The swingby mechanism analyzed with a new graphical formalism for 3D elastic collisions

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Abstract

We introduce a simple graphical formalism for analyzing 3D elastic collisions using energy and momentum conservation. We use the formalism to elucidate the physics of the swingby mechanism. As a concrete example we treat the encounter of Pioneer 10 with Jupiter in 1973.

I. INTRODUCTION

Some years ago, van Allen[1] made a promising suggestion for teaching celestial mechanics. He analyzed the swingby (or gravitational assist) maneuvre of Pioneer 10 during its encounter with Jupiter in 1973. Due to this operation, the spacecraft gained enough speed to finally leave the solar system. The swingby maneuvre is essential for any spacecraft mission to one of the outer planets. Simple energy considerations show that a "direct flight" beyond Jupiter it is not practical. It would be virtually impossible to lift the required amount of fuel from earth's surface. Instead, the spacecraft "steals" some energy from Mars or Jupiter to reach its destination.

In this article we show that Pioneer's encounter with Jupiter is a good example for teaching the physics of three-dimensional elastic collisions. The reasons are threefold: (1) It is a real-world problem with real data readily available. As van Allen pointed out, very detailed trajectory data for Pioneer 10 and Jupiter are freely available from JPL's HORIZONS server.[2] The availability of actual data and our ability to compare them with the theoretical predictions make the topic much more relevant than the usual abstract billiard-ball examples. (2) Students usually show great interest in any topic related to space flight. (3) The example clashes with some students' erroneous beliefs about the collisions: "But they do not collide at all". This misbelief gives us the opportunity to clarify the physical notion of a collision.

An obvious question to ask is: How large is the velocity of Pioneer after its encounter with Jupiter? While this is the subject of the present article, it was not in the focus of van Allen's interest. He was concerned with the resolution of an apparent paradox: How is the increase in Pioneer's speed consistent with energy conservation? He resolved the paradox by showing that Jupiter's kinetic energy decreases by the same amount that Pioneer's energy increases.

We want to use the Pioneer-Jupiter example to show how energy and momentum conservation allow predictions about the final state (i.e. the final velocities) in an elastic collision. In doing this, we are confronted with an immediate problem: It is trivial to write down the energy and momentum conservation equations, but it is difficult to solve them. Explicit equations tend to be clumsy (to say the least) and are never printed out in the textbook literature. Many textbooks present variants of a graphical method. Its most lucid version is found in the textbook of Landau and Lifshitz.[3] The method is quite intricate: It involves the addition of momentum vectors that have to be scaled by the reduced mass in part. This is certainly not the stuff our students are delighted about. The formalism is difficult to comprehend, difficult to apply and difficult to remember.

We have developed a new graphical method for applying energy and momentum in a 3D elastic collision. It will be presented in the first part of the paper (Sec. II.-III.).

The starting point in the development of the new formalism has been a paper by Millet[4] in which an exceedingly simple formula was derived for the 1D case (the formula is quoted below). While an analogously simple formula does not exist for the 3D case (just because several angles are involved) we found a graphical method that is much simpler than the traditional one. The method involves three easy steps. It allows to determine the final speed of the collision partners if the initial velocities are known as well as one of the final directions. It is difficult to imagine an even easier method for the solution of this relatively complex problem.

In the second part of the paper (Sec. IV-VI), we apply the method to the Pioneer-Jupiter encounter. Our result for the final speed of Pioneer 10 agrees reasonably well with the observed data.

II. ELASTIC COLLISION IN ONE DIMENSION

Before we approach the 3D case it is helpul to review the elastic collision of two bodies in one dimension. In this section, we essentially replicate the analysis of Millet.[4] It will serve us as a reference for the 3D case.

To fix the notation, let us recapitulate the three phases of a collision:

1. There is an *initial state* in which a number of bodies move without mutual interaction.
2. During the *interaction phase*, the bodies approach each other, interact and depart again.
3. In the *final state* the distance of the interaction partners is again so large that we can neglect their interaction.

In this article, we will treat the elastic collision of *two* bodies. We will use the following notation (Fig. 1): The indices 1 and 2 denote the first and the second body (with masses

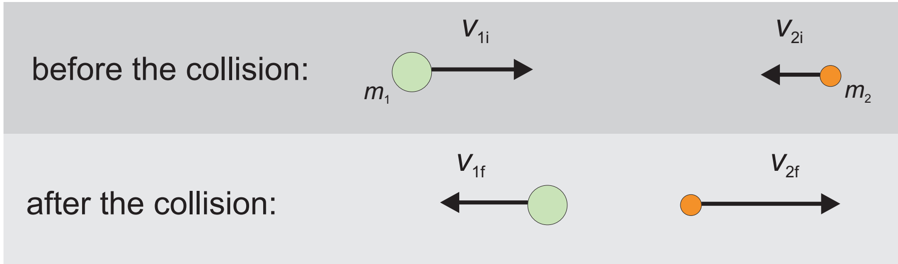

FIG. 1: Collision of two bodies in one dimension

m_1 and m_2). The indices i and f refer to the initial and final state, respectively. The velocity of the first body after the collision is denoted by $v_{1\mathrm{f}}$, for example. In the 1D case we have to specify a convention for the sign of the velocities. In the following, a positive velocity means that the corresponding body moves in the positive x direction, i. e, to the right in Fig. 1.

