\|x - y\| < \Delta'_1$.
Using (8.15), we get for all $t \geq 0$

$$d(\hat{X}^2_t(x) - \hat{X}^2_t(y))^2 = 2(\hat{X}^2_t(x) - \hat{X}^2_t(y))(d\hat{l}^1_t(x) - d\hat{l}^1_t(y)) \leq 0$$

and we can conclude that

$$|\hat{X}^2_t(x) - \hat{X}^2_t(y)| \leq |x^2 - y^2|, \qquad \forall t \geq 0,\ x, y \in S\backslash\{0\}.$$

Hence,

$$\sup_{s\leq q_1} |\hat{X}^2_s(x) - \hat{X}^2_s(y)| \leq |x^2 - y^2|. \tag{8.16}$$

By (8.13) and (8.14) we have:

$$\sup_{s\leq q_1} |\hat{l}^1_s(x) - \hat{l}^1_s(y)| \leq |x^2 - y^2| + \sup_{s\leq q_1} |\hat{X}^2_s(x) - \hat{X}^2_s(y)| \leq 2|x^2 - y^2| \tag{8.17}$$

and

$$\sup_{s\le q_1}|\hat X^1_s(x)-\hat X^1_s(y)|\le |x^1-y^1|+|a|\sup_{s\le q_1}|\hat l^1_s(x)-\hat l^1_s(y)|\le |x^1-y^1|+2|a||x^2-y^2|. \tag{8.18}$$

Since

$$\sup_{s\le q_1}|\hat Z_s(x)-\hat Z_s(y)|\le |m|\sup_{s\le q_1}|\hat X^1_s(x)-\hat X^1_s(y)|+\sup_{s\le q_1}|\hat X^2_s(x)-\hat X^2_s(y)|, \tag{8.19}$$

it becomes obvious that there exists a $\Delta'_1>0$ such that

$$\sup_{s\le q_1}|\hat Z_s(x)-\hat Z_s(y)|<d \qquad \text{for all } y\in S\backslash\{0\} \text{ such that } \|x-y\|<\Delta'_1.$$

Since $Z(x)\equiv \hat Z(x)$ on $[0,q_1]$, it follows for such y by our choice of d that $\hat Z_s(y)>0$ for all $s\in[0,q_1]$, which implies $l^2_{q_1}(y)=0$ and $Z(y)\equiv\hat Z(y)$ on $[0,q_1]$, i.e. (8.12) holds. Hence, $X(y)\equiv\hat X(y)$ and $Z(y)\equiv\hat Z(y)$ on $[0,q_1]$ if $\|x-y\|<\Delta'_1$, and by combining (8.16) - (8.19) it obviously follows that (8.11) holds for $n=1$.
Now assume that (8.11) holds for any $n\ge 1$. We consider again only the case $s(q_{n+1})=1$, i.e. $Z_s(x)>0$ for all $s\in[q_n,q_{n+1}]$. By a similar argument as above for $n=1$, we can find a random $\Delta'_n>0$ such that $Z_s(y)>0$ on $[q_n,q_{n+1}]$ for all $\|x-y\|<\Delta'_n$. Hence, for such y and $t\in[q_n,q_{n+1}]$ we have $dl^2_t(x)=dl^2_t(y)=0$, so that

$$X^1_t(x)=X^1_{q_n}(x)+w^1_t-w^1_{q_n}-a(l^1_t(x)-l^1_{q_n}(x)), \tag{8.20}$$
$$X^2_t(x)=X^2_{q_n}(x)+w^2_t-w^2_{q_n}+l^1_t(x)-l^1_{q_n}(x), \tag{8.21}$$

and $X_t(y)$ can be written in the same manner. Then, by (8.21)

$$d(X^2_t(x)-X^2_t(y))^2=2(X^2_t(x)-X^2_t(y))(dl^1_t(x)-dl^1_t(y))\le 0, \qquad \forall t\in[q_n,q_{n+1}],$$

and we obtain that

$$\sup_{t\in[q_n,q_{n+1}]}|X^2_t(x)-X^2_t(y)|\le|X^2_{q_n}(x)-X^2_{q_n}(y)|. \tag{8.22}$$

From (8.21) and (8.22) we conclude that

$$|l^1_t(x)-l^1_t(y)|\le|X^2_t(x)-X^2_t(y)|+|X^2_{q_n}(x)-X^2_{q_n}(y)|\le 2\,|X^2_{q_n}(x)-X^2_{q_n}(y)| \tag{8.23}$$

and from (8.20), (8.22) and (8.23) that

$$\begin{aligned}|X^1_t(x)-X^1_t(y)|&\le|X^1_{q_n}(x)-X^1_{q_n}(y)|+|a|\,|l^1_t(x)-l^1_t(y)|\\&\le|X^1_{q_n}(x)-X^1_{q_n}(y)|+2\,|a|\,|X^2_{q_n}(x)-X^2_{q_n}(y)|\end{aligned} \tag{8.24}$$

for all $t\in[q_n,q_{n+1}]$. By induction assumption, it is obvious from (8.22), (8.23) and (8.24) that (8.11) holds for $n+1$ in place of n. □

8.3 Computation of the Local Times

The local time $l^1(x)$ can be computed directly by applying SKOROHOD's Lemma (see Lemma 2.3) to equation (8.4). This yields

$$l^1_t(x) = \sup_{s\le t}\left[x^2 + w^2_s - b\,l^2_s(x)\right]^- = \left[-x^2 - \inf_{s\le t}\left(w^2_s - b\,l^2_s(x)\right)\right]^+, \qquad t\ge 0.$$

Fix any q_n. Since $X^2_{r_1(q_n)}(x)=0$ and $t\mapsto l^1_t(x)$ is increasing, we have for all $s\le r_1(q_n)$:

$$\begin{aligned} w^2_{r_1(q_n)} - b\,l^2_{r_1(q_n)}(x) &= -x^2 - l^1_{r_1(q_n)}(x) \le -x^2 - l^1_s(x) = -X^2_s(x) + w^2_s - b\,l^2_s(x)\\ &\le w^2_s - b\,l^2_s(x). \end{aligned} \tag{8.25}$$

Therefore, for all $t\in A_n$:

$$l^1_t(x) = l^1_{r_1(q_n)}(x) = \left[-x^2 - w^2_{r_1(q_n)} + b\,l^2_{r_1(q_n)}(x)\right]^+. \tag{8.26}$$

Note that by our restrictions to θ_2 in (8.1) one can easily check that

$$\operatorname{sign}(m)(m+b) > 0. \tag{8.27}$$

Recall that

$$Z_t(x) = \operatorname{sign}(m)\,\tilde x + \operatorname{sign}(m)\,\tilde w_t - \operatorname{sign}(m)\,(1+ma)\,l^1_t(x) + \operatorname{sign}(m)\,(m+b)\,l^2_t(x), \qquad t\ge 0,$$

and by SKOROHOD's Lemma we obtain

$$\operatorname{sign}(m)\,(m+b)\,l^2_t(x) = \left[-\operatorname{sign}(m)\,\tilde x - \inf_{s\le t}\left(\operatorname{sign}(m)\,\tilde w_s - \operatorname{sign}(m)\,(1+ma)\,l^1_s(x)\right)\right]^+, \quad t\ge 0.$$

Using $Z_{r_2(q_n)}(x)=0$, (8.27) and the fact that $t\mapsto l^2_t(x)$ is increasing, we get for all $s\le r_2(q_n)$:

$$\begin{aligned} \operatorname{sign}(m)\left(\tilde w_{r_2(q_n)} - (1+ma)\,l^1_{r_2(q_n)}(x)\right) &= -\operatorname{sign}(m)\,\tilde x - \operatorname{sign}(m)\,(m+b)\,l^2_{r_2(q_n)}(x)\\ &\le -\operatorname{sign}(m)\,\tilde x - \operatorname{sign}(m)\,(m+b)\,l^2_s(x)\\ &= -Z_s(x) + \operatorname{sign}(m)\left(\tilde w_s - (1+ma)\,l^1_s(x)\right)\\ &\le \operatorname{sign}(m)\left(\tilde w_s - (1+ma)\,l^1_s(x)\right), \end{aligned}$$

so that

$$\operatorname{sign}(m)\,(m+b)\,l^2_{r_2(q_n)}(x) = \left[-\operatorname{sign}(m)\,\tilde x - \operatorname{sign}(m)\,\tilde w_{r_2(q_n)} + \operatorname{sign}(m)\,(1+ma)\,l^1_{r_2(q_n)}(x)\right]^+.$$

Using again (8.27), we obtain for all $t\in A_n$:

$$l^2_t(x) = l^2_{r_2(q_n)}(x) = \left[\tfrac{1}{m+b}\left(-\tilde x - \tilde w_{r_2(q_n)} + (1+ma)\,l^1_{r_2(q_n)}(x)\right)\right]^+. \tag{8.28}$$

Next we compute the local times of the process with perturbed starting point. In the sequel we shall use the abbreviation $x_\varepsilon := x+\varepsilon\,e^j$, resp. $\tilde x_\varepsilon := \tilde x + \varepsilon\,e^j$, $\varepsilon\in\mathbb{R}$, $j\in\{1,2\}$, where $(e^j)_{j=1,2}$ denotes again the canonical basis of $\mathbb{R}^2$ and $|\varepsilon|$ is always supposed to be sufficiently small, so that x_ε resp. $\tilde x_\varepsilon$ lies in $S\backslash\{0\}$.

