Roller Coasters without Differential Equations – A Newtonian Approach to Constrained Motion

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Abstract

Within the context of Newton's equation, we develop a simple approach to the constrained motion of a body forced to move along a specified trajectory. Because the formalism uses a local frame of reference it is simpler than previous methods, making more complicated geometries accessible. No Lagrangian multipliers are necessary to determine the constraining forces. Although the method is able to deal with friction, it becomes particularly simple for conservative systems. We give an analytic expression for the constraining force for any two-dimensional frictionless trajectory that can be written in the form $y = f(x)$. The approach is illustrated with examples from roller coaster physics, e. g. the circular loop or the clothoid loop. We will find analytic expressions for the constraining force in both cases.

I. INTRODUCTION

When a body slides along a guiding track its motion is said to be constrained. Constrained motion is not easily described by Newton's equation of motion. In order to make the body follow a curved track, a force must act between the track and the body. This force is called *constraining force*. The trouble is that it is not a priori known. Thus, in Newton's equation $\vec{F} = m \cdot \vec{a}$, neither the total force $\vec{F}$ nor the acceleration $\vec{a}$ are known at any given time. It seems that Newton's equation does not suffice to solve the problem.

Of course, problems with constrained motion *can* be solved. Holonomous constraints[10] are routinely treated in textbooks of theoretical physics. Usually, one uses an approach where the governing equations are not Newton's but Lagrange's (see e. g. [1]). Generalized coordinates are chosen so that the number of dynamic equations is reduced to the number of independent degrees of freedom

Although the Lagrangian approach is perfectly valid in principle, there may be problems in practice. The guiding track may have a complicated shape. Then it is difficult to find a set of coordinates in which the constraint equations become simple (technically: one or more of the coordinates have to become "cyclic", i. e. they must not explicitly appear in the Lagrangian). Usually, one does not see the Lagrange formalism applied to a guiding track more complicated than a circle or a parabola.

Can problems with constrained motion be solved with Newton's equation? It is indeed possible to include the constraining force into Newton's law [2–4]. For a single particle forced to move under a constraint $f(\vec{r}, t) = 0$, Newton's equation takes the form

$$m \cdot \ddot{\vec{r}} = \vec{F}_{\text{other}} + \lambda \vec{\nabla} f, \tag{1}$$

where the last term on the right hand side represents the constraining force while $\vec{F}_{\text{other}}$ stands for other (external) forces. The gradient fixes the *direction* of the constraining force because it is orthogonal to the surfaces of constant f. The unknown constant λ enters because we do not know the *magnitude* of the constraining force. It is called a Lagrange multiplier.

The three equations (1) together with the constraint equation $f(\vec{r}, t) = 0$ are sufficient to solve for the four unknowns $\vec{r}(t)$ and λ. Thus, problems with constrained motion can in principle be solved with the Lagrange multiplier method. As with the approach described earlier, problems arise in practical calculations because in all but the simplest problems it is virtually impossible to find a coordinate system in which the constraint equation becomes tractable.

In the present paper we present an approach in which the practical difficulties that render the traditional methods almost useless are avoided. We consider the case of a particularly simple holonomic constraint: a body is forced to move on a fixed trajectory $\vec{r}(t)$. The key observation that characterizes our approach is the following: In order to formulate Newton's law for a body moving along a given trajectory we do not need a global set of coordinates which is adapted to the problem. It is sufficient to define coordinates in the vicinity of the given trajectory. In the language of General Relativity: We only need a local frame of reference. Elementary differential geometry provides the mathematical tools needed.

The paper is organized as follows. In Sec. II we review the basics of elementary differential geometry needed to formulate our approach. Secs. III–IV deal with velocity and acceleration for constrained motion and introduce the notion of constraining forces. Newton's law for constrained motion is formulated in Sec V. From Sec. VI on we will talk about conservative systems (without friction). In this case, not even a differental equation has to be solved to obtain the constraining force. We will be able to derive a closed expression for the constraining force for a fairly general class of trajectories (Sec. VII).

Roller coasters are an interesting physics application of the type of constrained motion considered here. In order to make the presentation more vivid we will talk about roller coasters from now on (a more detailed discussion of roller coaster physics can be found in [5, 6]). In example problems we will discuss some well-known (as well as some not-so-well-known) results from roller coaster physics. To illustrate the power of the formalism even in nontrivial situations we will treat the clothoid loop in Sec. IX.

II. THE GEOMETRY OF CONSTRAINED MOTION

We consider the type of motion illustrated in Fig. 1. A body is forced to move along a given trajectory in three dimensions. Before we start a physical analysis we will give a purely mathematical description of the motion. Elementary differential geometry provides a neat way for doing this. A curve in three-dimensional space is described by a set of three mutually orthogonal unit vectors at any point of the curve, the well-known Frénet vectors.

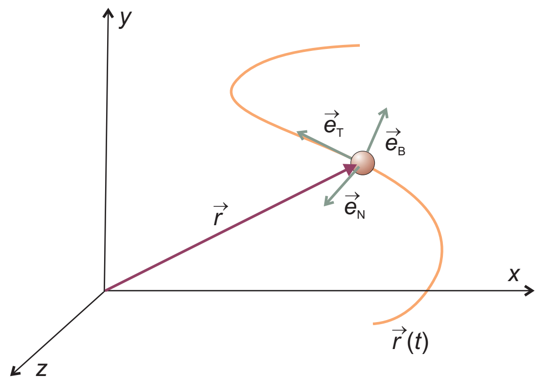

FIG. 1: A body is forced to move along a given trajectory. On each point of the trajectory, a set of three orthogonal unit vectors can be defined.

