A semiquantitative treatment of surface charges in DC circuits

Rainer Müller[a)]
Technische Universität Braunschweig, Physikdidaktik, Bienroder Weg 82, D-38106 Braunschweig, Germany

(Received 21 November 2011; accepted 13 June 2012)

Surface charges play a major role in DC circuits because they help generate the electric field and potential distributions necessary to move the charges around the circuit. Unfortunately, it is generally regarded as a difficult task to determine the surface charge distribution for all but the simplest geometries. In this paper, we develop a graphical method for the approximate construction of surface charge distributions in DC circuits. This method allows us to determine (approximately) the location and the amount of surface charge for almost any circuit geometry. The accuracy of this semi-quantitative method is limited only by one's ability to draw equipotential lines. We illustrate the method with several examples.
[http://dx.doi.org/10.1119/1.4731722]

I. INTRODUCTION

The simple DC circuit is a basic component of every physics curriculum, which leads one to think that all details about this common system are well known and understood. Yet several authors[1–12] have pointed out an important gap in the usual presentations of the subject. In discussions of the Drude model, for example, it is said that the electrons in a wire are guided by an electric field located inside the conductor with a direction parallel to the wire at any point. This statement might be puzzling to students who remember that, in the context of a Faraday cage, there is a proof that there is no electric field inside a conductor. If this is true, how can there be an electric field in a current-carrying wire? What kind of charges are generating it and where do they reside in a DC circuit?

For a physicist, it is evident that the Faraday cage argument does not apply to the DC circuit. The Faraday cage is in electrostatic equilibrium, whereas the wire is in a stationary non-equilibrium state. In their textbook, Chabay and Sherwood lucidly illustrate the transition from the electrostatic to the DC case.[1]

The origin of the electric field inside a long straight wire has been discussed by Sommerfeld.[2] He found that the field inside the conductor is generated by charges that are located at the surface of the wire and are therefore called *surface charges*. Jackson has identified three roles for these surface charges in real circuits:[3] (1) they maintain the potential around the circuit, (2) they provide the electric field in the space outside of the conductor, and (3) they assure the confined flow of current by generating an electric field that is parallel to the wire. The latter role can be nicely illustrated by a straight wire that is being bent while the current is flowing.[1] In this case, a simple feedback mechanism ensures that the electric field follows the wire even after it is bent: charges accumulate on the inner and outer edges of the bend until the additional field generated by the newly accumulated surface charges forces the flowing electrons to follow the wire. The accumulation process takes place very quickly—effectively instantaneous from a macroscopic perspective—and is complete as soon as the total electric field points along the wire at any place inside the conductor. The resulting pattern of surface charges, however, is quite complicated and cannot be determined by straightforward arguments.

Over the years, several authors have discussed different aspects of surface charges. Jefimenko has demonstrated their existence experimentally,[4] while Jackson,[3] Heald,[5] Hernandez and Assis,[6,7] and Davis and Kaplan[8] have found analytical solutions for several simple geometries. A qualitative approach to more complex geometries, including conductors with varying diameter and resistance, has been given by Haertel.[9] Meanwhile, Galili and Goibargh,[10] Harbola,[11] and Davis and Kaplan[8] have discussed the role of surface charges for the energy transport from the battery to a resistor, and Preyer has carried out numerical simulations to determine the distribution of surface charges.[12]

There seems to be general agreement that the distribution of surface charges is too complex to be determined by any simple rules.[3] Indeed, Heald spells out the difficulty as follows: the distribution of surface charges "depends on the detailed geometry of the circuit itself and even of its surroundings. For instance, we would have to specify exactly how the pieces of hookup wire are bent. And since most real-world circuits have rather complicated geometries, the mathematical difficulty of making this calculation is forbidding."[5]

In this paper, we show that the situation is not as hopeless as the above statement might suggest. We specify a simple method for the graphical construction of surface charge distributions in two-dimensional DC circuits. Using this method, we can determine the location and the amount of surface charge in a semi-quantitative manner for almost arbitrarily complex circuit geometries. The accuracy of the method is limited only by one's ability to draw equipotential lines.

II. TWO TYPES OF SURFACE CHARGES

Surface charges occur in two different ways that must be treated separately. We distinguish surface charges of types I and II as described below. It must be stressed that both types of surface charges act together to fulfill the three roles mentioned earlier.

Type-I surface charges occur at the boundary of two conductors with different resistivities (e.g., at the interface between a wire and a resistor). In order to keep the current constant, the electric field must be larger inside the resistor. Therefore, type-I surface charges accumulate at the interface between the two materials and contribute to the extra field inside the resistor. (A more accurate name would be interface charges.)