Lemma 8.4. *For all q_n, $n \in \mathbb{N}$, there exists a random $\Delta_n > 0$ such that for all $|\varepsilon| < \Delta_n$ a.s.:*

i) $l^1_{q_n}(x_\varepsilon) = \left[-x^2_\varepsilon - w^2_{r_1(q_n)} + b\, l^2_{r_1(q_n)}(x_\varepsilon)\right]^+$,

ii) $l^2_{q_n}(x_\varepsilon) = \left[\frac{1}{m+b}\left(-\tilde{x_\varepsilon} - \tilde{w}_{r_2(q_n)} + (1+ma)\, l^1_{r_2(q_n)}(x_\varepsilon)\right)\right]^+$.

Proof. We prove only i) (the proof of ii) is completely analogous). Using again SKOROHOD's Lemma, we obtain for all ε and q_n:

$$\begin{aligned} l^1_{q_n}(x_\varepsilon) &= \left[-x^2_\varepsilon - \inf_{s \le q_n}\left(w^2_s - b\, l^2_s(x_\varepsilon)\right)\right]^+ = \left[-x^2_\varepsilon - \inf_{s \le q_n}\left(w^2_s - b\, l^2_s(x) + b\left(l^2_s(x) - l^2_s(x_\varepsilon)\right)\right)\right]^+ \\ &= \left[-x^2_\varepsilon - \inf_{s \le q_n}\left(f(s) + g_\varepsilon(s)\right)\right]^+, \end{aligned}$$

where

$$f(s) := w^2_s - b\, l^2_s(x), \quad g_\varepsilon(s) := b\left(l^2_s(x) - l^2_s(x_\varepsilon)\right).$$

From the calculation of $l^1(x)$ above we know that

$$\inf_{s \le q_n} f(s) = f(r_1(q_n)),$$

and we have to show that for sufficiently small $|\varepsilon|$:

$$\inf_{s \le q_n}\left(f(s) + g_\varepsilon(s)\right) = f(r_1(q_n)) + g_\varepsilon(r_1(q_n)). \tag{8.29}$$

This is clear, if $q_n < \inf C^2$ or $q_n < \inf C^1$. Namely, in the first case we have $Z_s(x) > 0$ for all $s \in [0, q_n]$ and by Lemma 8.3 we can find a $\Delta_n > 0$ such that $Z(x_\varepsilon) > 0$ on $[0, q_n]$ for all $|\varepsilon| < \Delta_n$, which implies $l^2_s(x) = l^2_s(x_\varepsilon) = 0$ and thus $g_\varepsilon(s) = 0$ for all $s \in [0, q_n]$. In the second case we can conclude analogously that there exists a $\Delta_n > 0$ such that $X^2_s(x_\varepsilon) > 0$ and $l^1_s(x) = l^1_s(x_\varepsilon) = 0$ for all $s \in [0, q_n]$, if $|\varepsilon| < \Delta_n$, i.e. i) holds.
Moreover, we may assume that $s(q_n) = 1$. Otherwise, setting $q'_n = \max\{q_l \le q_n : s(q_l) = 1\}$, we have $X^2(x) > 0$ on $[q'_n, q_n]$ and, applying again Lemma 8.3, $X^2_s(x_\varepsilon) > 0$ for all $s \in [q'_n, q_n]$ and for $|\varepsilon|$ small enough, so that $l^1_{q'_n}(x_\varepsilon) = l^1_{q_n}(x_\varepsilon)$.
Therefore it is enough to consider the case $q_n > \max\{\inf C^1, \inf C^2\}$ and $s(q_n) = 1$, which implies $l^1_{q_n}(x) > 0$. Then, we know that $Z_s(x) > 0$ for all $s \in [q_{n-1}, q_n]$. We apply again Lemma 8.3 and find a $\Delta'_n > 0$ such that $Z_s(x_\varepsilon) > 0$ for all $s \in [q_{n-1}, q_n]$ and $|\varepsilon| < \Delta'_n$. Hence, for such ε it follows that g_ε is constant on $[q_{n-1}, q_n]$, so that

$$\inf_{s \in [q_{n-1}, q_n]}\left(f(s) + g_\varepsilon(s)\right) = \inf_{s \in [q_{n-1}, q_n]} f(s) + g_\varepsilon(r_1(q_n)) = f(r_1(q_n)) + g_\varepsilon(r_1(q_n)). \tag{8.30}$$

Since $q_n > \inf C^1$, $s(q_n) = 1$ and C^1 is the support of $l^1(x)$, we have $l^1_s(x) < l^1_{q_n}(x)$ for all $s \in [0, q_{n-1}]$, which implies $d := l^1_{q_n}(x) - l^1_{q_{n-1}}(x) > 0$, and, proceeding as in (8.25), we get for all $s \in [0, q_{n-1}]$:

$$\begin{aligned} w^2_{r_1(q_n)} - b\, l^2_{r_1(q_n)}(x) &= -x^2 - l^1_{r_1(q_n)}(x) = -x^2 - l^1_{q_n}(x) = -x^2 - l^1_{q_{n-1}}(x) - d \le -x^2 - l^1_s(x) - d \\ &= -X^2_s(x) + w^2_s - b\, l^2_s(x) - d \le w^2_s - b\, l^2_s(x) - d. \end{aligned}$$

Hence,

$$\inf_{s\le q_{n-1}} f(s) - f(r_1(q_n)) \ge d. \tag{8.31}$$

By Lemma 8.3 there exists a random $\Delta_n'' > 0$ such that a.s.

$$\sup_{s\le q_n} |g_\varepsilon(s)| < \frac{d}{2}, \qquad \forall |\varepsilon| < \Delta_n''. \tag{8.32}$$

Now using (8.31) and (8.32) we obtain for such ε:

$$\begin{aligned}\inf_{s\le q_{n-1}} (f(s) + g_\varepsilon(s)) &\ge \inf_{s\le q_{n-1}} f(s) - \sup_{s\le q_{n-1}} |g_\varepsilon(s)| > d + f(r_1(q_n)) - \frac{d}{2} \\ &= f(r_1(q_n)) + \frac{d}{2} > f(r_1(q_n)) + g_\varepsilon(r_1(q_n)).\end{aligned} \tag{8.33}$$

Combining (8.30) and (8.33) shows that (8.29) holds for all $|\varepsilon| < \Delta_n := \min(\Delta_n', \Delta_n'')$, and the claim follows.

□

The following lemma will be useful in the next subsection, when we compute the difference quotients of X. Recall the definition of (q_{n_k}) in (8.8) and that $c := b(ma+1)/(m+b)$.