If no external forces are acting on the system, momentum conservation tells us that the center-of-mass velocity

$$v_{\mathrm{CM}} = \frac{1}{m_1 + m_2}\left(m_1 v_1 + m_2 v_2\right) \tag{1}$$

remains unchanged during the collision. In particular v_{CM} is the same in the initial and the final state, and Eq. 1 holds with indices i or f attached to v_1 and v_2.

In the calculation below, we will use the lab frame of reference as well as the center-of-mass frame. *Primed* quantities refer to the CM frame, whereas *unprimed* quantities refer to the lab frame. For example, v denotes the velocity of a body that is measured by an observer in the lab frame; v' is the velocity of the same body as measured by an observer in the CM frame. Both are connected by

$$v' = v - v_{\mathrm{CM}}. \tag{2}$$

Let us now analyze the collision using energy and momentum conservation. In the CM system, momentum conservation implies that the total momentum is zero (insert $v_{\mathrm{CM}} = 0$ into Eq. (1)). This holds for all times, especially in the initial and the final state:

$$m_1 v'_{1\mathrm{i}} = -m_2 v'_{2\mathrm{i}} \tag{3}$$

$$m_1 v'_{1\mathrm{f}} = -m_2 v'_{2\mathrm{f}}. \tag{4}$$

The momenta of the two collision partners have equal magnitude and opposite sign.

Conservation of energy states that the total kinetic energy is equal in the initial and the final state:

$$\frac{1}{2}m_1 v_{1\mathrm{i}}'^2 + \frac{1}{2}m_2 v_{2\mathrm{i}}'^2 = \frac{1}{2}m_1 v_{1\mathrm{f}}'^2 + \frac{1}{2}m_2 v_{2\mathrm{f}}'^2. \tag{5}$$

We insert (3) and (4) to eliminate the variables of body 2. We find

$$\frac{1}{2}m_1 v_{1\mathrm{i}}'^2 + \frac{1}{2}\frac{m_1^2}{m_2} v_{1\mathrm{i}}'^2 = \frac{1}{2}m_1 v_{1\mathrm{f}}'^2 + \frac{1}{2}\frac{m_1^2}{m_2} v_{1\mathrm{f}}'^2. \tag{6}$$

Factoring out leads to:

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1\mathrm{i}}'^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1\mathrm{f}}'^2. \tag{7}$$

Accordingly, the following relation holds between the velocities of body 2 before and after the collision:

$$v_{1\mathrm{i}}'^2 = v_{1\mathrm{f}}'^2 \qquad \text{or} \qquad v_{1\mathrm{i}}' = \pm v_{1\mathrm{f}}'. \tag{8}$$

The upper sign can be discarded because it represents a non-collision. Thus, in the CM system the magnitude of the velocity of body 1 is the same before and after the collision, only the sign is reversed. Since the same equation holds for body 2 we omit the corresponding labels from now on. We write for both

$$v_\mathrm{f}' = -v_\mathrm{i}' \tag{9}$$

The transformation to the lab frame is carried out with the help of Eq. (2). We insert $v_\mathrm{f}' = v_\mathrm{f} - v_\mathrm{CM}$ and $v_\mathrm{i}' = v_\mathrm{i} - v_\mathrm{CM}$ to obtain an equation that holds for both bodies independently:

$$v_\mathrm{f} = 2\, v_\mathrm{CM} - v_\mathrm{i}. \tag{10}$$

This simple equation (given in Ref. [4]) solves the collision problem for the 1D case. We first calculate the CM velocity with Eq. (1) from the initial conditions. With Eq. (10) we then obtain the final velocity for each collision partner. Of course, the solution of the 1D elastic collision problem can be found in any textbook. The particular form of Eq. 10, however, is quite apt for the generalization to the 3D case.

III. ELASTIC COLLISION IN THREE DIMENSIONS

In this section, we will derive a generalization of Eq. (10) to the three-dimensional case (Fig. 2). Again, we invoke energy and momentum conservation. Most of the equations look

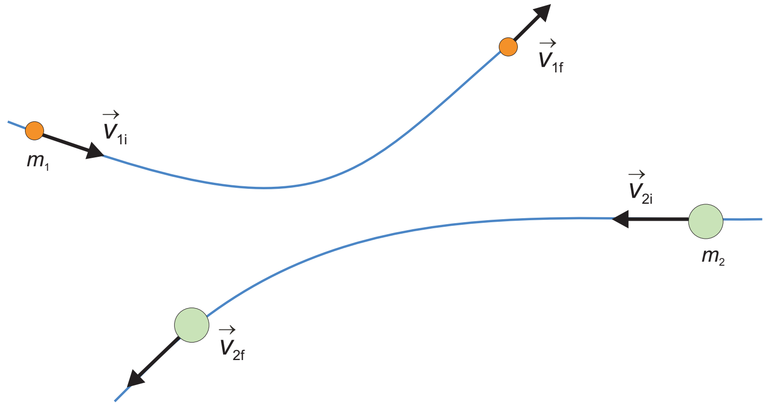

FIG. 2: Elastic collision in 3D

similar to those of the previous section. For example, the center-of-mass velocity is given by a formula that is a direct generalization of Eq. (1):

$$\vec{v}_{\mathrm{CM}} = \frac{1}{m_1 + m_2} \left(m_1 \, \vec{v}_1 + m_2 \, \vec{v}_2 \right). \tag{11}$$