Type-II surface charges reside at the surface of conductors; they sit at the boundary between a conductor and the surrounding medium (usually air). The charges that accumulate at the edges of a bent wire are an example of this type of surface charge.

A. The field discontinuity and the kink in the equipotential lines

Let us adapt some well-known facts from electrostatics to the case of DC circuits. We will use the microscopic Maxwell equations throughout this paper so that we are only dealing with the electric field. Figure 1 shows two regions separated by a sheet of area S carrying a surface charge density $\sigma = q/S$. As shown in any textbook on electrodynamics, Gauss' law interrelates the electric fields in both regions. While the tangential components of the fields are continuous across the sheet, there is a discontinuity in the normal component whose magnitude is governed by the surface charge density. If the sheet extends in the yz-plane, we have

$$E_{2,x} - E_{1,x} = \frac{\sigma}{\epsilon_0}, \tag{1}$$

$$E_{2,y} - E_{1,y} = 0, \tag{2}$$

$$E_{2,z} - E_{1,z} = 0, \tag{3}$$

where ϵ_0 is the permittivity of free space.

The field can be written as the gradient of the electric potential ϕ. Although the field is discontinuous at the sheet, the potential is continuous everywhere (this can be shown from Maxwell's equations, cf. Ref. 13); at any point, the field vector is perpendicular to the equipotential lines (gray lines in Fig. 1). Because of the field discontinuity, there is a kink in the equipotential lines at the location of the surface charges. A kink in the equipotential lines can thus be used as an indicator for a sheet with surface charges. This observation, which can be expressed quantitatively with the help of Eq. (1), will be the key to the formulation of our surface charge rules.

B. The magnitude of type-I surface charge densities

Type-I surface charges are prototypically represented by the "resistor" shown in Fig. 2. Two conductors with different

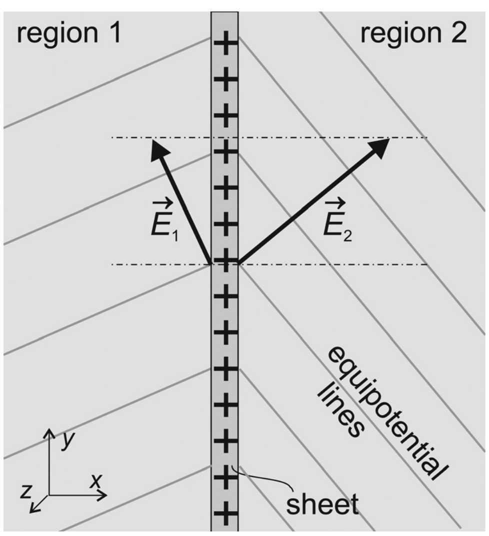


Fig. 1. Electric field at a boundary with surface charges.

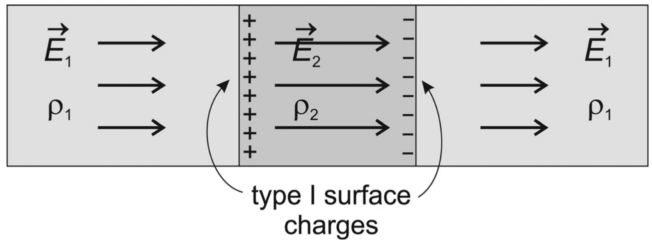


Fig. 2. Surface charges at the boundary between two adjoining conductors.

resistivities ρ_1 and ρ_2 adjoin to a common boundary with cross-sectional area A. Using Ohm's law $E = j\rho$, where j is the volume current density, we can relate the current I to the electric field by

$$I = jA = \frac{A}{\rho}E. \tag{4}$$

Because the current is constant in the circuit, we have

$$\frac{A}{\rho_1}E_1 = \frac{A}{\rho_2}E_2, \tag{5}$$

and applying Eq. (1) to the left boundary in Fig. 2 gives

$$\sigma = \epsilon_0 \frac{I}{A}(\rho_2 - \rho_1). \tag{6}$$

We get the same expression with opposite sign for the right boundary in Fig. 2. A similar result has been found by Jefimenko.[14]

Equation (6) is a quantitative expression for the density of type-I surface charges. The physical interpretation of this equation has already been stated—a constant current requires a larger electric field inside the resistor than in the wire. A portion of this field is generated by the surface charges described by Eq. (6). It should be noted that in the discussion of Fig. 2, we disregarded charges at the outer surfaces of the conductors even though in this example, there will be charges at the conductor/air boundaries. These type-II surface charges are examined in Sec. II C.