Lemma 8.5. *For all q_n there exists a random $\Delta_n > 0$ such that for all $|\varepsilon| < \Delta_n$ a.s.:*

$$\text{i)} \quad l^1_{q_n}(x_\varepsilon) - l^1_{q_n}(x) = (x^2 - x^2_\varepsilon) \sum_{i=1}^{N_{12}(q_n)} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon) \tfrac{b}{m+b} \sum_{j=1}^{N_{21}(q_n)} c^{j-1}, \qquad \text{if } s(q_n) = 2$$

$$\text{ii)} \quad l^2_{q_n}(x_\varepsilon) - l^2_{q_n}(x) = (x^2 - x^2_\varepsilon) \tfrac{ma+1}{m+b} \sum_{i=1}^{N_{12}(q_n)} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon) \tfrac{1}{m+b} \sum_{j=1}^{N_{21}(q_n)} c^{j-1}, \quad \text{if } s(q_n) = 1.$$

Proof. i) Let q_n be such that $s(q_n) = 2$ and $k := N_{12}(q_n) + N_{21}(q_n)$ be the number of crossings through the wedge before time q_n. If $k = 0$, i.e. $q_n < \inf C^1$, by Lemma 8.3 we have $l^1_{q_n}(x_\varepsilon) = l^1_{q_n}(x) = 0$ for sufficiently small $|\varepsilon|$, i.e. i) holds trivially. Therefore, we can assume $k \ge 1$. Then, note that $q_{n_k} = \max\{q_{n_l} : q_{n_l} < q_n\}$. Now applying Lemma 8.3 and Lemma 8.4, we can choose $\Delta_n > 0$ such that for all $|\varepsilon| < \Delta_n$:

- $l^i_{q_l}(x) = l^i_{q_l}(x_\varepsilon) = 0$, if $q_l < \inf C^i$, and both of them are strictly positive, if $q_l > \inf C^i$, for all $l \le n$, $i = 1, 2$,
- the formulae i) and ii) in Lemma 8.4 hold with q_{n_l} in place of q_n for all $l \le k$,
- $l^2(x_\varepsilon)$ is constant on $[q_{n_{l-1}}, q_{n_l}]$ for all q_{n_l}, $2 \le l \le k$, satisfying $s(q_{n_l}) = 1$, (note that $l^2(x)$ is constant on $[q_{n_{l-1}}, q_{n_l}]$ by definition of (q_{n_l})),
- $l^1(x_\varepsilon)$ is constant on $[q_{n_{l-1}}, q_{n_l}]$ for all q_{n_l}, $2 \le l \le k$, satisfying $s(q_{n_l}) = 2$,
- $l^1(x)$ and $l^1(x_\varepsilon)$ are constant on $[q_{n_k}, q_n]$.

The last requirement ensures that it suffices to prove formula i) with q_{n_k} in place of q_n in the left hand side. Now we prove this by induction over k.
Since $s(q_n) = 2$, for $k = 1$ we have $N_{12}(q_n) = 1$, $N_{21}(q_n) = 0$ and $q_{n_1} < \inf C^2$ so that by (8.26) and Lemma 8.4 i):

$$l^1_{q_{n_1}}(x_\varepsilon) - l^1_{q_{n_1}}(x) = x^2 - x_\varepsilon^2 + b\left(l^2_{r_1(q_{n_1})}(x_\varepsilon) - l^2_{r_1(q_{n_1})}(x)\right) = x^2 - x_\varepsilon^2.$$

For $k = 2$, i.e. $N_{12}(q_n) = N_{21}(q_n) = 1$ and $q_{n_1} < \inf C^1$, we can use again (8.26) and Lemma 8.4 i) and afterwards (8.28) and Lemma 8.4 ii) to obtain:

$$\begin{aligned} l^1_{q_{n_2}}(x_\varepsilon) - l^1_{q_{n_2}}(x) &= x^2 - x_\varepsilon^2 + b\left(l^2_{r_1(q_{n_2})}(x_\varepsilon) - l^2_{r_1(q_{n_2})}(x)\right) = x^2 - x_\varepsilon^2 + b\left(l^2_{q_{n_1}}(x_\varepsilon) - l^2_{q_{n_1}}(x)\right) \\ &= x^2 - x_\varepsilon^2 + \tfrac{b}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon + (ma+1)\left(l^1_{r_1(q_{n_1})}(x_\varepsilon) - l^1_{r_1(q_{n_1})}(x)\right)\right) \\ &= x^2 - x_\varepsilon^2 + \tfrac{b}{m+b}(\tilde{x} - \tilde{x}_\varepsilon). \end{aligned}$$

Now assume that we have exactly $k+2$ crossings through the wedge before time q_n and that the induction assumption

$$l^1_{q_{n_k}}(x_\varepsilon) - l^1_{q_{n_k}}(x) = (x^2 - x_\varepsilon^2)\sum_{i=1}^{N_{12}(q_n)-1} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon)\tfrac{b}{m+b}\sum_{j=1}^{N_{21}(q_n)-1} c^{j-1}$$

holds a.s. (note that before time q_{n_k} we have exactly two crossings less than before time $q_{n_{k+2}}$). Then, proceeding as for $k = 2$ and using the induction assumption, it follows that

$$\begin{aligned} l^1_{q_{n_{k+2}}}(x_\varepsilon) - l^1_{q_{n_{k+2}}}(x) =& x^2 - x_\varepsilon^2 + b\left(l^2_{q_{n_{k+1}}}(x_\varepsilon) - l^2_{q_{n_{k+1}}}(x)\right) \\ =& x^2 - x_\varepsilon^2 + \tfrac{b}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon + (ma+1)\left(l^1_{q_{n_k}}(x_\varepsilon) - l^1_{q_{n_k}}(x)\right)\right) \\ =& x^2 - x_\varepsilon^2 + \tfrac{b(ma+1)}{m+b}(x^2 - x_\varepsilon^2)\sum_{i=1}^{N_{12}(q_n)-1} c^{i-1} \\ &+ \tfrac{b}{m+b}(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{b(ma+1)}{m+b}(\tilde{x} - \tilde{x}_\varepsilon)\tfrac{b}{m+b}\sum_{j=1}^{N_{21}(q_n)-1} c^{j-1} \\ =& (x^2 - x_\varepsilon^2)\sum_{i=1}^{N_{12}(q_n)} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon)\tfrac{b}{m+b}\sum_{j=1}^{N_{21}(q_n)} c^{j-1}, \end{aligned}$$

since $c = b(ma+1)/(m+b)$ by definition, which completes the proof of i).
ii) We shall proceed exactly as in i). Let q_n be such that $s(q_n) = 1$ and set again $k := N_{12}(q_n) + N_{21}(q_n)$. Analogously to i), the case $k = 0$ is trivial, so we can suppose $k \geq 1$ and choose $\Delta_n > 0$ as in i), where in the last requirement l^1 should be replaced by l^2. Hence, we can prove ii) with q_{n_k} in place of q_n in the left hand side of ii) by induction over k: Let $|\varepsilon| < \Delta_n$. For $k = 1$ we have $N_{12}(q_n) = 0$, $N_{21}(q_n) = 1$ and $q_{n_1} < \inf C^1$. We use (8.28) and Lemma 8.4 ii) to obtain:

$$l^2_{q_{n_1}}(x_\varepsilon) - l^2_{q_{n_1}}(x) = \tfrac{1}{m+b}(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\left(l^1_{r_2(q_{n_1})}(x_\varepsilon) - l^1_{r_2(q_{n_1})}(x)\right) = \tfrac{1}{m+b}(\tilde{x} - \tilde{x}_\varepsilon).$$

For $k = 2$, which implies $N_{12}(q_n) = N_{21}(q_n) = 1$ and $q_{n_1} < \inf C^2$, applying again (8.28) and Lemma 8.4 ii) and afterwards (8.26) and Lemma 8.4 i), we get

$$\begin{aligned} l^2_{q_{n_2}}(x_\varepsilon) - l^2_{q_{n_2}}(x) &= \tfrac{1}{m+b}\,(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\left(l^1_{q_{n_1}}(x_\varepsilon) - l^1_{q_{n_1}}(x)\right) \\ &= \tfrac{1}{m+b}\,(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\,(x^2 - x^2_\varepsilon) + \tfrac{b\,(ma+1)}{m+b}\left(l^2_{r_1(q_{n_1})}(x_\varepsilon) - l^2_{r_1(q_{n_1})}(x)\right) \\ &= \tfrac{1}{m+b}\,(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\,(x^2 - x^2_\varepsilon). \end{aligned}$$