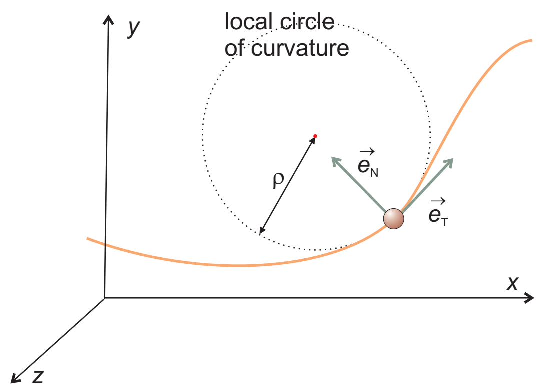

FIG. 2: The normal vector is directed to the center of the local circle of curvature.

Although the topic is covered in many mathematical texts [7] as well as some physics books [8] we find it useful to review some properties of the Frénet vectors. At any point of the curve, the *tangent vector* $\vec{e}_{\mathrm{T}}$ is directed tangential to the curve (Fig. 1). Its direction thus changes from point to point, whereas is length remains fixed to 1 by definition.

The direction of the *normal vector* $\vec{e}_{\mathrm{N}}$ is chosen so that it points to the local center of curvature (Fig. 2). It is orthogonal to the tangent vector and has unit length. Finally, the third of the Frénet vectors, the *binormal vector* $\vec{e}_{\mathrm{B}}$ is defined to be orthogonal to both $\vec{e}_{\mathrm{B}}$ and $\vec{e}_{\mathrm{N}}$. It has unit length, too.

In principle, the curve can be parametrized by an arbitrary variable t. A particularly advantageous parametrization uses the *arc length* s, defined by

$$s = \int_0^{t_0} v(t)\,\mathrm{d}t, \tag{2}$$

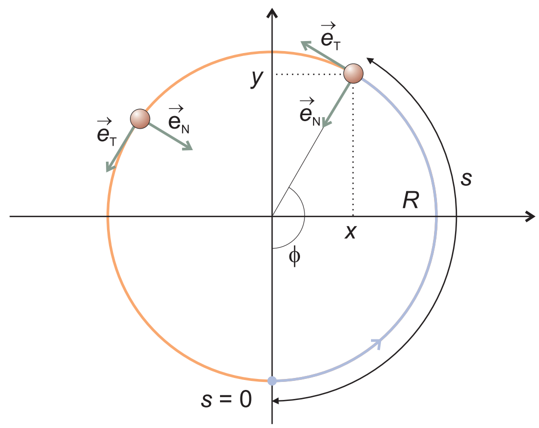

FIG. 3: Circular trajectory

where

$$v(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}. \tag{3}$$

Differentiation of Eq. (2) leads to the relation

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \dot{s}. \tag{4}$$

Explicit expressions for the tangent vector and the normal vector can be obtained as follows. For a given curve $\vec{r}(s)$, the *tangent vector* is calculated by differentiation:

$$\vec{e}_{\mathrm{T}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}. \tag{5}$$

If s denotes the arc length, this vector is normalized to unity automatically. The *normal vector* is defined by

$$\vec{e}_{\mathrm{N}} = \rho \cdot \frac{\mathrm{d}\vec{e}_{\mathrm{T}}}{\mathrm{d}s}. \tag{6}$$

The direction of $\vec{e}_{\mathrm{N}}$ is fixed so that the normalizing factor ρ becomes positive. ρ can be interpreted geometrically as the radius of the local circle of curvature (Fig. 2). Often, the reciprocal of ρ is used: the *local curvature* $\kappa = 1/\rho$.

Problem 1: Circular trajectory

Determine tangent vector, normal vector, and radius of curvature for the circular trajectory shown in Fig. 3:

$$x(t) = R\sin\omega t, \qquad y(t) = -R\cos\omega t, \qquad z(t) = 0. \tag{7}$$

Solution:

In Eq. (7) the trajectory is parametrized by the time t. To use the arc length instead, we calculate with Eq. (3):

$$v = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} = \omega R, \tag{8}$$

where the relation $\sin^2 \omega t + \cos^2 \omega t = 1$ was used. s und t are therefore related by

$$s = \int_0^t \omega R \, \mathrm{d}t = \omega R t. \tag{9}$$

Thus, when parametrized by the arc length, Eq. (7) becomes

$$\begin{aligned} \vec{r}(s) &= \big(x(s),\ y(s),\ z(s)\big) \\ &= \left(R \sin\left(\frac{s}{R}\right),\ -R\cos\left(\frac{s}{R}\right),\ 0\right). \end{aligned} \tag{10}$$

The tangent vector is obtained by differentiation of Eq. (10):

$$\vec{e}_{\mathrm{T}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s} = \left(\cos\left(\frac{s}{R}\right),\ \sin\left(\frac{s}{R}\right),\ 0\right). \tag{11}$$

As expected, the length of this vector is unity. Similarly, the normal vector is calculated according to Eq. (6):

$$\frac{\mathrm{d}\vec{e}_{\mathrm{T}}}{\mathrm{d}s} = \underbrace{\frac{1}{R}}_{=1/\rho} \cdot \underbrace{\left(-\sin\left(\frac{s}{R}\right),\ \cos\left(\frac{s}{R}\right),\ 0\right)}_{=\vec{e}_{\mathrm{N}}}. \tag{12}$$

It is directed to the center of the circle. Not unexpectedly, the local radius of curvature is constant along the trajectory and is equal to the radius of the circle R.