C. The magnitude of type-II surface charge densities

Consider a piece of conductor where the conductor–air interface lies in the yz-plane, as shown in Fig. 3. Inside the conductor the electric field is directed parallel to the current flow, say in y-direction. Using Eq. (1) and the fact that $E_{1,x} = 0$, we deduce the normal field component just outside the conductor is

$$E_{2,x} = \frac{\sigma}{\epsilon_0}. \tag{7}$$

According to Eq. (2), the tangential component does not change at the boundary so

$$E_{1,y} = E_{2,y} = E. \tag{8}$$

We already mentioned that a kink in the equipotential lines is an indicator of surface charges. Let α denote the kink angle of an equipotential line at the conductor's surface (see

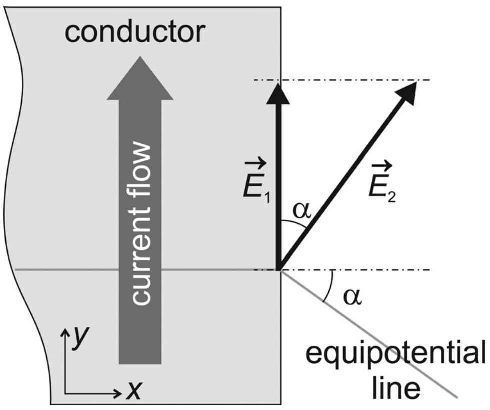


Fig. 3. The kink angle α can be related to the surface charge via the electric field discontinuity.

Fig. 3). Because α is also the angle between the outside electric field vector and the surface, we can relate this angle to the components of the electric field: $\tan\alpha = E_{2,x}/E_{2,y}$. Equation (7) then gives

$$\sigma = \epsilon_0 E \tan\alpha. \tag{9}$$

Because the electric field E inside the conductor cannot be easily measured, we can use Eq. (4) to write

$$\sigma = \epsilon_0 \frac{I\rho}{A} \tan\alpha. \tag{10}$$

This equation connects the surface charge density with the kink angle of the equipotential lines. Note that the sign of the surface charge density can immediately be read off from the orientation of the kink with respect to the direction of current flow (as demonstrated in Fig. 4):

- If $\alpha = 0$, there is no kink in the equipotential line and hence no surface charge.
- If $\alpha > 0$ (i.e., if the "arrowhead" formed by the kink points in the direction of the current flow), then σ is positive. This situation is seen in the left portion of Fig. 4.
- If $\alpha < 0$ (i.e., if the kink's "arrowhead" points in the opposite direction to the current flow), then σ is negative. This situation is seen in the left portion of Fig. 4.

III. FORMULATION OF THE SURFACE CHARGE RULES

Equation (10) allows us to determine the distribution of type-II surface charges if the kink angle of the equipotential lines is known. Thus, we have effectively reduced the surface charge problem to the task of finding the equipotential lines for a DC circuit. Although exact solutions for the potential have been found for simple geometries,[2,3,5–8] obtaining solutions in general is a difficult problem. It is, however, possible to determine the equipotential lines approximately using a graphical approach. We provide a set of simple rules for this approach below.

Let us consider an illustrative example. Figure 5 shows a circuit consisting of a 20-V battery and a single wire with uniform resistivity throughout; the poles of the battery are marked with "+" and "−." Some features of the circuit's equipotential lines can be determined easily:

(1) Inside the wire, the electric potential can be determined using Ohm's law. Because the resistivity of the wire is uniform throughout, Ohm's law implies that the electric potential varies linearly along the wire. The potential drops steadily from the plus pole to the minus pole. Thus, the equipotential lines pass through the wire at regular intervals (as shown in Fig. 5).
(2) Outside the wire, the electric potential drops from a maximum to a minimum between the poles of the battery. Accordingly, all equipotential lines must pass between the poles of the battery.

Using the two features above, we can specify a practical method to find the distribution of surface charges in a given DC circuit. The rules are formulated for two-dimensional circuits, but they could be easily generalized to three dimensions. What follows is a step-by-step procedure for this method.