Now assume that we have exactly $k+2$ crossings before time q_n and that the induction assumption

$$l^2_{q_{n_k}}(x_\varepsilon) - l^2_{q_{n_k}}(x) = (x^2 - x^2_\varepsilon)\,\tfrac{ma+1}{m+b}\sum_{i=1}^{N_{12}(q_n)-1} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon)\,\tfrac{1}{m+b}\sum_{j=1}^{N_{21}(q_n)-1} c^{j-1}$$

holds a.s.
Arguing as for $k = 2$ and using the induction assumption, we obtain:

$$\begin{aligned} &l^2_{q_{n_{k+2}}}(x_\varepsilon) - l^2_{q_{n_{k+2}}}(x) \\ =&\tfrac{1}{m+b}\,(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\left(l^1_{q_{n_{k+1}}}(x_\varepsilon) - l^1_{q_{n_{k+1}}}(x)\right) \\ =&\tfrac{1}{m+b}\,(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\,(x^2 - x^2_\varepsilon) + \tfrac{b\,(ma+1)}{m+b}\left(l^2_{q_{n_k}}(x_\varepsilon) - l^2_{q_{n_k}}(x)\right) \\ =&\tfrac{1}{m+b}\,(\tilde{x} - \tilde{x}_\varepsilon) + \tfrac{ma+1}{m+b}\,(x^2 - x^2_\varepsilon) \\ &+ \tfrac{b\,(ma+1)}{m+b}\left((x^2 - x^2_\varepsilon)\,\tfrac{ma+1}{m+b}\sum_{i=1}^{N_{12}(q_n)-1} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon)\,\tfrac{1}{m+b}\sum_{j=1}^{N_{21}(q_n)-1} c^{j-1}\right) \\ =&(x^2 - x^2_\varepsilon)\,\tfrac{ma+1}{m+b}\sum_{i=1}^{N_{12}(q_n)} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon)\,\tfrac{1}{m+b}\sum_{j=1}^{N_{21}(q_n)} c^{j-1}, \end{aligned}$$

and the claim follows. □

8.4 Computation of the Derivatives

Let $t \in [0, \tau_0)\backslash C$ be fixed and n such that $t \in A_n$. We choose $\Delta_n > 0$ such that for all $|\varepsilon| < \Delta_n$:

- $l^i_{q_l}(x) = l^i_{q_l}(x_\varepsilon) = 0$, if $q_l < \inf C^i$, and both of them are strictly positive, if $q_l > \inf C^i$, for all $l \leq n$, $i = 1, 2$,
- the formulae i) and ii) in Lemma 8.5 and in Lemma 8.4 hold,
- $l^2(x_\varepsilon)$ is constant on $[r_1(q_n), q_n]$ if $s(q_n) = 1$,
- $l^1(x_\varepsilon)$ is constant on $[r_2(q_n), q_n]$ if $s(q_n) = 2$,
- $l^i_t(x_\varepsilon) = l^i_{q_n}(x_\varepsilon)$, $i = 1, 2$.

Now let $0 < |\varepsilon| < \Delta_n$.

Derivatives of X^1

From (8.3) we get immediately

$$X_t^1(x_\varepsilon) - X_t^1(x) = x_\varepsilon^1 - x^1 - a\left(l_{q_n}^1(x_\varepsilon) - l_{q_n}^1(x)\right) + l_{q_n}^2(x_\varepsilon) - l_{q_n}^2(x). \tag{8.34}$$

Case 1: $s(t) = 0$. Then, $X_t^1(x_\varepsilon) - X_t^1(x) = x_\varepsilon^1 - x^1$, so that obviously

$$\frac{\partial X_t^1(x)}{\partial x^1} = 1, \qquad \frac{\partial X_t^1(x)}{\partial x^2} = 0.$$

Case 2: $s(t) = 1$. We insert formula (8.26), Lemma 8.4 i) and afterwards Lemma 8.5 ii) into (8.34) to obtain:

$$\begin{aligned} X_t^1(x_\varepsilon) - X_t^1(x) =& x_\varepsilon^1 - x^1 - a\left(x^2 - x_\varepsilon^2 + b\left(l_{r_1(q_n)}^2(x_\varepsilon) - l_{r_1(q_n)}^2(x)\right)\right) + l_{q_n}^2(x_\varepsilon) - l_{q_n}^2(x) \\ =& x_\varepsilon^1 - x^1 - a\,(x^2 - x_\varepsilon^2) + (1 - ab)\left(l_{q_n}^2(x_\varepsilon) - l_{q_n}^2(x)\right) \\ =& x_\varepsilon^1 - x^1 - a\,(x^2 - x_\varepsilon^2) \\ &+ (1 - ab)\left((x^2 - x_\varepsilon^2)\,\tfrac{ma+1}{m+b}\sum_{i=1}^{N_{12}(t)} c^{i-1} + (\tilde{x} - \tilde{x}_\varepsilon)\,\tfrac{1}{m+b}\sum_{j=1}^{N_{21}(t)} c^{j-1}\right). \end{aligned}$$

i) Let $x_\varepsilon = x + \varepsilon\, e^1$, which implies $x_\varepsilon^2 - x^2 = 0$ and $\tilde{x}_\varepsilon - \tilde{x} = m\,\varepsilon$. Recall that $c := b(ma + 1)/(m + b)$. Then, we get

$$\frac{1}{\varepsilon}\left(X_t^1(x_\varepsilon) - X_t^1(x)\right) = 1 - \tfrac{m(1-ab)}{m+b}\sum_{j=1}^{N_{21}(t)} c^{j-1} = 1 - (1 - c)\sum_{j=1}^{N_{21}(t)} c^{j-1} = c^{N_{21}(t)},$$

and, since the right hand side does not depend on ε,

$$\frac{\partial X_t^1(x)}{\partial x^1} = c^{N_{21}(t)}.$$

ii) If $x_\varepsilon = x + \varepsilon\, e^2$, i.e. $x_\varepsilon^2 - x^2 = \varepsilon$ and $\tilde{x}_\varepsilon - \tilde{x} = -\varepsilon$, we obtain

$$\begin{aligned} \frac{1}{\varepsilon}\left(X_t^1(x_\varepsilon) - X_t^1(x)\right) &= a - \tfrac{1-ab}{m+b}\,(ma + 1)\sum_{i=1}^{N_{12}(t)} c^{i-1} + \tfrac{1-ab}{m+b}\sum_{j=1}^{N_{21}(t)} c^{j-1} \\ &= a - \tfrac{ma+1}{m}\,(1 - c)\sum_{i=1}^{N_{12}(t)} c^{i-1} + \tfrac{1}{m}\,(1 - c)\sum_{j=1}^{N_{21}(t)} c^{j-1} \\ &= a - \tfrac{ma+1}{m}(1 - c^{N_{12}(t)}) + \tfrac{1}{m}(1 - c^{N_{21}(t)}) = \tfrac{ma+1}{m}c^{N_{12}(t)} - \tfrac{1}{m}c^{N_{21}(t)}. \end{aligned}$$

Hence,

$$\frac{\partial X_t^1(x)}{\partial x^2} = \frac{1}{m}\left((ma + 1)\,c^{N_{12}(t)} - c^{N_{21}(t)}\right).$$