Problem 2: General trajectory of the form $y = f(x)$

Determine tangent vector, normal vector, and radius of curvature for any trajectory in the x-y plane that can be written in the form $y = f(x)$.

Solution:

This problem encompasses a large class of trajectories. Its solution will therefore open up a broad range of applications. In a roller coaster context, for example, arbitrarily shaped curves, hills, and valleys are covered. Loops and spirals, however, are not, because they cannot be represented by a single-valued function $f(x)$.

The position vector reads

$$\vec{r}(t) = \big(x(t),\ f(x(t)),\ 0\big). \tag{13}$$

Using the chain rule we write

$$\dot{y}(t) = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = f'(x) \cdot \dot{x}, \tag{14}$$

so that

$$v = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} = \sqrt{\dot{x}^2(t)\,(1 + f'^2(x))} \tag{15}$$

and

$$s = \int_0^t \sqrt{1 + f'^2(x)} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_{x(0)}^{x(t)} \sqrt{1 + f'^2(x)}\,\mathrm{d}x. \tag{16}$$

To calculate the tangent vector, we use the chain rule again:

$$\vec{e}_{\mathrm{T}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}s}. \tag{17}$$

By differentiation of Eq. (16) we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{\sqrt{1 + f'^2(x)}}. \tag{18}$$

With $\vec{r}$ from Eq. (13) and $\mathrm{d}\vec{r}/\mathrm{d}x = (1,\ f'(x),\ 0)$, this yields for the tangent vector:

$$\vec{e}_{\mathrm{T}} = (1,\ f'(x),\ 0) \cdot \frac{\mathrm{d}x}{\mathrm{d}s} \quad = \quad \frac{1}{\sqrt{1 + f'^2(x)}} \cdot (1,\ f'(x),\ 0)\,. \tag{19}$$

Finally we calculate the normal vector, again using the chain rule:

$$\begin{aligned} \frac{\mathrm{d}\vec{e}_{\mathrm{T}}}{\mathrm{d}s} &= \frac{\mathrm{d}\vec{e}_{\mathrm{T}}}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}s} \\ &= \underbrace{\frac{|f''(x)|}{(1 + f'^2(x))^{\frac{3}{2}}}}_{=\,1/\rho} \cdot \underbrace{\frac{\mathrm{sign}(f''(x))}{\sqrt{1 + f'^2(x)}} \cdot (-f'(x),\ 1,\ 0)}_{=\,\vec{e}_{\mathrm{N}}}\,. \end{aligned} \tag{20}$$

The local radius of curvature for a trajectory of the form $y = f(x)$ is thus given by:

$$\rho = \frac{(1 + f'^2(x))^{\frac{3}{2}}}{|f''(x)|}. \tag{21}$$

This formula (which is a standard result in differential geometry) completes the problem.

III. VELOCITY AND ACCELERATION

Up to now, everything has been mathematics. No physics was involved, even if we have given suggestive names to some quantities (like v). Let us now interpret $\vec{r}(t)$ as the physical

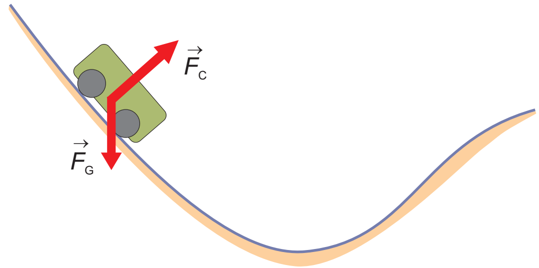

FIG. 4: The constraining force is directed normal to the track

trajectory of a moving body. The kinematical variables of its motion can be expressed with the help of the unit vectors considered above. The velocity, for example, can be written

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}}_{=\vec{e}_{\rm T}} \cdot \underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{=v} . \tag{22}$$

so that

$$\vec{v} = v \cdot \vec{e}_{\rm T} \qquad \text{with} \qquad v = \dot{s}. \tag{23}$$

Likewise, the acceleration can be obtained by differentiating Eq. (23):

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(v \cdot \vec{e}_{\rm T} \right) . \tag{24}$$

With the product rule we get:

$$\begin{aligned} \vec{a} &= \dot{v} \cdot \vec{e}_{\rm T} + v \cdot \frac{\mathrm{d}\vec{e}_{\rm T}}{\mathrm{d}t} \\ &= \dot{v} \cdot \vec{e}_{\rm T} + v \cdot \underbrace{\frac{\mathrm{d}\vec{e}_{\rm T}}{\mathrm{d}s}}_{=\vec{e}_{\rm N}/\rho} \cdot \underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{=v} . \end{aligned} \tag{25}$$

The acceleration of a moving body can thus be expressed solely in terms of the normal vector and the tangent vector (cf. e. g. [8]:

$$\vec{a} = \dot{v} \cdot \vec{e}_{\rm T} + \frac{v^2}{\rho} \cdot \vec{e}_{\rm N}. \tag{26}$$

By definition of the Frénet vectors, there is no component of the acceleration in the direction of the binormal vector:

IV. CONSTRAINING FORCES

In a roller coaster, the car is guided along a specified track (Fig. 4). If the track is curved, forces must act between track and car in order to change the direction of motion. The forces that accomplish these direction changes are called *constraining forces*.