Step 1: Draw the circuit. Using Ohm's and Kirchhoff's laws, determine the current and the value of the potential at each point of the conducting elements (wires, resistors, etc.).
Step 2: Mark equal potential differences on the conductors. Divide the voltage of the battery into 20–30 equal parts and mark the corresponding locations on the conductors, using the results of step 1. In general, these equipotential marks will be straight lines parallel to the cross-section of the conductor. On a wire with uniform resistivity, the marks will be equally spaced (cf. Fig. 5); inside a resistor, the spacing will

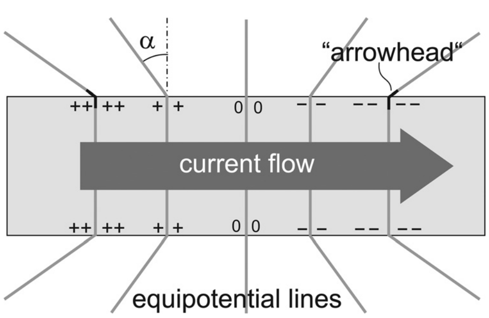


Fig. 4. The sign of the surface charges can be determined from the orientation of the kink with respect to the direction of current flow.

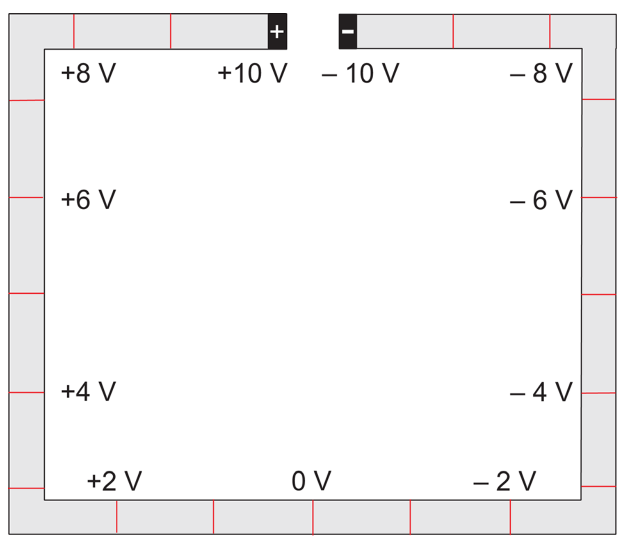


Fig. 5. Potential distribution in a homogeneous DC circuit. The voltage of the battery is assumed to be 20 V.

be smaller. This step forms the basis for the construction of the equipotential curves.

Step 3: Draw an equal number (20–30) of starting points for equipotential curves between the two poles of the battery. (As discussed, all equipotential lines must pass the region between the poles of the battery.)

Step 4: Finish the construction of the equipotential curves. Connect the starting points with the marks on the conductors, taking into account the following rules:

- Equipotential curves never cross.
- Equipotential curves cross conductors only at the points that have been determined in step 2. Otherwise, they must pass around all conductors.
- When drawing equipotential curves, it is helpful to imagine they are elastic bands that repel each other.[15]
- Do not try to draw a smooth transition at the surface of the conductor; in general there will be a kink here.
- Far away from the circuit, the equipotential curves merge into those of an electric dipole.

Step 5: Use Fig. 4 to determine the sign of the (type-II) surface charges wherever there is a kink in the equipotential curves. The magnitude of σ can be determined using Eq. (10). In the drawing, it might be helpful to use symbols like $++$, $+$, 0, $-$, $--$ to indicate relative amounts of surface charge.

Step 6: Determine the magnitude of the (type-I) surface charges at the interface between two conductors with different resistivities using Eq. (6).

This method is illustrated in Fig. 6 where the surface charges on a sinuous wire are constructed using paper and pencil. The resulting surface-charge pattern is quite complex even for this relatively simple circuit. Chabay and Sherwood[1,17] consider a similar geometry in their discussion of the field buildup mechanism. Using purely qualitative arguments, they find an approximate distribution that hardly resembles the complicated pattern seen here. Preyer discusses a similarly shaped wire and a comparison with his numerical results (Fig. 8 in Ref. 12) shows that our method reproduces the correct surface charge distribution quite well.