The transformation between CM and lab frame now reads (cf. (2)):

$$\vec{v}' = \vec{v} - \vec{v}_{\mathrm{CM}}. \tag{12}$$

Energy and momentum conservation equations are direct generalizations of Eqs.(3)–(5)), too. Momentum conservation reads

$$m_1 \vec{v}'_{1\mathrm{i}} = -m_2 \vec{v}'_{2\mathrm{i}} \tag{13}$$

$$m_1 \vec{v}'_{1\mathrm{f}} = -m_2 \vec{v}'_{2\mathrm{f}}. \tag{14}$$

The first equation defines the CM frame, the second gives three equations for the six unknown final velocity components. Energy conservation gives a fourth equation:

$$\frac{1}{2} m_1 \vec{v}'^2_{1\mathrm{i}} + \frac{1}{2} m_2 \vec{v}'^2_{2\mathrm{i}} = \frac{1}{2} m_1 \vec{v}'^2_{1\mathrm{f}} + \frac{1}{2} m_2 \vec{v}'^2_{2\mathrm{f}}. \tag{15}$$

In contrast to the 1D case the problem is underdetermined by energy and momentum conservation because we have only four equations for six unknown variables. Additional input is needed to solve for the final state. Usually, the final direction (two angles) of one of the collision partners is given. If angular momentum conservation holds, the whole motion is confined to a plane and the problem effectively reduces to two dimensions. The graphical construction below will be easier if we assume this.

The analog of Eq. (7) is obtained in the same way as in the 1D case:

$$\vec{v}_{\mathrm{f}}'^{2} = \vec{v}_{\mathrm{i}}'^{2}. \tag{16}$$

Again, this equation holds for both collision partners independently. We use Eq. (12) to transform Eq. (16) to the lab frame. We get

$$\left(\vec{v}_{\mathrm{f}} - \vec{v}_{\mathrm{CM}}\right)^2 = \left(\vec{v}_{\mathrm{i}} - \vec{v}_{\mathrm{CM}}\right)^2. \tag{17}$$

This equation has a remarkably simple form. We will soon find a geometrical interpretation. Three more equations result from transforming Eq. (14) to the lab frame:

$$\vec{v}_{2\mathrm{f}} - \vec{v}_{\mathrm{CM}} = -\frac{m_1}{m_2}\left(\vec{v}_{1\mathrm{f}} - \vec{v}_{\mathrm{CM}}\right). \tag{18}$$

With this equation, we can calculate the final velocity of body 2 if we know that of body 1.

A graphical solution is the easiest way to approach Eq. (17). Mathematically, an equation of the form $r^2 =$ const. defines a sphere in 3D (or a circle in 2D). Our graphical solution is based on this observation. For the sake of simplicity, we assume the 2D case in the following.

The input data we need are the initial velocities of both bodies and the final direction of one of them. The magnitude of the final velocity is determined graphically in the following way:

1. Determine the CM velocity with Eq. (11).

2. Draw a coordinate system and plot the vectors $\vec{v}_{\mathrm{CM}}$ and $\vec{v}_{1\mathrm{i}}$ (Fig. 3).

3. The difference vector $\vec{v}_{1\mathrm{i}} - \vec{v}_{\mathrm{CM}}$ points from the head of $\vec{v}_{\mathrm{CM}}$ to the head of $\vec{v}_{1\mathrm{i}}$ (Fig. 4). The right-hand side of $\left(\vec{v}_{1\mathrm{f}} - \vec{v}_{\mathrm{CM}}\right)^2 = \left(\vec{v}_{1\mathrm{i}} - \vec{v}_{\mathrm{CM}}\right)^2$ therefore defines a circle around the head of $\vec{v}_{\mathrm{CM}}$ that touches the head of $\vec{v}_{1\mathrm{i}}$. Draw this circle

4. Because the right-hand side of Eq. (17) must be equal to the left-hand side, the head of the unknown vector $\vec{v}_{1\mathrm{f}}$ must lie on that circle, too. We assumed that the direction of $\vec{v}_{1\mathrm{f}}$ is known. Thus we can plot this vector and read off its magnitude (Fig. 5). We now know the final velocity of body 1.

5. Calculate the final velocity of body 2 with Eq. (18). The problem is solved.

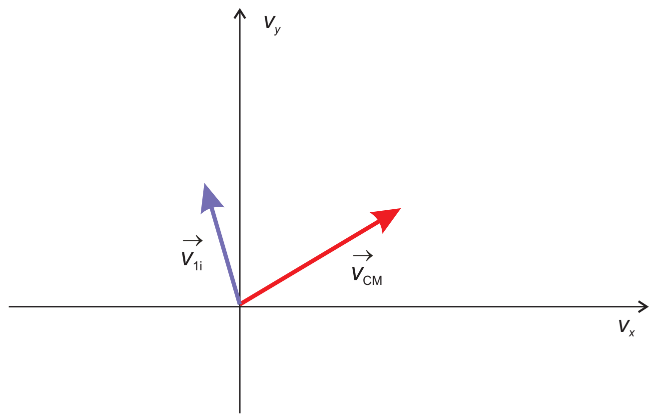

FIG. 3: The initial velocity and the CM velocity are known. The vectors are plotted in a coordinate system.