Case 3: $s(t) = 2$. We insert formula (8.28), Lemma 8.4 ii) and finally Lemma 8.5 i) into (8.34). This yields:

$$\begin{aligned}
X_t^1(x_\varepsilon) - X_t^1(x) =& x_\varepsilon^1 - x^1 - a\left(l_{q_n}^1(x_\varepsilon) - l_{q_n}^1(x)\right) \\
&+ \tfrac{1}{m+b}\left[\tilde{x} - \tilde{x}_\varepsilon + (ma+1)\left(l_{r_2(q_n)}^1(x_\varepsilon) - l_{r_2(q_n)}^1(x)\right)\right] \\
=& x_\varepsilon^1 - x^1 + \tfrac{1}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon\right) + \left(\tfrac{ma+1}{m+b} - a\right)\left(l_{q_n}^1(x_\varepsilon) - l_{q_n}^1(x)\right) \\
=& x_\varepsilon^1 - x^1 + \tfrac{1}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon\right) \\
&+ \tfrac{1-ab}{m+b}\left(x^2 - x_\varepsilon^2\right)\sum_{i=1}^{N_{12}(t)} c^{i-1} + \left(\tilde{x} - \tilde{x}_\varepsilon\right)\tfrac{1-ab}{m+b}\,\tfrac{b}{m+b}\sum_{j=1}^{N_{21}(t)} c^{j-1}.
\end{aligned}$$

i) Let $x_\varepsilon = x + \varepsilon\, e^1$. Then,

$$\begin{aligned}
\frac{1}{\varepsilon}\left(X_t^1(x_\varepsilon) - X_t^1(x)\right) &= 1 - \tfrac{m}{m+b} - \tfrac{(1-ab)m}{m+b}\,\tfrac{b}{m+b}\sum_{j=1}^{N_{21}(t)} c^{j-1} = \tfrac{b}{m+b} - \tfrac{b}{m+b}(1-c)\sum_{j=1}^{N_{21}(t)} c^{j-1} \\
&= \tfrac{b}{m+b}\, c^{N_{21}(t)},
\end{aligned}$$

which implies

$$\frac{\partial X_t^1(x)}{\partial x^1} = \tfrac{b}{m+b}\, c^{N_{21}(t)}.$$

ii) If $x_\varepsilon = x + \varepsilon\, e^2$, we obtain

$$\begin{aligned}
\frac{1}{\varepsilon}\left(X_t^1(x_\varepsilon) - X_t^1(x)\right) &= \tfrac{1}{m+b} - \tfrac{1-ab}{m+b}\sum_{i=1}^{N_{12}(t)} c^{i-1} + \tfrac{1-ab}{m+b}\,\tfrac{b}{m+b}\sum_{j=1}^{N_{21}(t)} c^{j-1} \\
&= \tfrac{1}{m+b} - \tfrac{1}{m}(1-c)\sum_{i=1}^{N_{12}(t)} c^{i-1} + \tfrac{b}{m(m+b)}(1-c)\sum_{j=1}^{N_{21}(t)} c^{j-1} \\
&= \tfrac{1}{m+b} - \tfrac{1}{m}\left(1 - c^{N_{12}(t)}\right) + \tfrac{b}{m(m+b)}\left(1 - c^{N_{21}(t)}\right) \\
&= \tfrac{1}{m}\left(c^{N_{12}(t)} - \tfrac{b}{m+b}\, c^{N_{21}(t)}\right),
\end{aligned}$$

so that

$$\frac{\partial X_t^1(x)}{\partial x^2} = \tfrac{1}{m}\left(c^{N_{12}(t)} - \tfrac{b}{m+b}\, c^{N_{21}(t)}\right).$$

Derivatives of X^2

From (8.4) we deduce directly

$$X_t^2(x_\varepsilon) - X_t^2(x) = x_\varepsilon^2 - x^2 + l_{q_n}^1(x_\varepsilon) - l_{q_n}^1(x) - b\left(l_{q_n}^2(x_\varepsilon) - l_{q_n}^2(x)\right). \tag{8.35}$$

Case 1: $s(t) = 0$. Clearly,

$$\frac{\partial X_t^2(x)}{\partial x^1} = 0, \qquad \frac{\partial X_t^2(x)}{\partial x^2} = 1.$$

Case 2: $s(t) = 1$. We insert formula (8.26) and Lemma 8.4 i) into (8.35) to obtain:

$$\begin{aligned} X_t^2(x_\varepsilon) - X_t^2(x) &= x_\varepsilon^2 - x^2 + x^2 - x_\varepsilon^2 + b\left(l^2_{r_1(q_n)}(x_\varepsilon) - l^2_{r_1(q_n)}(x)\right) - b\left(l^2_{q_n}(x_\varepsilon) - l^2_{q_n}(x)\right) \\ &= 0, \end{aligned}$$

and it follows that

$$\frac{\partial X_t^2(x)}{\partial x^1} = \frac{\partial X_t^2(x)}{\partial x^2} = 0.$$

Case 3: $s(t) = 2$. We use again formula (8.28), Lemma 8.4 ii) and Lemma 8.5 i) in (8.35). This yields:

$$\begin{aligned} X_t^2(x_\varepsilon) - X_t^2(x) =& x_\varepsilon^2 - x^2 + l^1_{q_n}(x_\varepsilon) - l^1_{q_n}(x) \\ & - \tfrac{b}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon + (ma+1)\left(l^1_{r_2(q_n)}(x_\varepsilon) - l^1_{r_2(q_n)}(x)\right)\right) \\ =& x_\varepsilon^2 - x^2 - \tfrac{b}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon\right) \\ & + \left(x^2 - x_\varepsilon^2\right)(1-c)\sum_{i=1}^{N_{12}(q_n)} c^{i-1} + \left(\tilde{x} - \tilde{x}_\varepsilon\right)\tfrac{b}{m+b}(1-c)\sum_{j=1}^{N_{21}(q_n)} c^{j-1} \\ =& x_\varepsilon^2 - x^2 - \tfrac{b}{m+b}\left(\tilde{x} - \tilde{x}_\varepsilon\right) + \left(x^2 - x_\varepsilon^2\right)\left(1 - c^{N_{12}(t)}\right) + \left(\tilde{x} - \tilde{x}_\varepsilon\right)\tfrac{b}{m+b}\left(1 - c^{N_{21}(t)}\right) \\ =& -\left(x^2 - x_\varepsilon^2\right)c^{N_{12}(t)} - \left(\tilde{x} - \tilde{x}_\varepsilon\right)\tfrac{b}{m+b}\, c^{N_{21}(t)}. \end{aligned}$$

If $x_\varepsilon = x + \varepsilon\, e^1$, we obtain

$$\frac{\partial X_t^2(x)}{\partial x^1} = \tfrac{mb}{m+b}\, c^{N_{21}(t)},$$

and, if $x_\varepsilon = x + \varepsilon\, e^2$, we get

$$\frac{\partial X_t^2(x)}{\partial x^2} = c^{N_{12}(t)} - \tfrac{b}{m+b}\, c^{N_{21}(t)}.$$

The proof of Theorem 8.1 is now complete.

8.5 Orthogonale Reflection

We consider the case of orthogonale reflection to illustrate the results of Theorem 8.1:

Example 8.6 (Orthogonale Reflection)**.** For $\xi \in (0,\pi)\backslash\{\frac{\pi}{2}\}$ we set $\theta_1 = \theta_2 = 0$, which implies $a = 0$ and from (8.2) we get

$$c = \cos^2\xi, \qquad \frac{b}{m+b} = c = \cos^2\xi, \qquad \frac{m}{m+b} = 1 - \frac{b}{m+b} = \sin^2\xi.$$

Then, for any $t \in [0,\tau_0)\backslash C$ satisfying $s(t) = 1$ we obtain, using (8.9):

$$\begin{aligned} \frac{\partial X_t^1(x)}{\partial x^2} &= \frac{1}{m}\left(c^{N_{12}(t)} - c^{N_{21}(t)}\right) = \frac{1-c}{m}\, c^{N_{12}(t)}\, \mathbb{1}_{\{N_{21}(t) = N_{12}(t)+1\}} \\ &= \frac{\sin^2\xi}{\tan\xi}\,(\cos\xi)^{2N_{12}(t)}\, \mathbb{1}_{\{\inf C^2 < \inf C^1\}} = \sin\xi\,(\cos\xi)^{2N_{12}(t)+1}\, \mathbb{1}_{\{\inf C^2 < \inf C^1\}}, \end{aligned}$$

and if $s(t) = 2$ we get by (8.10):

$$\begin{aligned}\frac{\partial X_t^1(x)}{\partial x^2} &= \frac{1}{m}\left(c^{N_{12}(t)} - c^{N_{21}(t)+1}\right) = \frac{1-c}{m}\, c^{N_{12}(t)}\, \mathbb{1}_{\{N_{12}(t)=N_{21}(t)\}} \\ &= \sin\xi\,(\cos\xi)^{2N_{12}(t)+1}\,\mathbb{1}_{\{\inf C^2<\inf C^1\}}.\end{aligned}$$