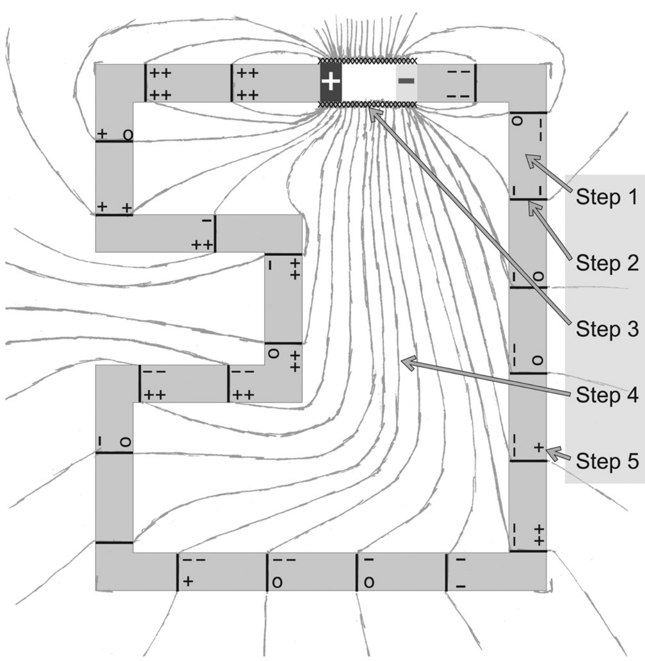


Fig. 6. Paper-and-pencil construction of surface charges. The steps indicated in the figure refer to the corresponding steps described in the text.

IV. TESTING THE METHOD WITH AN ACCURATE NUMERICAL STUDY

Equipotential lines in DC circuits are rarely discussed in textbooks. It may therefore be desirable to guide our intuition with some examples. In what follows, we show an accurate depiction of equipotential lines for several circuits. The calculations are performed numerically as described below and the results are related to the surface charge rules stated above.

First we consider the DC circuit with uniform resistivity previously discussed in Fig. 5. A numerical calculation of surface charges for a similar geometry has already been carried out by Preyer.[12] Because there is no interface between different conducting materials, we are only dealing with type-II surface charges. In our calculations, we assume a potential of $\pm 10\,\mathrm{V}$ at the two poles of the battery. The "wire" has a uniform resistivity of 0.25 Ωm and a total length of 234 cm. For ease of computation and visualization, we consider a 2D situation where the z-component of the current and the electric field are equal to zero everywhere.

The electric field and the potential are calculated using the commercial finite-element software package ANSYS MAXWELL.[16] It is used in the two-dimensional DC conduction mode, where the tangential component of the field and the normal component of the current are assumed to be continuous along boundaries. The calculation is performed using a mesh of about 40 000 triangles. To determine the surface charge distribution, the electric field is exported onto a regular grid and the surface charge density is calculated using $\nabla \cdot \vec{E} = \rho/\epsilon_0$ by numerically differentiating the electric field. The results of the calculation are shown in Fig. 7. We note the following features.

Electric field: The arrows denote the direction and magnitude of the electric field. The total field shown is the sum of the field generated by the battery plus the field of the surface charges. To reduce clutter, the region around the battery is omitted because the field is so large. As expected, the electric field inside the wire points along the wire at all locations. It is worth noting that, as opposed to electrostatics, the field is not perpendicular to the surface of the conductor because the latter is no longer an equipotential surface.

Equipotential lines: As stated above, all equipotential lines pass between the two poles of the battery. This feature may also be linked to the fact that in the 2D geometry considered here, the equipotential lines indicate the direction of energy flow[12] (i.e., the Poynting vector is perpendicular to both the electric field and the z direction).

Surface charges: At first sight, the distribution of surface charges appears somewhat complicated. A closer inspection, however, reveals the following typical features:

(1) At the bends of the wire, a quadrupole-like charge distribution guides the electric field around the bend. This function of surface charges is often mentioned in texts but the corresponding surface charge distribution is hardly ever discussed in detail. The structure of this "corner distribution" becomes more apparent as shown in Fig. 8. Note that in this diagram, only the part of the electric field generated by the surface charges is shown. Students may gain