The motion of the car is described by Newton's law $\vec{F} = m \cdot \vec{a}$. Some effort is required, however, to apply this law to constrained motion. First, we formally separate the constraining forces $\vec{F}_{\mathrm{C}}$ (which are a priori unknown) from all other forces (which we assume to be known):

$$\vec{F} = \vec{F}_{\mathrm{C}} + \vec{F}_{\mathrm{other}}. \tag{27}$$

We assume that the constraining forces have no component in tangential direction. They do not change the speed but only the direction of the car. Strictly speaking, this is not an assumption but a definition. In general, the force between track and car *will* have a component in tangential direction. However, we usually prefer to call these force components friction forces and assign them to the "other forces" mentioned above. The physics is unaffected by this formal separation; our formalism will be able to deal with friction forces.

The components of the constraining forces in the direction of $\vec{e}_{\mathrm{N}}$ and $\vec{e}_{\mathrm{B}}$ will be denoted by F_{C} and F'_{C}, respectively. Newton's law now reads

$$m \cdot \vec{a} = \vec{F}_{\mathrm{other}} + F_{\mathrm{C}} \cdot \vec{e}_{\mathrm{N}} + F'_{\mathrm{C}} \cdot \vec{e}_{\mathrm{B}}. \tag{28}$$

It cannot be solved in the usual manner because the constraining force components F_{C} and F'_{C} are still unknown. In the usual Newtonian approach, they are treated as Lagrange multipliers. We will find a simpler way to deal with them.

V. COMPONENTS OF NEWTON'S LAW

The problem of the unknown constraining force components can be solved by separating the tangential, normal, and binormal component of Newton's law. We insert $\vec{a}$ from Eq. (26) to get

$$m \cdot \dot{v} \cdot \vec{e}_{\mathrm{T}} + \frac{mv^2}{\rho} \cdot \vec{e}_{\mathrm{N}} = \vec{F}_{\mathrm{other}} + F_{\mathrm{C}} \cdot \vec{e}_{\mathrm{N}} + F'_{\mathrm{C}} \cdot \vec{e}_{\mathrm{B}}. \tag{29}$$

We multiply this equation by $\vec{e}_{\mathrm{T}}$ to obtain its tangential component. Using the orthonormality of the Frénet vectors we get

$$m \cdot \dot{v} = \vec{F}_{\mathrm{other}} \cdot \vec{e}_{\mathrm{T}}. \tag{30}$$

This equation can be regarded as *Newton's law along a specified track*. It is remarkable that the unknown constraining force components do not appear. The motion along the track is governed only by the tangential component of the other forces $\vec{F}_{\text{other}} \cdot \vec{e}_{\text{T}}$. Eq. (30) is a scalar equation for the change of the "speed" v along the specified track. As it stands, it can be solved for $v(t)$ and $s(t)$ by any of the standard methods.

The binormal component of Newton's law is particularly simple. By definition, there is no binormal component of the acceleration. Therefore the constraining force F'_{C} just cancels any component of the other forces $\vec{F}_{\text{other}}$ acting in this direction:

$$F'_{\text{C}} = -\vec{F}_{\text{other}} \cdot \vec{e}_{\text{B}}. \tag{31}$$

Finally, we obtain the normal component of Newton's law by multiplying Eq. (29) with $\vec{e}_{\text{N}}$:

$$\frac{mv^2}{\rho} = \vec{F}_{\text{other}} \cdot \vec{e}_{\text{N}} + F_{\text{C}}. \tag{32}$$

The interpretation of this equation is straightforward. The left hand side denotes the local centripetal force. In order to guide the car along the track, the total force acting on the car in normal direction must have the magnitude mv^2/ρ (it must act as centripetal force). The constraining force F_{C} adjusts itself so that this equality is fulfilled.

Once we have solved Eq. (30), we can insert $v(t)$ into the equation above to determine the constraining force component F_{C}:

$$F_{\text{C}} = \frac{mv^2}{\rho} - \vec{F}_{\text{other}} \cdot \vec{e}_{\text{N}}. \tag{33}$$

Our problem is solved. In addition to the kinematical variables $s(t)$ and $v(t)$, we are able to calculate the constraining force components F_{C} and F'_{C} that guide the car through the track.

VI. CONSTRAINED MOTION WITHOUT FRICTION

With the approach described above, problems with constrained motion along a given trajectory can be treated in a familiar manner, i. e. by solving ordinary differential equations. However, we can do even better for frictionless problems. If friction can be neglected, the total energy of the car is conserved. We assume that only kinetic and potential energy are relevant so that $E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \text{const}$. In this case, we need not solve the differential equation Eq. (30) for $v(t)$. It can be obtained much easier from energy conservation:

$$v(t) = \sqrt{\frac{2}{m}\left(E_{\text{tot}} - E_{\text{pot}}(t)\right)}. \tag{34}$$

FIG. 5: A swingboat as an example of a pendulum (Photo courtesy of Heidepark, Germany)

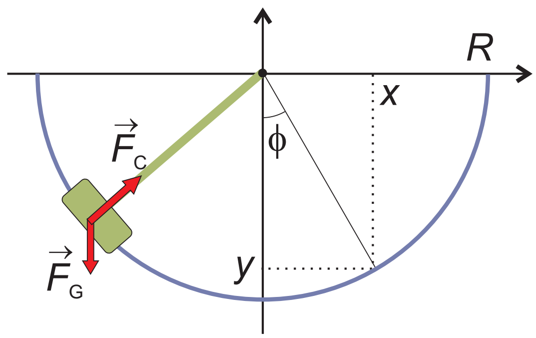

FIG. 6: Geometry of the pendulum

For roller coasters, the potential energy is $m \cdot g \cdot y$. The total energy is conveniently specified by the height of the lifthill y_0: $E_{\text{tot}} = m \cdot g \cdot y_0$. The top of the lifthill is the highest point of the track where the car is disengaged from the lifting cable. We assume the kinetic energy to be negligible there.