Hence, the results in Theorem 8.1 become

$$\frac{\partial X_t^1(x)}{\partial x^1} = \begin{cases} 1 & \text{if } s(t)=0, \\ (\cos\xi)^{2N_{21}(t)} & \text{if } s(t)=1, \\ (\cos\xi)^{2N_{21}(t)+2} & \text{if } s(t)=2, \end{cases}$$

$$\frac{\partial X_t^1(x)}{\partial x^2} = \begin{cases} 0 & \text{if } s(t)=0, \\ \sin\xi\,(\cos\xi)^{2N_{12}(t)+1}\,\mathbb{1}_{\{\inf C^2<\inf C^1\}} & \text{if } s(t)=1, \\ \sin\xi\,(\cos\xi)^{2N_{12}(t)+1}\,\mathbb{1}_{\{\inf C^2<\inf C^1\}} & \text{if } s(t)=2, \end{cases}$$

and

$$\frac{\partial X_t^2(x)}{\partial x^1} = \begin{cases} 0 & \text{if } s(t)=0, \\ 0 & \text{if } s(t)=1, \\ \sin\xi\,(\cos\xi)^{2N_{21}(t)+1} & \text{if } s(t)=2, \end{cases}$$

$$\frac{\partial X_t^2(x)}{\partial x^2} = \begin{cases} 1 & \text{if } s(t)=0, \\ 0 & \text{if } s(t)=1, \\ \sin^2\xi\,(\cos\xi)^{2N_{12}(t)}\,\mathbb{1}_{\{\inf C^2<\inf C^1\}} & \text{if } s(t)=2. \end{cases}$$

8.6 The Case $\xi = \frac{\pi}{2}$

The case $\xi = \frac{\pi}{2}$ can be treated very similarly. The only reason, why we consider this case seperately, is that the model has to be modified slightly, since the boundary of the wedge has to be represented in a different manner. We shall use the same notation as before:
For $\xi = \frac{\pi}{2}$ we have the wedge $S = \mathbb{R}_+^2$ with the two sides $\partial S_1 := \left\{(x^1, x^2) \in \mathbb{R}_+^2 : x^2 = 0\right\}$ and $\partial S_2 := \left\{(x^1, x^2) \in \mathbb{R}_+^2 : x^1 = 0\right\}$. Furthermore,

$$v_1 = (-\tan\theta_1, 1) =: (-a, 1), \quad v_2 = (1, -\tan\theta_2) =: (1, -b), \qquad \theta_1, \theta_2 \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}),$$

and we assume again $\alpha := (\theta_1 + \theta_2)/\xi < 2$ to ensure existence and uniqueness in law. Let τ_0 be as above in (8.6) so that (8.7) holds.
We need to consider the following system of Skorohod SDEs:

i) $X_t^1(x) = x^1 + w_t^1 - a\, l_t^1(x) + l_t^2(x),$
$X_t^2(x) = x^2 + w_t^2 + l_t^1(x) - b\, l_t^2(x), \qquad t \geq 0,$

ii) $X_t^i(x) \geq 0,\ dl_t^i(x) \geq 0,\ t \geq 0,\ i \in \{1, 2\}, \quad \int_0^\infty X_t^2(x)\, dl_t^1(x) = 0, \quad \int_0^\infty X_t^1(x)\, dl_t^2(x) = 0$

for all $x \in S\backslash\{0\}$ and a Brownian motion w as before.

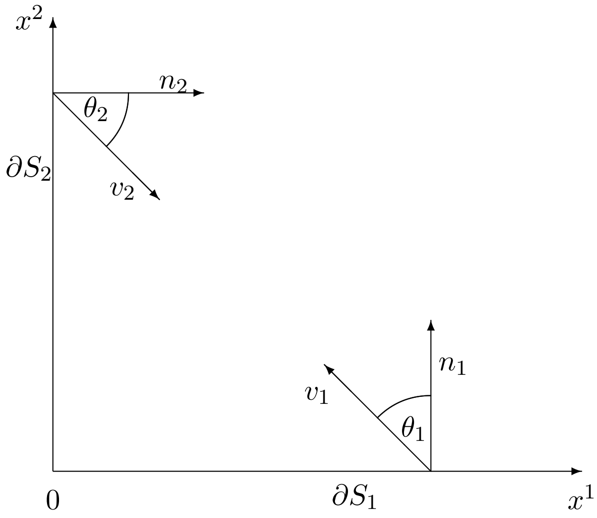

Figure 8.2: Wegde with $\xi = \frac{\pi}{2}$, $\theta_1 > 0$ and $\theta_2 > 0$

Theorem 8.7. *The mapping $x \mapsto X_t(x)$, $x \in S\backslash\{0\}$, is differentiable a.s. for all $t \in [0, \tau_0)\backslash C$, and the derivatives are given by:*

$$\frac{\partial X_t^1(x)}{\partial x^1} = \begin{cases} 1 & \text{if } s(t) = 0, \\ (ab)^{N_{21}(t)} & \text{if } s(t) = 1, \\ 0 & \text{if } s(t) = 2, \end{cases} \qquad \frac{\partial X_t^1(x)}{\partial x^2} = \begin{cases} 0 & \text{if } s(t) = 0, \\ a(ab)^{N_{12}(t)} & \text{if } s(t) = 1, \\ 0 & \text{if } s(t) = 2, \end{cases} \tag{8.36}$$

and

$$\frac{\partial X_t^2(x)}{\partial x^1} = \begin{cases} 0 & \text{if } s(t) = 0, \\ 0 & \text{if } s(t) = 1, \\ b(ab)^{N_{21}(t)} & \text{if } s(t) = 2, \end{cases} \qquad \frac{\partial X_t^2(x)}{\partial x^2} = \begin{cases} 1 & \text{if } s(t) = 0, \\ 0 & \text{if } s(t) = 1, \\ (ab)^{N_{12}(t)} & \text{if } s(t) = 2. \end{cases} \tag{8.37}$$

Remark 8.8. If we consider the case of orthogonale reflection at the boundary, i.e. $\theta_1 = \theta_2 = 0$, which implies $a = b = 0$, this result corresponds to that of Theorem 2.2.

Sketch of proof. Let A_n and q_n be as before. For fixed q_n, using again SKOROHOD's Lemma as at the beginning of Section 8.3, it easily follows that for all $t \in A_n$:

$$l_t^1(x) = l_{r_1(q_n)}^1(x) = \left[-x^2 - \inf_{s \le r_1(q_n)} \left(w_s^2 - b\, l_s^2(x)\right)\right]^+ = \left[-x^2 - w_{r_1(q_n)}^2 + b\, l_{r_1(q_n)}^2(x)\right]^+ \tag{8.38}$$

and

$$l_t^2(x) = l_{r_2(q_n)}^2(x) = \left[-x^1 - \inf_{s \le r_2(q_n)} \left(w_s^1 - a\, l_s^1(x)\right)\right]^+ = \left[-x^1 - w_{r_2(q_n)}^1 + a\, l_{r_2(q_n)}^1(x)\right]^+. \tag{8.39}$$

Analogously to Lemma 8.4 and Lemma 8.5 one can show that for suffiently small $|\varepsilon|$

$$l_{q_n}^1(x_\varepsilon) = \left[-x_\varepsilon^2 - \inf_{s \le r_1(q_n)} \left(w_s^2 - b\, l_s^2(x_\varepsilon)\right)\right]^+ = \left[-x_\varepsilon^2 - w_{r_1(q_n)}^2 + b\, l_{r_1(q_n)}^2(x_\varepsilon)\right]^+ \tag{8.40}$$

and

$$l^2_{q_n}(x_\varepsilon) = \left[-x^1_\varepsilon - \inf_{s \le r_2(q_n)} \left(w^1_s - a\, l^1_s(x_\varepsilon)\right)\right]^+ = \left[-x^1_\varepsilon - w^1_{r_2(q_n)} + a\, l^1_{r_2(q_n)}(x_\varepsilon)\right]^+, \tag{8.41}$$

and furthermore that

$$l^1_{q_n}(x_\varepsilon) - l^1_{q_n}(x) = (x^2 - x^2_\varepsilon) \sum_{i=1}^{N_{12}(q_n)} (ab)^{i-1} + (x^1 - x^1_\varepsilon)\, b \sum_{j=1}^{N_{21}(q_n)} (ab)^{j-1}, \qquad \text{if } s(q_n) = 2, \tag{8.42}$$

as well as

$$l^2_{q_n}(x_\varepsilon) - l^2_{q_n}(x) = (x^2 - x^2_\varepsilon)\, a \sum_{i=1}^{N_{12}(q_n)} (ab)^{i-1} + (x^1 - x^1_\varepsilon) \sum_{j=1}^{N_{21}(q_n)} (ab)^{j-1}, \qquad \text{if } s(q_n) = 1. \tag{8.43}$$