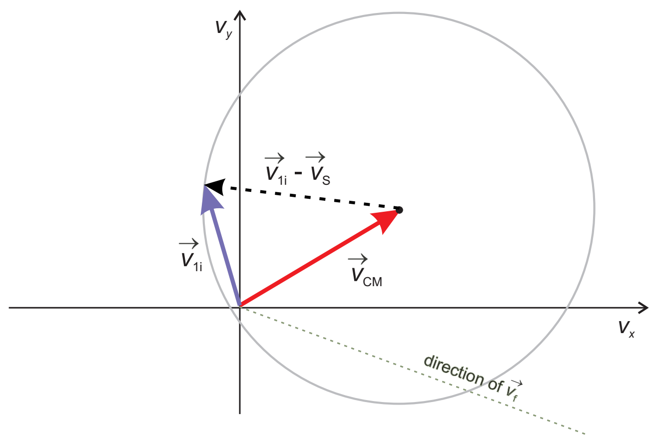

FIG. 4: Draw a circle around the head of $\vec{v}_{\rm CM}$ so that the head of $\vec{v}_{\rm 1i}$ lies on the circle.

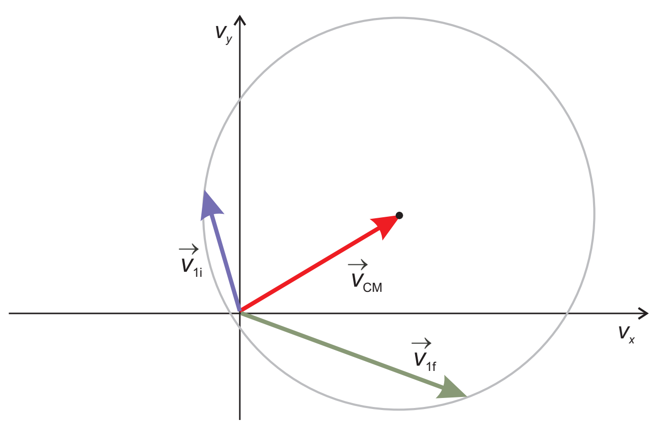

FIG. 5: The head of $\vec{v}_{\rm 1f}$ must lie on this circle, too.

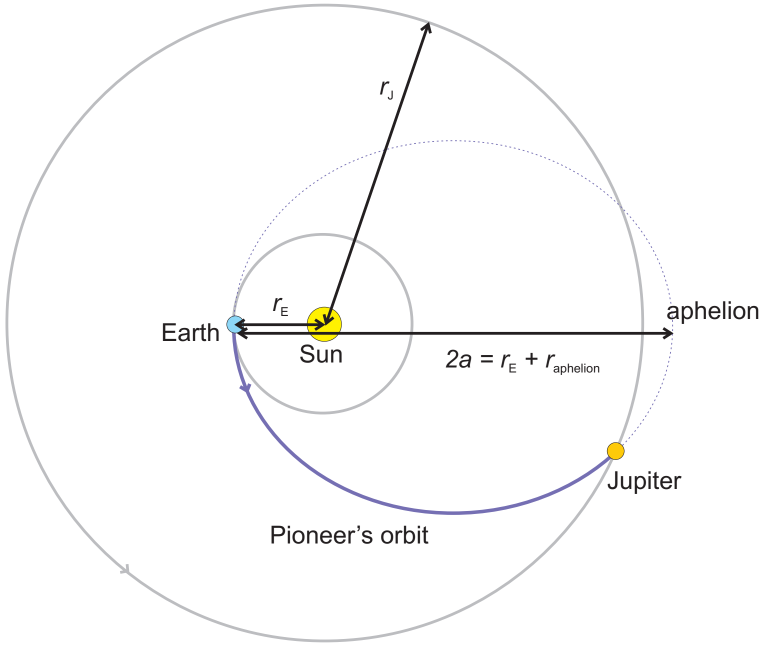

FIG. 6: The route of Pioneer 10 to Jupiter (not to scale)

IV. TRADITIONAL TREATMENT OF THE SWINGBY MECHANISM

Traditionally, the swingby maneuvre is not modeled as a collision. Instead, it is treated either numerically or as a succession of trajectory "patches" that can be analyzed with elementary celestial mechanics (hence the name "patched conic trajectories" for this approach). In our example, the first patch extends from the earth to the vicinity of Jupiter. In this part of the trajectory, Jupiter's gravity is neglected to a good approximation, and the spacecraft moves along a Kepler ellipse (Fig. 6).

Close to Jupiter, the planet's gravity takes over. For the few days of the swingby manoeuvre's duration, the influence of the sun is neglected. The trajectory in this patch is a Kepler hyperbola in the *rest frame of Jupiter*. The spacecraft's position and velocity at the end of the first patch have to be transformed to that frame. They serve as initial conditions for the second patch. Fig. 7 shows a plot of the observed Pioneer data in the vicinity of Jupiter. Pioneer's trajectory in the Jupiter rest frame is indeed a perfect Kepler hyperbola. Note that the initial speed and the final speed are the same in this frame.