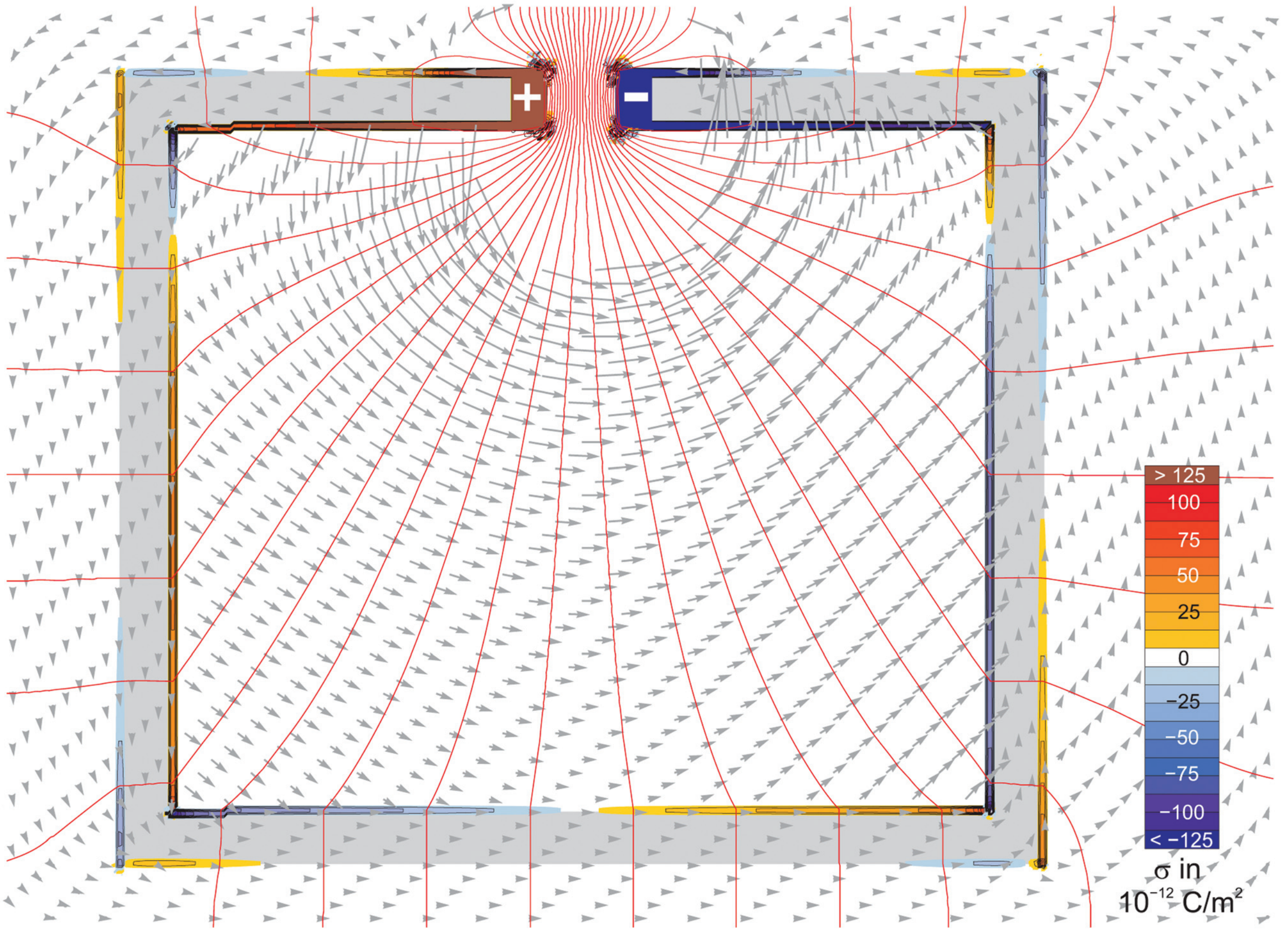


Fig. 7. Electric field, equipotential lines, and surface charges for a simple circuit with uniform resistivity. The density of surface charges is shaded from white to black (red to violet) representing most positive to most negative, respectively. The scale gives the surface charge density in 10^{-12} C/m^2.

some confidence by verifying that the field "lines" actually run from positive to negative surface charges.

(2) There is a tendency for the surface charges to be more positive closer to the positive pole of the battery and more negative closer to the negative pole. This feature—which leads to a dipole-like character of the field far from the circuit—is a remnant of the linearly varying surface charge density on Sommerfeld's infinite wire.[2] Most of the qualitative accounts seem to focus strongly on this result;[1,9,10,17] a linear variation of surface charge is the dominant feature in most schematic diagrams in the published literature. Figure 7 shows that the actual surface charge distribution is much more complex.

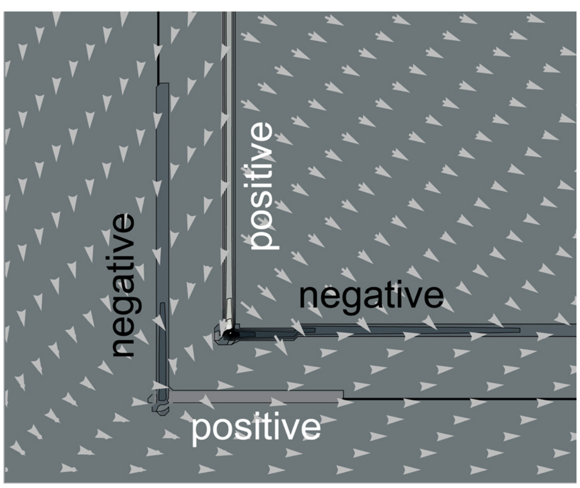


Fig. 8. A detailed look at the surface charge distribution at a bend of the wire from Fig. 7. In this figure, the arrows show only the part of the electric field generated by the surface charges.