If we use Eq. (34) together with Eq. (33), we can determine the constraining forces without ever solving a differential equation:

$$F_{\text{C}} = \frac{2\,(E_{\text{tot}} - E_{\text{pot}})}{\rho} - \vec{F}_{\text{other}} \cdot \vec{e}_{\text{N}}. \tag{35}$$

This fact is a particularly appealing feature of the approach. In the following problem, we will apply the formalism to determine the constraining force acting in the rope of a pendulum. In keeping with the roller coaster spirit we will consider a swingboat as a specific example.

Problem 3: Pendulum

The swingboat shown in Fig. 5 has a total height of 15 m and and a pendulum length of

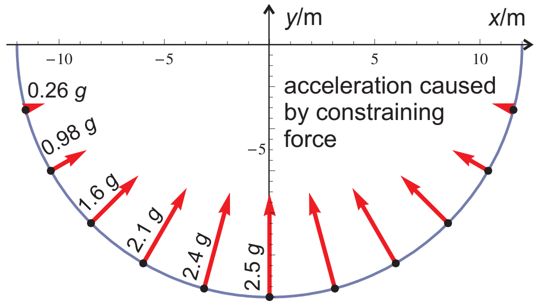

FIG. 7: Constraining acceleration (i. e. F_C/m) for the swingboat pendulum

12 m. It swings up to a maximum angle of 75°. Assume that it can be modeled as a point mass and determine the constraining force as well as the total force acting through the "pendulum rope" onto the body of the boat.

Solution:

The geometry of the problem is shown in Fig. 6. The constraining force F_C is calculated with Eq. (35). In Problem 1, we have already determined $\vec{e}_N$ for a circular trajectory so that we can write immediately (using $\phi = s/R$):

$$\vec{F}_G \cdot \vec{e}_N = -m \cdot g \cdot \cos\phi. \tag{36}$$

If we further use

$$y = -R \cdot \cos\phi \tag{37}$$

to simplify the potential energy, we can write down the following expression for the constraining force for the pendulum

$$F_C = \frac{2E_{tot}}{R} + 3\, mg\, \cos\phi. \tag{38}$$

Note that no small-angle approximation was required to obtain this result.

For the swingboat under consideration, the constraining force calculated with Eq. (38) is shown in Fig. 7 for various points of the trajectory. The constraining force is largest at the lowest point of the trajectory where the velocity has its maximum value (and gravity has to be cancelled, too). Because the passengers move together with the boat, our analysis applies also to the force acting from the seats onto the passengers. We see from Fig. 7 that they have to withstand a maximum acceleration of 2.5 g for the Heidepark swingboat.

The total force, shown in Fig. 8, is the vector sum of the constraining force and the gravitational force (i. e. the right-hand side of Eq. (28)). It is remarkable that for a simple

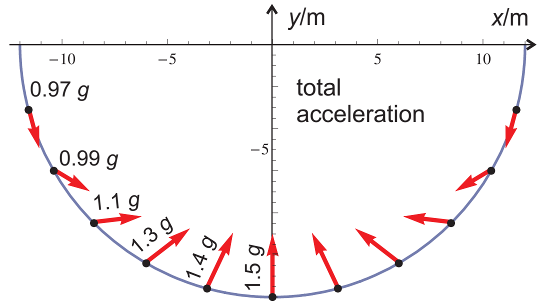

FIG. 8: Total acceleration for the swingboat pendulum

system like a pendulum the magnitude as well as the direction of the total force change in a relatively complicated way along the trajectory.

VII. THE ROLLER COASTER FORMULA

We now will derive a closed formula for the constraining force for any trajectory that can be represented in the form $y = f(x)$. We will assume y to be the vertical, hence the gravitational force has the components $\vec{F}_{\mathrm{G}} = (0,\ -m \cdot g,\ 0)$. In Problem 2 we have already calculated the normal vector for this type of motion (Eq. (20)) so that we can write

$$\vec{F}_{\mathrm{G}} \cdot \vec{e}_{\mathrm{N}} = -\frac{\mathrm{sign}(f''(x))}{\sqrt{1+f'^2(x)}} \cdot m \cdot g. \tag{39}$$

We have also determined the local radius of curvature in Eq. (21). As usual $v(t)$ is obtained from energy conservation:

$$v = \sqrt{\frac{2}{m}E_{\mathrm{tot}} - 2gf(x)}. \tag{40}$$

By inserting these expressions into Eq. (35) we obtain a formula for the constraining force under these very general circumstances. We will call it the *roller coaster formula*:

$$F_{\mathrm{C}} = \frac{2\,|f''(x)|}{(1+f'^2(x))^{\frac{3}{2}}}\ (E_{\mathrm{tot}} - m \cdot g \cdot f(x)) + \frac{\mathrm{sign}(f''(x))}{\sqrt{1+f'^2(x)}} \cdot m \cdot g. \tag{41}$$

Problem 4: Parabola-shaped hill

The parabola-shaped hill in Fig. 9 can be represented by the function $y = -c \cdot x^2$ (with $c > 0$). With the roller coaster formula, calculate the constraining force acting on the passengers in a car moving over the hill. What happens for $F_{\mathrm{C}} > 0$? How can a roller coaster constructor achieve "airtime" (i. e. weightlessness)?