The third and final patch starts when Jupiter's gravitation can again be neglected in comparison with the sun's. Again, position and velocity have to be transformed from the Jupiter-centered frame to the heliocentric frame. It turns out that the spacecraft's velocity is

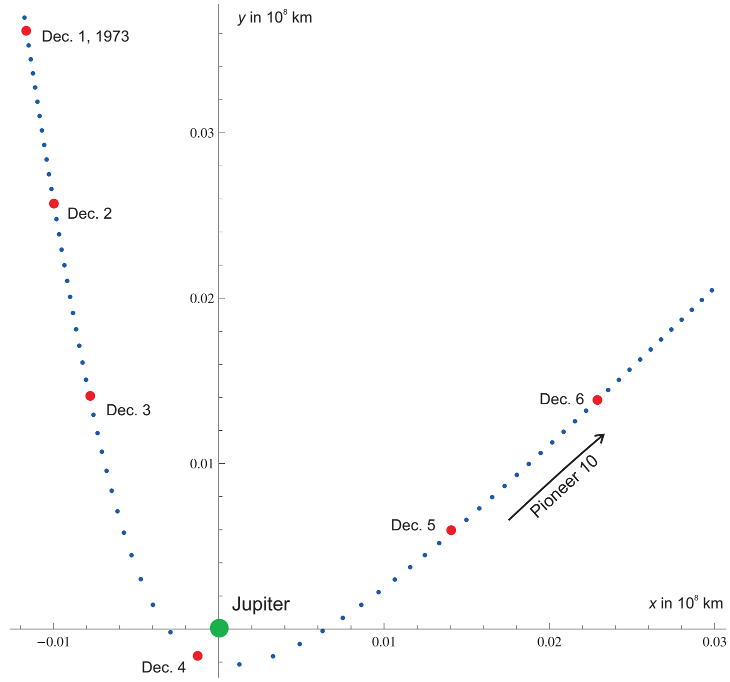

FIG. 7: Pioneer's encounter with Jupiter as seen from the rest frame of Jupiter. Dots denote the position of the spacecraft at intervals of 2 hours. Big dots mark Pioneer's position at 2:00 each day.

now so large that it travels on a Kepler hyperbola with respect to the sun. As a consequence, it will leave the solar system.

The problem with the patched conic trajectories approach lies in the words "It turns out" in the preceding paragraph. Not much can be learned about the physics of the swingby mechanism from this approach. Pioneer's speed in the rest frame of Jupiter is the same before and after the encounter (i. e. at the end of the second patch). At the beginning of the third patch, however, its heliocentric velocity is already higher than the solar system escape velocity. It seems that something crucial happens during the coordinate transformation. But a mere coordinate transformation cannot transfer energy to the spacecraft. It seems that we have missed something.

In the following section we will show how Pioneer's final speed after the encounter with Jupiter can be determined with the formalism introduced above. It should be mentioned that the collision analysis in itself should not be considered as a replacement for the patched conic trajectories approach. The latter yields the full trajectory data including the final direction of the spacecraft. The collision approach needs the final direction as additional

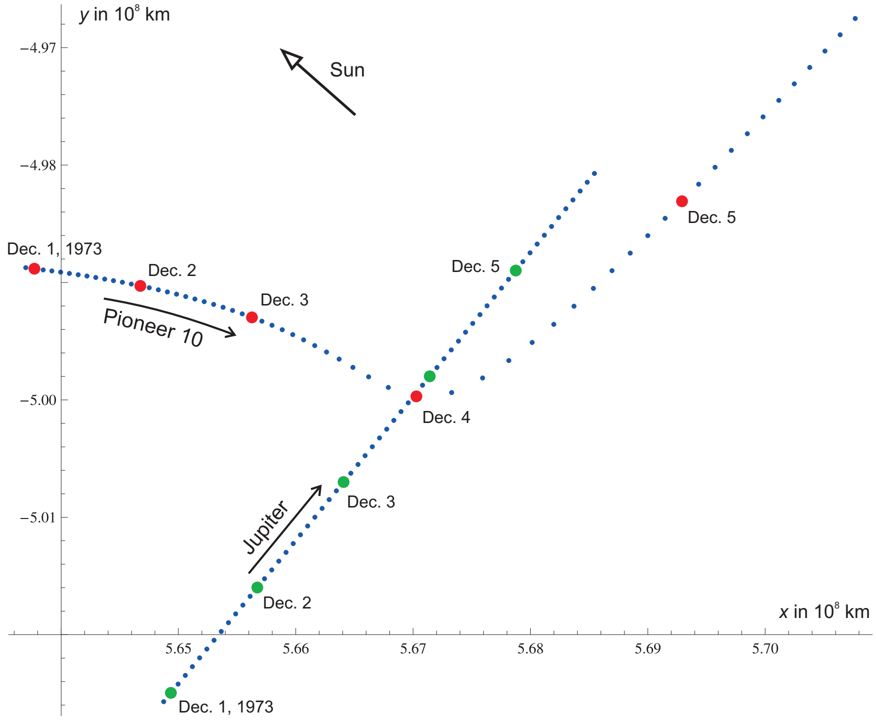

FIG. 8: Pioneer's encounter with Jupiter in a heliocentric system. Dots denote the position of two bodies at intervals of 2 hours. Big dots mark their position at 2:00 each day.

input. It is shown in Problem 2 how this information can be obtained.

The present approach gives insight into the physics behind the swingby mechanism (for more details, see Ref. 5). We will see below that it is possible to give a simple interpretation of the swingby mechanism in terms of familiar everyday phenomena.

V. PIONEER'S SWINGBY AS AN ELASTIC COLLISION

To discuss the collision we need initial data. They are readily available from the JPL server. We reduce the problem to 2D by assuming that the whole collision takes places in the ecliptic plane. This is a good approximation.