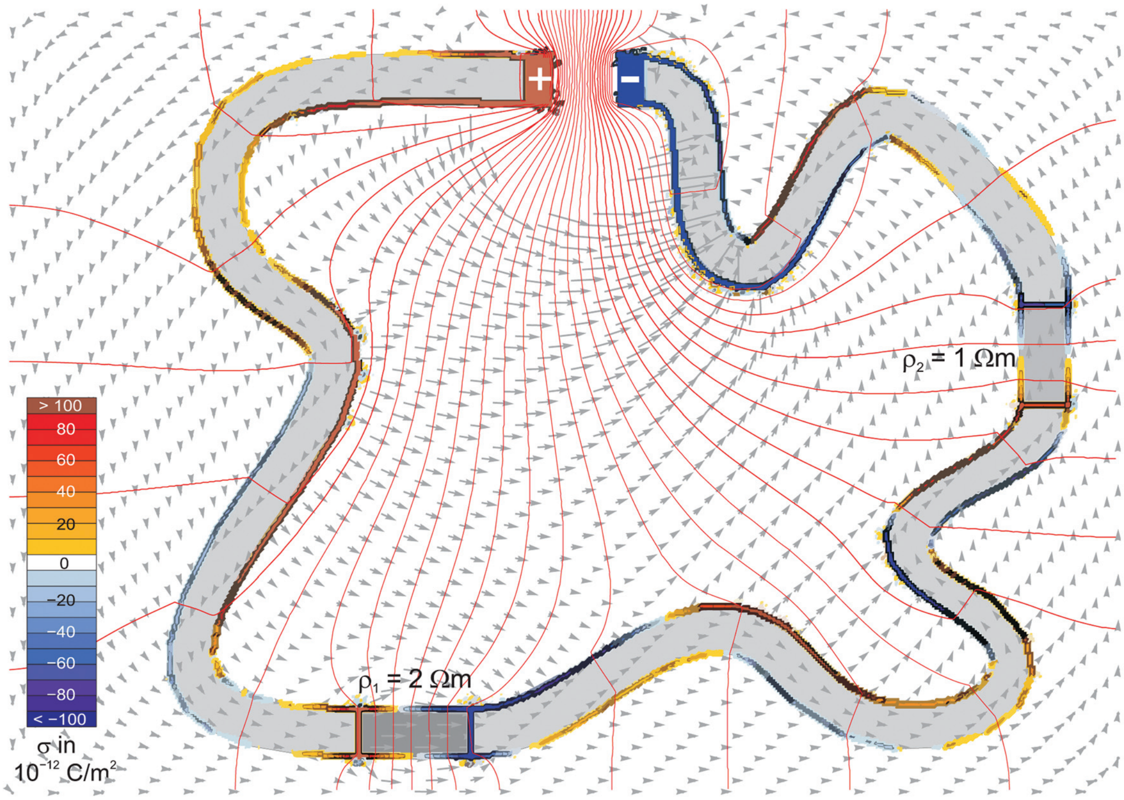
We can make a quantitative estimate for the magnitude of the surface charge using Eq. (9). Inside the homogeneous wire, the electric field is constant and given by $E = V/L$, where V is the voltage on the poles of the battery and L is the length of the wire. The surface charge density is thus

$$\sigma = \epsilon_0 \frac{V}{L} \tan\alpha. \tag{11}$$

Using $V = 20$ V and $L = 2.34$ m, we find

$$\sigma = (7.6 \times 10^{-11}) \tan\alpha \, \mathrm{C/m^2}. \tag{12}$$

Therefore, on a location where the kink angle is 45°, the surface charge density is 7.6×10^{-11} C/m^2, a value consistent with the results obtained numerically (cf. Fig. 7). Such a surface charge density corresponds to approximately 500 electrons per square millimeter.