For small $|\varepsilon| > 0$ we can now compute the derivatives of X by a similar proceeding as in Section 8.4. We consider a fixed q_n and $t \in A_n$. Then,

$$X^1_t(x_\varepsilon) - X^1_t(x) = x^1_\varepsilon - x^1 - a\,\left(l^1_{q_n}(x_\varepsilon) - l^1_{q_n}(x)\right) + l^2_{q_n}(x_\varepsilon) - l^2_{q_n}(x),$$

which is equal to $x^1_\varepsilon - x^1$ if $s(t) = 0$. Otherwise, if $s(t) = 1$, we use (8.38), (8.40) and (8.43) to obtain

$$\begin{aligned} X^1_t(x_\varepsilon) - X^1_t(x) &= x^1_\varepsilon - x^1 - a(x^2 - x^2_\varepsilon) + (1 - ab)\left(l^2_{q_n}(x_\varepsilon) - l^2_{q_n}(x)\right) \\ &= x^1_\varepsilon - x^1 - a(x^2 - x^2_\varepsilon) + a(x^2 - x^2_\varepsilon)(1 - (ab)^{N_{12}(t)}) + (x^1 - x^1_\varepsilon)(1 - (ab)^{N_{21}(t)}) \\ &= a(ab)^{N_{12}(t)}(x^2_\varepsilon - x^2) + (ab)^{N_{21}(t)}(x^1_\varepsilon - x^1), \end{aligned}$$

and if $s(t) = 2$, inserting (8.39), (8.41) yields $X^1_t(x_\varepsilon) - X^1_t(x) = 0$. Hence, it becomes obvious that (8.36) holds.
Analogously, we obtain the derivatives of X^2 in (8.37), where we simply apply (8.39), (8.41) and (8.42) in the case $s(t) = 2$. □

Finally, comparing the results of Theorem 8.1 and Theorem 8.7, we observe that the derivatives are continuous in ξ, since $m \uparrow \infty$ as $\xi \uparrow \frac{\pi}{2}$ (resp. $m \downarrow -\infty$ as $\xi \downarrow \frac{\pi}{2}$) and $c \to ab$ as $\xi \to \frac{\pi}{2}$.

8.7 The Neumann Condition

Corollary 8.9. *Suppose $\alpha \le 0$. Then, for all bounded continuous f and $t > 0$, the transition semigroup $P_t f(x) := \mathbb{E}[f(X_t(x))]$, $x \in S\backslash\{0\}$, satisfies the* NEUMANN *condition at ∂S:*

$$x \in \partial S_i \Longrightarrow D_{v_i} P_t f(x) = 0, \qquad i = 1, 2,$$

where $D_{v_i} := v_i \cdot \nabla$ is the directional derivative operator associated with the direction of reflection v_i on the side ∂S_i, $i = 1, 2$.

Sketch of proof. Let $i = 1$ (the case $i = 2$ can be treated analogously). By a density argument it is sufficient to consider bounded functions f, which are continuously differentiable and have bounded derivatives. Recall that we have $\tau_0 = \infty$ a.s., since $\alpha \le 0$. Let $t > 0$ and $x \in \partial S_1$, which implies immediately $\inf C^1 < \inf C^2$, so that by (8.9) and (8.10)

$$N_{21}(t) = N_{12}(t), \text{ if } s(t) = 1 \quad \text{and} \quad N_{12}(t) = N_{21}(t) + 1, \text{ if } s(t) = 2. \tag{8.44}$$

Since $v_1 = (-a, 1)$, we have

$$D_{v_1} P_t f(x) = -a \frac{\partial}{\partial x^1} \mathbb{E}[f(X_t(x))] + \frac{\partial}{\partial x^2} \mathbb{E}[f(X_t(x))].$$

Using the properties of reflected BROWNian motions, a.s. the set C has zero LEBESGUE measure and $t \notin C$ a.s. Hence

$$D_{v_1} P_t f(x) = -a \frac{\partial}{\partial x^1} \mathbb{E}[f(X_t(x)) \mathbb{1}_{\{t \in [0,\infty) \setminus C\}}] + \frac{\partial}{\partial x^2} \mathbb{E}[f(X_t(x)) \mathbb{1}_{\{t \in [0,\infty) \setminus C\}}].$$

By the results in Theorem 8.1 resp. in Theorem 8.7 the partial derivatives of $X_t(x)$ depend only on constants and the number of crossings before time t. Using the fact that the paths are continuous w.r.t. the sup-norm by Lemma 8.3, we conclude that the derivatives of $x \mapsto X_t(x)$ are at least locally bounded. Since the derivatives of f are also bounded, we may write the differentials in the expectation by dominated convergence (see e.g. Korollar 16.3 in [1]) and obtain by chain rule:

$$\begin{aligned} D_{v_1} P_t f(x) =& \mathbb{E}\Bigg[-a \left(\frac{\partial f}{\partial x^1}(X_t(x)) \frac{\partial X_t^1(x)}{\partial x^1} + \frac{\partial f}{\partial x^2}(X_t(x)) \frac{\partial X_t^2(x)}{\partial x^1} \right) \mathbb{1}_{\{t \in [0,\infty) \setminus C\}} \\ &+ \left(\frac{\partial f}{\partial x^1}(X_t(x)) \frac{\partial X_t^1(x)}{\partial x^2} + \frac{\partial f}{\partial x^2}(X_t(x)) \frac{\partial X_t^2(x)}{\partial x^2} \right) \mathbb{1}_{\{t \in [0,\infty) \setminus C\}} \Bigg] \\ =& \mathbb{E}\Bigg[\frac{\partial f}{\partial x^1}(X_t(x)) \left(-a \frac{\partial X_t^1(x)}{\partial x^1} + \frac{\partial X_t^1(x)}{\partial x^2} \right) \mathbb{1}_{\{t \in [0,\infty) \setminus C\}} \\ &+ \frac{\partial f}{\partial x^2}(X_t(x)) \left(-a \frac{\partial X_t^2(x)}{\partial x^1} + \frac{\partial X_t^2(x)}{\partial x^2} \right) \mathbb{1}_{\{t \in [0,\infty) \setminus C\}} \Bigg]. \end{aligned}$$

By inserting the formulae for the derivatives of X, established in Theorem 8.1 resp. in Theorem 8.7, and using (8.44), one can easily check that for each $t \in [0,\infty) \setminus C$ a.s.

$$-a \frac{\partial X_t^1(x)}{\partial x^1} + \frac{\partial X_t^1(x)}{\partial x^2} = 0 \qquad \text{and} \qquad -a \frac{\partial X_t^2(x)}{\partial x^1} + \frac{\partial X_t^2(x)}{\partial x^2} = 0.$$

□

Remark 8.10. For an arbitrary α, it follows by the same reasoning that

$$x \in \partial S_i \Longrightarrow D_{v_i} \mathbb{E}\left[f(X_t(x)) \mathbb{1}_{\{t < \tau_0\}} \right] = 0, \qquad i = 1, 2,$$

i.e. the transition semigroup associated with the process, which is absorbed at the corner, satisfies the NEUMANN condition.