We will use the heliocentric frame throughout. The trajectory data of Pioneer 10 and Jupiter in this frame are shown in Fig. 8. Pioneer's increase in velocity can already be read off from this diagram: The dots representing the position at equal intervals of time are spaced more closely in the incoming part of the trajectory (left) than in the outgoing (right).

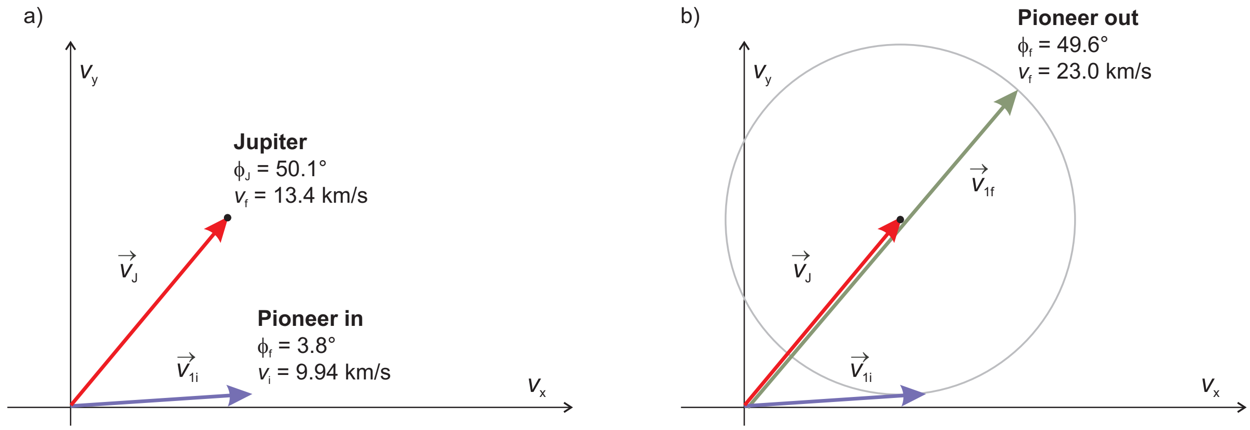

FIG. 9: Construction of the final velocity vector of Pioneer 10

We emphasize that the diagram shows the *same* data as Fig. 7, only the reference frame has changed. Throughout the process, Jupiter moves with the approximately constant velocity of 13,44 km/s. Jupiter's velocity vector encloses an angle $\phi_J = 50.1°$ with the x axis. To a very good approximation, the center of mass of the Jupiter-Pioneer system lies in Jupiter's center. The CM velocity is therefore identical to Jupiter's velocity.

The initial and the final state must be chosen so that Jupiter's influence on the spacecraft is small. For the initial state, we chose (somewhat arbitrarily) Nov. 22, 1973 at 0:00. At this time, Pioneer's speed was 9.94 km/s and the angle of its velocity vector with respect to the x axis was $\phi_i = 3.8°$ counterclockwise.

For the final state, we chose Dec. 22, 1973 at 0:00. Pioneer's velocity vector had an angle $\phi_f = 49.6°$ with respect to the x axis at this time. We will calculate the magnitude of its velocity in the following.

We must keep in mind, however, that neither the initial nor the final state are well defined. Because of the sun's gravity, Pioneer and Jupiter do not move on straight trajectories even if their mutual gravitational influence can be neglected. In the definition of the initial and the final state one always has to find a compromise so that the sun's influence as well as the collision partner's interaction can both be considered small.

To apply the graphical formalism of Sec. III, we draw a diagram with the initial velocities of Pioneer $\vec{v}_{1i}$ and Jupiter $\vec{v}_{CM} = \vec{v}_J$ (Fig. 9). We construct a circle around the head of $\vec{v}_J$, that touches the head of $\vec{v}_{1i}$. With our knowledge of Pioneer's final direction ϕ_f we can construct its final velocity vector $\vec{v}_{1f}$. The result of the construction (which is carried out best on millimeter paper) is a speed of 23.0 km/s. The agreement with the measured value

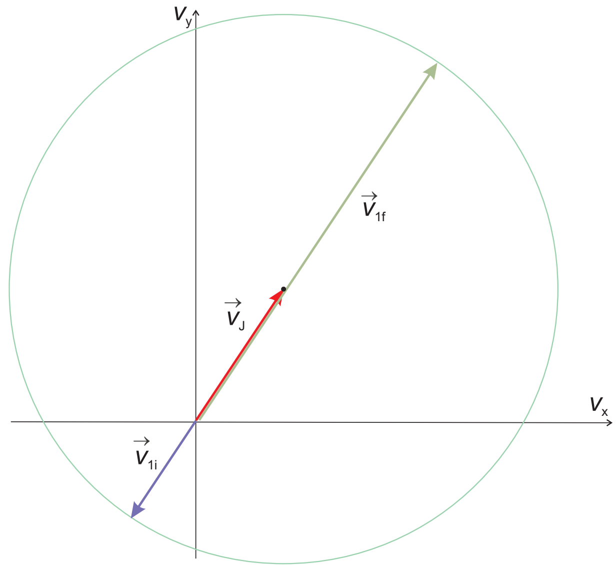

FIG. 10: The largest final velocity results from the "reflection" of the spacecraft, i. e. its deflection by an angle of 180 degrees

of 22.7 km/s is quite satisfactory. Pioneer has more than doubled its velocity during the encounter with Jupiter.