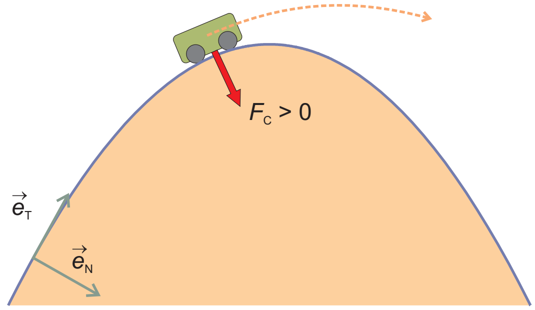

FIG. 9: Parabola-shaped hill

Solution:

We need the derivatives $f'(x) = -2cx$ and $f''(x) = -2c$. Hence, $\mathrm{sign}(f''(x)) = -1$ everywhere. From Eq. (41), we obtain:

$$F_\mathrm{C} = \frac{4c}{(1+4c^2x^2)^{\frac{3}{2}}}\left(E_\mathrm{tot} + mgcx^2\right) - \frac{mg}{\sqrt{1+4c^2x^2}}. \tag{42}$$

At the top of the hill, at $x = 0$, this reduces to

$$F_\mathrm{C}(\mathrm{top}) = 4c \cdot E_\mathrm{tot} - m \cdot g. \tag{43}$$

Energy conservation tells us, that the car does not reach the top if $E_\mathrm{tot} < 0$. If the total energy is positive and smaller than $mg/(4c)$, the constraining force ist negative at the top of the hill. It is directed upward, i. e. opposite to $\vec{e}_\mathrm{N}$. According to Eq. (43), it is smaller than the gravitational force. Thus, the magnitude of the force that the seat exerts on a passenger is reduced from its usual value mg. The passengers feel somewhat lighter as they roll over the hill.

If $E_\mathrm{tot} > mg/(4c)$, the constraining force becomes positive at the top of the hill, i. e. it will be directed downward. An ordinary rail track cannot exert a downward force on the car. The car will tend to leave the track and follow the dashed line in Fig. 9. To avoid this unwanted behavior, the car must have additional counter wheels below the track. In addition the passengers must be tied to their seats by a safety harness.

"Airtime" is a particular attraction in which the passengers experience weightlessness for a short time. We expect this to be the case if the car and the passengers follow a free-fall trajectory. The track must have the shape of a free-fall parabola. Fig. 10 shows an impressive example of parabola-shaped hills. If v_0 denotes the velocity at the top of the hill,

FIG. 10: Parabola-shaped hills can give airtime. Photo courtesy of Europapark, Germany

the free-fall trajectory has the form

$$y = -\frac{g}{2v_0^2} \cdot x^2. \tag{44}$$

We thus have to insert into Eq. (42)

$$c = \frac{g}{2v_0^2} \qquad \text{and} \qquad E_{\text{tot}} = \frac{1}{2} m v_0^2, \tag{45}$$

which leads to

$$F_{\text{C}}(x) = 0 \quad \text{for all } x. \tag{46}$$

The constraining force is zero everywhere. As expected, the car (as well as the passengers) follow the track without the necessity of any constraining force, resulting in weightlessness.

VIII. CIRCULAR LOOP

Why do we never see circular loops in roller coasters? Real loops always have an inverted-teardrop shape (the form of the clothoid loop we will encounter in the next Section). There is a physical reason that can be discussed with our approach. The constraining force for circular motion has already been derived in Problem 3 (Eq. (38)):

$$F_{\text{C}} = \frac{2E_{\text{tot}}}{R} + 3\, mg\, \cos\phi. \tag{47}$$

From Problem 4 we know that at any point of the track $F_{\text{C}} > 0$ must hold if there are no counter wheels. Otherwise, the car will fall out of the track. Applied to Eq. (47), this gives a condition for E_{tot}:

$$E_{\text{tot}} > -\frac{3}{2} mgR\, \cos\phi. \tag{48}$$

If the car is to make it through the loop, this condition must be fulfilled everywhere, especially at the top of the loop where $\phi = \pi$. Thus, there is a minimum total energy the car must have at the beginning of the loop:

$$E_{\text{tot}} > \frac{3}{2} mgR. \tag{49}$$

Conveniently, this expression is written as a condition for the minimum height of the lifthill:

$$y_0 > 2.5\, R. \tag{50}$$

The lifthill must be higher than the loop by at least half a loop radius.

We can now understand the problem of the circular loop. If the condition Eq. (49) is fulfilled, the constraining forces at the beginning and the end of the loop become forbiddingly large. If we insert E_{tot} from Eq. (49) into Eq. (47), we obtain:

$$F_{\text{C}} = 3\, mg + 3\, mg \, \cos\phi. \tag{51}$$

At the loop entry and exit ($\phi = 0$ bzw. $\phi = 2\pi$), this becomes

$$F_{\text{C,max}} = 6\, mg. \tag{52}$$

As displayed in Fig. 11 the passengers suddenly have to stand six times their own weight at the entry of the loop. This is unacceptable from a medical point of view. Note that there is no design parameter to resolve the situation because the value of $F_{\text{C,max}}$ is independent of the loop radius.