List of Frequently Used Notation

$:=$	equal to by definition
$\equiv$	identification of two functions
I	finite set of indices
$\vert I\vert$	the cardinality of I, i.e. the number of elements contained in I
$\mathbb{R}^I$	the $\vert I\vert$-dimensional Euclidian space; $\mathbb{R} = \mathbb{R}^1$
$\mathbb{R}_+$	the nonnegative real numbers
$\mathbb{R}_+^I$	$:= [0,\infty)^I$, the set of $\vert I\vert$-tuples of nonnegative real numbers
$\mathbb{N}$	$:= \{1, 2, \ldots\}$, the natural numbers
$\mathbb{Q}$	the set of rational numbers
$\Vert . \Vert$	the Euclidian norm on $\mathbb{R}^I$; $\Vert x\Vert^2 = \sum_{i\in I}(x^i)^2$
v^T	the transposed of the vector v
$a \wedge b$	$:= \min\{a, b\}$, the minimum of $a, b \in \mathbb{R}$
$a \vee b$	$:= \max\{a, b\}$, the maximum of $a, b \in \mathbb{R}$
a^+	$:= \max\{a, 0\}$ if $a \in \mathbb{R}$
a^-	$:= \max\{-a, 0\}$ if $a \in \mathbb{R}$
$C(U)$	the continuous functions from U into $\mathbb{R}$
$C^k(U)$	the functions in $C(U)$ with continuous derivatives up to order k
$D(U, V)$	the space of functions from U into V which are right continuous and have left limits (i.e. cadlag functions)
$\operatorname{supp}\mu$	the support of the measure μ
$f\vert_K$	the restriction of the function f to the set K
∇	the gradient: $\nabla f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n})$ for $f = f(x^1, \ldots, x^n)$
D_v	$:= v \cdot \nabla := \sum_{i=1}^n v_i \frac{\partial}{\partial x^i}$, the directional derivative operator associated with the direction $v = (v_1, \ldots, v_n)$
e	$:= \exp(1)$, the EULER number
δ_{ij}	the KRONECKER delta; $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$
$\mathbb{1}_A$	the indicator function of the set A; $\mathbb{1}_A(x) = 1$ if $x \in A$, $\mathbb{1}_A(x) = 0$ if $x \notin A$
∂A	the boundary of the set A
$\mathcal{P}(A)$	the power set of the set A
$\operatorname{sign}(x)$	$:= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$
e^i	$(e^i)_{i\in I}$ denotes the canonical basis of $\mathbb{R}^I$, i.e. $e^i(j) = \delta_{ij}$
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$	the underlying filtered probability space
$\mathbb{E}$	the expectation operator w.r.t. $\mathbb{P}$
$\mathbb{E}[X\vert\mathcal{A}]$	the conditional expectation of the random variable X w.r.t. the σ-algebra $\mathcal{A}$
$P_{s,i}^c$, $P_{s,i}$, $\tilde{P}_{0,i}^{s,t}$, $\hat{P}_{s,i}^T$	probability measures on the space of the I-valued cadlag functions such that the coordinate process is MARKOVian with time-dependent generator
$E_{s,i}^c$, $E_{s,i}$, $\tilde{E}_{0,i}^{s,t}$, $\hat{E}_{s,i}^T$	the expectation operators w.r.t. $P_{s,i}^c$, $P_{s,i}$, $\tilde{P}_{0,i}^{s,t}$ and $\hat{P}_{s,i}^T$
L_t^c, L_t	time-dependent generator of the coordinate process under $P_{s,i}^c$ and $P_{s,i}$
P_t, $P_{s,t}$	transition semigroup of homogeneous and inhomegeneous MARKOV processes
$\mu \otimes \nu$	the product measure of μ and ν
$\sigma(X)$	the smallest σ-algebra with respect to which the random variable X is measurable
$\sigma(X_s;\, 0 \leq s \leq t)$	the smallest σ-algebra with respect to which the random variable X_s is measurable for all $s \in [0, t]$

$\langle X \rangle$	the quadratic variation process of X
$\langle X, Y \rangle$	the quadratic covariation process of X and Y
M^τ	the stopped martingale process M for any stopping time τ
o	the LANDAU symbol
w.r.t.	with respect to
a.s., a.e.	almost surely, almost everywhere
$\square$	end of proof

"increasing" is used with the same meaning as "non-decreasing" and "decreasing" with the same meaning as "non-increasing". In strict cases "strictly increasing" resp. "strictly decreasing" are used.

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Zusammenfassung

Das Hauptziel dieser Diplomarbeit ist zu zeigen, dass die Lösungen von Systemen stochastischer Differentialgleichungen (SDE) mit Reflektionsterm, sog. SKOROHOD SDEs, bezüglich ihres deterministischen Anfangswertes pfadweise differenzierbar sind.
Vorgelegt sei also für eine endliche Indexmenge I das folgende System solcher SKOROHOD SDEs:

$$\begin{aligned} X_t^i(x) &= x^i + \int_0^t b^i(X_r(x))\,dr + l_t^i(x) + \int_0^t \sigma^i(X_r^i(x))\,dw_r^i, \quad t \geq 0,\, i \in I, \\ X_t^i(x) &\geq 0, \quad dl_t^i(x) \geq 0, \quad \int_0^\infty X_t^i(x)\,dl_t^i(x) = 0, \qquad i \in I, \end{aligned} \tag{A.1}$$

für alle $x \in \mathbb{R}_+^I$, wobei die Koeffizientenfunktionen $b^i : \mathbb{R}_+^I \to \mathbb{R}$ und $\sigma^i : \mathbb{R}_+ \to \mathbb{R}$, $i \in I$, stetig differenzierbar und LIPSCHITZ stetig seien. $(w^i)_{i\in I}$ bezeichne dabei eine Familie unabhängiger BROWNscher Bewegungen auf einem Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, \mathbb{P})$.
Das Ziel ist nun zu zeigen, dass die Lösungen $X_t(x)$, $t \geq 0$, fast sicher bzgl. x^j, $j \in I$, partiell differenzierbar sind, und probabilistische Darstellungen für die Ableitungen zu finden.
Dieses Problem ist bereits für den Fall $\sigma^i \equiv 1$, $i \in I$, in [3] gelöst worden: Das Hauptresultat besagt, dass sich die Ableitungen durch eine gewöhnliche Differentialgleichung beschreiben lassen, wenn der Prozess sich im Innern des $\mathbb{R}_+^I$ befindet, und dass sie Null werden, wenn der Prozess den Rand trifft. Aufgrund der komplexen Struktur der Menge von Zeitpunkten, wo der Prozess den Rand trifft, wird diese Darstellunng ziemlich kompliziert. Jedoch ergibt sich für die Ableitungen eine einfache Darstellung mit Hilfe eines Random Walks mit Werten in der Indexmenge I.
In dieser Diplomarbeit wird nun versucht, dieses Resultat auf das System (A.1) zu verallgemeinern unter stärkeren Voraussetzungen an die Koeffizientenfunktionen. Mit Hilfe einer Variablensubstitution, der sog. LAMPERTI Methode, wird dazu das System (A.1) in ein System mit konstanten Diffusionskoeffizienten überführt und dann die Differenzierbarkeit wie in [3] gezeigt.
Als weitere wesentliche Ergänzung der Resultate in [3] wird der Prozess einer BROWNschen Bewegung in einem Keil (engl. "wedge") mit schiefer Reflektion, welcher sich auch mit Hilfe eines zweidimensionalen Systems von SKOROHOD Gleichungen beschreiben lässt, auf pfadweise Differenzierbarkeit nach dem Anfangswert untersucht. Die erhaltenen Ableitungen sind konstant auf jedem Zeitintervall, wo sich der Prozess im Inneren des Keils befindet, und hängen von der Anzahl der bisherigen Überquerungen des Keils ab.
Die Arbeit ist wie folgt gegliedert: In Abschnitt 3 wird gezeigt, dass die Lösungen von (A.1) stetig vom Anfangswert abhängen, sogar in dem Fall, wenn die Diffusionskoeffizienten von allen Komponenten von X abhängen, bevor dann in Abschnitt 4 die Lösungen auf Differenzierbarkeit untersucht werden.
In Abschnitt 6 wird eine alternative Beweismöglichkeit der Random Walk Darstellung im Fall $\sigma^i \equiv 1$, $i \in I$, diskutiert, wobei vorausgesetzt wird, dass die Drift-Koeffizienten nicht negativ sind. Dabei wird ein "Penalizations"-Ansatz benutzt, um Approximationen der Lösungen zu konstruieren, und eine Random Walk Darstellung für diese Approximationen hergeleitet. Als Vorbereitung werden in einem Exkurs in Abschnitt 5 FEYNMAN-KAC Formeln für MARKOV Prozesse mit zeitabhängigem Generator behandelt.
In Abschnitt 7 werden dann die partiellen Ableitungen der Übergangshalbgruppe von X berechnet, wobei mittels Maßwechsel und GIRSANOV-Transformation die Komponenten von X entkoppelt werden.
Abschließend behandelt Abschnitt 8 die pfadweise Differenzierbarkeit einer BROWNschen Bewegung in einem Keil.