The small discrepancy between our prediction and the measured data can be traced to the gravitational influence of the sun, i. e. to the above-mentioned inability to properly define a non-interacting initial and final state

VI. PHYSICAL INTERPRETATION OF THE SWINGBY MECHANISM

To understand the physics behind the swingby mechanism more deeply we ask under which circumstances the velocity gain is largest. The geometrical construction tells us that the magnitude of $\vec{v}_{1f}$ is largest when the radius of the circle has its maximum possible value. This happens if the spacecraft's initial velocity is just *opposite* to Jupiter's (Fig. 10). The swingby effect is largest if the spacecraft is "reflected" by 180 degrees. Its final velocity is then $2|\vec{v}_J| + |\vec{v}_{1i}|$ (cf. Eq. (10)).

Thus, with the maximally efficient swingby we are back to the physics of 1D elastic collisions. The problem is the same as with the baseball and the bat. In both cases, a small body gains speed by an elastic reflection on a large body. In discussing this simple case it becomes immediately clear why the motion of Jupiter is so important. Without it, the

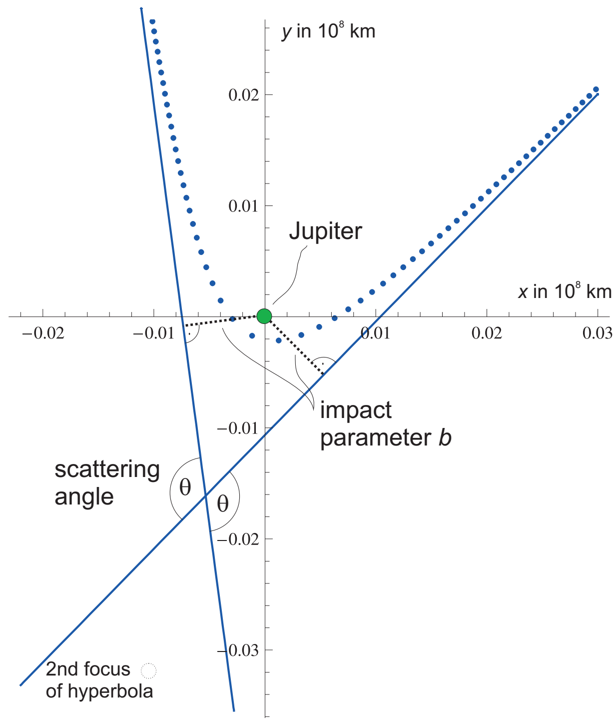

FIG. 11: The scattering geometry in the jupiter-centric frame (same data points as in Fig. 7).

spacecraft would gain no more speed than a baseball thrown against a wall at rest.

Why didn't NASA's engineers choose the most efficient angle for Pioneer's approach to Jupiter? The answer can be read off Fig. 6. The earth and Jupiter orbit the sun in the same direction. To utilize Earth's orbital velocity of 30 km/s all spacecraft start in this direction, too (this is another interesting topic in celestial mechanics, see Ref. 5). Hence Pioneer did approach Jupiter approximately from behind. The actual trajectory chosen is a compromise that ensures an acceptable velocity gain.

VII. CONCLUSION

We have introduced a simple graphical formalism for treating elastic collisions in two or three dimensions. We have successfully applied the method to the encounter of Pioneer 10 with Jupiter in 1973. The agreement with the observed data from NASA's JPL archives is quite good. By asking under which circumstances the swingby effect is largest we were able to give a simple physical interpretations of the mechanism.

VIII. PROBLEMS

1. Pioneer 10 started from earth at $r = 0.99$ AU with an initial velocity of 39.1 km/s, moving in the same direction as the earth (Fig. 6). From Earth to Jupiter, the spacecraft followed a Kepler ellipse in the Sun's gravitational field. Calculate the semimajor axis of this ellipse. Determine the maximal distance from the Sun that Pioneer 10 could have reached without the swingby maneuvre.

   Solution: The semimajor axis of Pioneer's orbit can be determined from its total energy in the gravitational field of the sun by the relation

   $$E_{\mathrm{tot}} = -\frac{Gmm_{\mathrm{S}}}{2a} \tag{19}$$

   (solar mass $m_S = 1,99 \cdot 10^{30}$ kg). We first calculate the kinetic and potential energy:

   $$E_{\mathrm{kin}} = \frac{1}{2}m \cdot (39,1\,\mathrm{km/s})^2 = m \cdot 764 \cdot 10^6\,\frac{\mathrm{J}}{\mathrm{kg}} \tag{20}$$

   $$E_{\mathrm{pot}}(r = 0,99\,\mathrm{AU}) = -\frac{Gm \cdot m_{\mathrm{S}}}{0,99\,\mathrm{AU}} = -m \cdot 894 \cdot 10^6\,\frac{\mathrm{J}}{\mathrm{kg}}. \tag{21}$$

   Accordingly, the total energy is

   $$E_{\mathrm{tot}} = -m \cdot 130 \cdot 10^6 \frac{\mathrm{J}}{\mathrm{kg}}. \tag{22}$$

   The total energy is smaller than zero, i. e. the spacecraft is bound to the sun and cannot leave the solar system.