IX. CLOTHOID LOOP

Up to now, we have considered relatively simple geometries in our concrete examples. To show the power of the formalism we will turn to something more difficult now: the clothoid loop. The clothoid is a particularly challenging geometry because it can be mathematically described only by a parametric representation. Of course, the problem is also physically interesting because real loops *have* a clothoid shape.

The parametric representation of the clothoid (Fig.12) reads:

$$\begin{aligned} x(t) &= R \cdot C(t), \\ y(t) &= R \cdot S(t). \end{aligned} \tag{53}$$

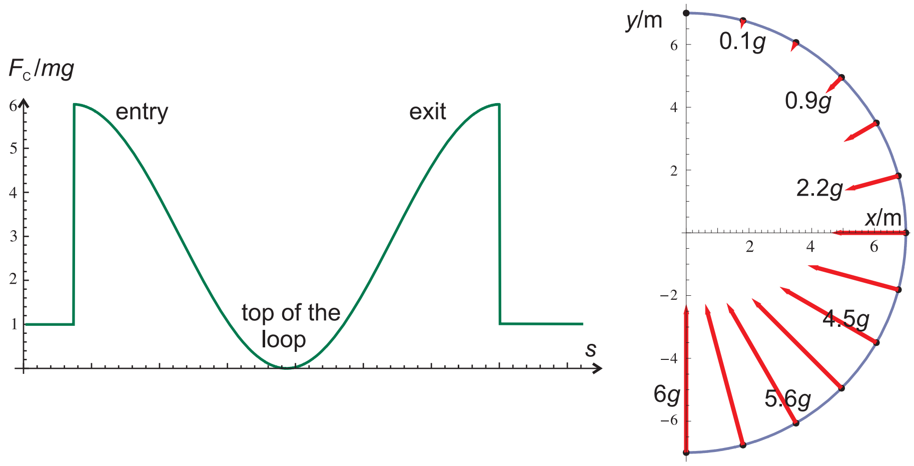

FIG. 11: Constraining acceleration in a circular loop. At the entry, the acceleration suddenly rises from 1 g to 6 g. Only half of the loop is shown in the diagram on the right.

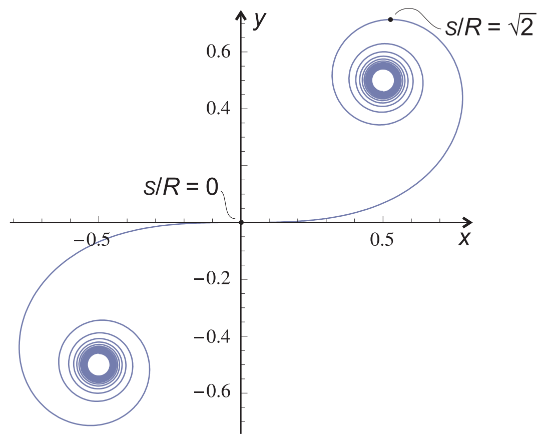

FIG. 12: The clothoid

The two functions $C(t)$ and $S(t)$ are known as Fresnel integrals. They cannot be represented by elementary functions. Instead they are defined by definite integrals (cf. [9]):

$$C(t) = \int_0^t \cos\left(\frac{\pi}{2}u^2\right) \mathrm{d}u,$$
$$S(t) = \int_0^t \sin\left(\frac{\pi}{2}u^2\right) \mathrm{d}u. \tag{54}$$

To calculate the arc length for the clothoid we need

$$\frac{\mathrm{d}}{\mathrm{d}t}C(t) = \cos\left(\frac{\pi}{2}t^2\right),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = \sin\left(\frac{\pi}{2}t^2\right). \tag{55}$$

FIG. 13: The characteristic shape of a roller coaster loop. Photograph: istockphoto/Marcio Silva

According to Eq. (2):

$$
\begin{aligned}
s &= \int_0^t \sqrt{\dot{x}^2(u) + \dot{y}^2(u)}\,\mathrm{d}u \\
&= \int_0^t R \cdot \underbrace{\left(\cos^2\left(\frac{\pi}{2}u^2\right) + \sin^2\left(\frac{\pi}{2}u^2\right)\right)}_{=1}^{\frac{1}{2}} \mathrm{d}u \\
&= R \cdot t. \qquad (56)
\end{aligned}
$$

The arc length and the parameter t are related via $t = s/R$.

An actual roller coaster loop is shown in Fig. 13. As a comparison with Fig. 14 shows, this shape can be constructed from two mirror-symmetrical sections of the clothoid, extending from $s/R = 0$ to $s/R = \sqrt{2}$. Because of the mirror symmetry, we will consider only the first half of the loop in the following.

In the parametric representation (53), the position vector along the track reads

$$\vec{r}(s) = \left(R \cdot C\left(\frac{s}{R}\right),\ R \cdot S\left(\frac{s}{R}\right)\right). \qquad (57)$$

The tangent vector is obtained by differentiation:

$$\vec{e}_\mathrm{T} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s} = \left(C'\left(\frac{s}{R}\right),\ S'\left(\frac{s}{R}\right)\right). \qquad (58)$$

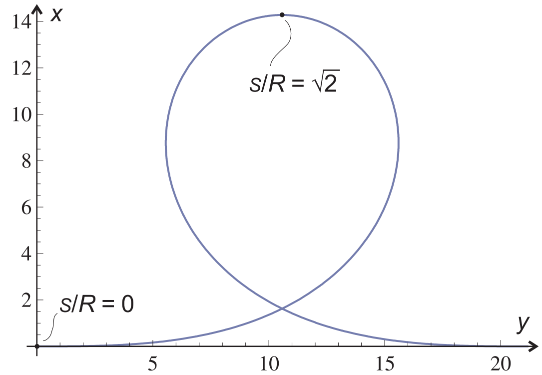

FIG. 14: A roller coaster loop is constructed from two clothoid section.

With Eq. (55) we get

$$\vec{e}_{\mathrm{T}} = \left(\cos\left(\frac{\pi}{2}\frac{s^2}{R^2}\right),\ \sin\left(\frac{\pi}{2}\frac{s^2}{R^2}\right)\right). \tag{59}$$

Another differentiation leads to the normal vector and to the local radius of curvature:

$$\vec{e}_{\mathrm{N}} = \left(-\sin\left(\frac{\pi}{2}\frac{s^2}{R^2}\right),\ \cos\left(\frac{\pi}{2}\frac{s^2}{R^2}\right)\right) \tag{60}$$

and

$$\rho = \frac{R^2}{\pi s}. \tag{61}$$

This formula illustrates a defining feature of the clothoid: its curvature (the inverse of ρ) increases linearly with s. As usual, the velocity of the car is calculated from energy conservation:

$$\begin{aligned} v(s) &= \sqrt{v_0^2 - 2g \cdot y(s)} \\ &= \sqrt{v_0^2 - 2gR \cdot S\left(\frac{s}{R}\right)}, \end{aligned} \tag{62}$$

where v_0 is the velocity at the loop entry.

We have now all ingredients needed to calculate the constraining force from Eq. (33):

$$F_{\mathrm{C}} = \frac{2\pi s}{R^2} E_{\mathrm{tot}} - \frac{2\pi s}{R} mg\, S\left(\frac{s}{R}\right) + mg\, \cos\left(\frac{\pi}{2}\frac{s^2}{R^2}\right). \tag{63}$$

This is our result: an analytic expression for the constraining force in a clothoid loop. It will allow us to discuss the reasons why a clothoid loop is less harmful to the passengers than a circular loop.

As with the circular loop we first have to calculate the minimum velocity at the entry of the loop. Again, it is determined by the condition $F_{\mathrm{C}} > 0$ everywhere. The most critical

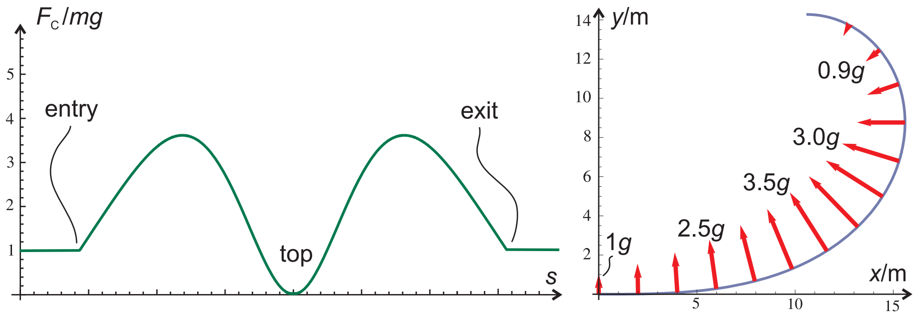

FIG. 15: Constraining acceleration for a clothoid loop (height 14 m) with minimal entry velocity.

point is the top of the loop: $s/R = \sqrt{2}$. To evaluate the constraining force there, we insert the numerical value of the Fresnel integral $S(\sqrt{2}) = 0.71$. We obtain the condition

$$E_{\text{tot}} > 0.83\, mgR, \tag{64}$$

or, refering to the top of the lifthill:

$$y_0 > 0.83\, R. \tag{65}$$

Since the total height of the loop is $R \cdot S(\sqrt{2}) = 0.71R$ (cf. Eq. (53)), the lifthill has to be only 16% higher than the top of the loop. Compare this to 25% for the circular loop. Here is a first advantage of the clothoid loop: a lower entry speed is required, corresponding to lower centripetal forces.

The main advantage of the clothoid loop can be understood with the help of Fig. 15. It shows the constraining acceleration for a loop with $R = 20$ m (corresponding to a height of 14 m) with the minimum entry velocity specified by Eq. (64). We see that there is no hard onset of the constraining force at the entry of the loop. Instead, the force gradually increases from 1 g to a maximum value of 3.6 g and then decreases again near the top. The linear increase is the reason that of all geometrical shapes a clothoid is chosen. As we have seen above, its characteristic feature is the linear increase of curvature along the track. The clothoid makes loops in roller coasters feasible. Accelerations of 3–4 g are not problematic for healthy passengers.

X. DISCUSSION

We have developed a Newtonian approach to constrained motion along a specified trajectory that is simpler than the traditional methods. Because it uses only local coordinates it makes complicated geometries easier accessible. The approach is based on elementary differential geometry, especially the Frénet vectors. The approach becomes especially simple for conservative motion. We have been able to give an analytic expression for the constraining force for any two-dimensional trajectory that can be written in the form $y = f(x)$.

As a physically interesting application of the approach we considered various roller coaster tracks. Especially, we calculated the constraining forces for a circular loop and a clothoid loop. It was possible to find an analytic expression for the constraining force in both cases.

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