   With Eq. (19), we calculate the semimajor axis

   $$a = -\frac{Gm \cdot m_{\mathrm{S}}}{2E_{\mathrm{tot}}} = 3,41\,\mathrm{AU}. \tag{23}$$

   From Fig. 6 we read off the relation between the semimajor axis and the aphelion of Pioneer's trajectory:

   $$2a = r_{\mathrm{E}} + r_{\mathrm{aphelion}}. \tag{24}$$

   We obtain

   $$r_{\mathrm{aphelion}} = 2a - r_{\mathrm{E}} = 5,8\,\mathrm{AU}. \tag{25}$$

   Without the encounter with Jupiter ($r = 5.06$ AU), the spacecraft would not reach distances from the sun greater than 5.8 AU.

2. In the calculation of Pioneer's final velocity, its final direction had to be known in advance. Apply your knowledge of Rutherford scattering to determine the "scattering angle" from the initial conditions. Use a jupiter-centered frame of reference with initial velocity $v'_{1\mathrm{i}} = 9.02$ km/s and impact parameter $b = 744500$ km.

   Solution: The geometry of the problem is shown in Fig. 11. We want to determine the angle θ between the incoming and the outgoing asymptotes of Pioneer's Kepler hyperbola. We need a relation between the initial velocity, the impact parameter, and the scattering angle. For the $1/r^2$ force law, this relation can be found in any derivation of the Rutherford scattering cross section (e. g. Ref. [6]). There are only two differences between repulsive Coulomb scattering and attractive gravitational scattering: (1) The role of the hyperbola's two foci is exchanged, see Fig. 11; (2) The constant $q_1 q_2/(4\pi\epsilon_0)$ has to be replaced by $G m_1 m_2$. Thus, the desired relation is

$$\cot\left(\frac{\theta}{2}\right) = \frac{b \cdot v'^2_{1\mathrm{i}}}{G m_{\mathrm{J}}}, \tag{26}$$

   where $m_{\mathrm{J}} = 1.9 \cdot 10^{27}$ kg is Jupiter's mass. If we insert the values given above we find that the scattering angle in the jupiter-centered frame is $\theta = 128.9°$. This can be transformed to the heliocentric frame by vector addition. The result for the heliocentric outgoing angle is 49,0°. The agreement with the measured value 49,6° used above is satisfactory.

3. From the initial conditions given in Problem 1, calculate Pioneer's velocity on its arrival at Jupiter (on Nov. 22, 1973, when it distance from the sun was $r = 5.0$ AU).
   Solution: We have already calculated the total energy of Pioneer in the gravitational field of the sun ($-m \cdot 130 \cdot 10^6$J/kg). Thus, the kinetic energy at $r = 5.0$ AU is given by

$$E_{\mathrm{kin}} = E_{\mathrm{tot}} - E_{\mathrm{pot}} = -m \cdot 130 \cdot 10^6 \frac{\mathrm{J}}{\mathrm{kg}} + \frac{G m m_{\mathrm{S}}}{5\,\mathrm{AU}}. \tag{27}$$

$$E_{\mathrm{kin}} = m \cdot 47,6 \cdot 10^6 \frac{\mathrm{J}}{\mathrm{kg}} \tag{28}$$

   so that

$$v = 9,76\,\frac{\mathrm{km}}{\mathrm{s}}. \tag{29}$$

   This can be compared with the measured value 9,94 km/s used above. The discrepancy is due to the gravitational influence of Jupiter.

4. Calculate Pioneer's total energy after its encounter with Jupiter.

   Solution: On Dec. 22, 1973, after the swingby, Pioneer's solar distance was 5,06 AU whereas its velocity had the above-mentioned value of 22,7 km/s. For the total energy we obtain

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{kin}} = \frac{1}{2}\, m\ (22,7\text{km/s})^2\ -\ \frac{Gmm_{\text{S}}}{5,06\,\text{AE}}. \tag{30}$$

$$E_{\text{tot}} = m \cdot 81,4 \cdot 10^6 \frac{\text{J}}{\text{kg}}. \tag{31}$$

   Before the encounter with Jupiter, Pioneer's total energy was negative so that it was bound to the sun. Its positive sign after the swingby means that Pioneer can now leave the solar system. At the time of the last contact in 2005, Pioneer 10 was 89.7 AU away from the Sun.

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APPENDIX A: FIGURE CAPTIONS

Fig. 1: Collision of two bodies in one dimension

Fig. 2: Elastic collision in 3D

Fig. 3: The initial velocity and the CM velocity are known. The vectors are plotted in a coordinate system.

Fig. 4: Draw a circle around the head of $\vec{v}_{\mathrm{CM}}$ so that the head of $\vec{v}_{1\mathrm{i}}$ lies on the circle.

Fig. 5: The head of $\vec{v}_{1\mathrm{f}}$ must lie on this circle, too.

Fig. 6: The route of Pioneer 10 to Jupiter (not to scale)

Fig. 7: Pioneer's encounter with Jupiter as seen from the rest frame of Jupiter. Dots denote the position of the spacecraft at intervals of 2 hours. Big dots mark Pioneer's position at 2:00 each day.

Fig. 8: Pioneer's encounter with Jupiter in a heliocentric system. Dots denote the position of two bodies at intervals of 2 hours. Big dots mark their position at 2:00 each day.

Fig. 9: Construction of the final velocity vector of Pioneer 10

Fig. 10: The largest final velocity results from the "reflection" of the spacecraft, i. e. its deflection by an angle of 180 degrees.

Fig. 11: The scattering geometry in the jupiter-centric frame (same data points as in Fig. 7).