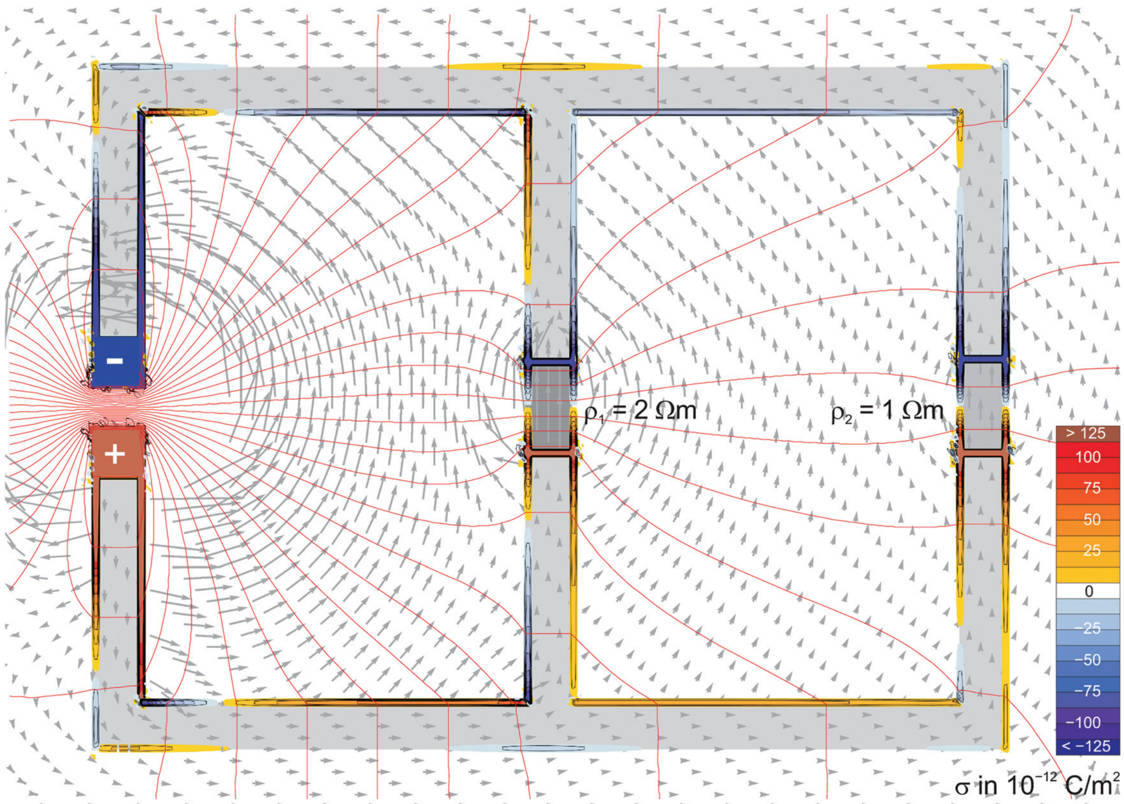


Fig. 9. Electric field, equipotential lines, and surface charges for a series connection of two resistors with arbitrarily twisted pieces of hookup wire (top) and a parallel connection of two different resistors (bottom). The same shading/coloring scheme is used as in Fig. 7. The surface charge density is given in 10^{-12} C/m^2.

V. FURTHER EXAMPLES

Figure 9 shows the surface charges for two more complicated circuits. In the top part of the figure, two resistors are connected in series to a battery with arbitrarily curved pieces of hookup wire. We can see that even in this "forbiddingly difficult" geometry,[5] the equipotential lines are no harder to construct than in the previous example. The shapes of the equipotential lines is predetermined to a large extent by their uniform distance along the hookup wire. On the other hand, the kink angle—and accordingly the amount of surface charge—sensibly depends on the orientation of the wire. This observation substantiates Jackson's statement that in real circuits the surface charge distribution depends strongly on the precise location of all parts of the circuit.[3]

At the interface between the wire and the resistors the resistivity changes. Here, we come across type-I surface charges of for the first time. There is a larger amount of charge on the left resistor because its resistivity is twice as large as that of the right resistor (2 Ωm compared to 1 Ωm). The wire's resistivity is 0.25 Ωm.

The same two resistors are connected in parallel in the lower part of Fig. 9. The pattern of equipotential lines reflects the more complicated voltage distribution in the circuit. In all previous examples, the current was the same in all parts of the circuit. This is not the case in a parallel connection. For this reason, the relative amount of surface charge can no longer be estimated from the kink angle alone, although the signs of the charges are given correctly. According to Eq. (10), the current I at the respective locations has to be specified as well. In the parallel connection shown here, the current is largest between the poles of the battery and before the circuit separates into two branches.

VI. DISCUSSION

It is generally thought that the determination of surface charge distributions in all but the simplest circuits borders on the impossible. In this paper, we have shown that this is not the case. We have outlined a relatively simple scheme for the semi-quantitative construction of surface charge distributions that can be applied to almost any circuit geometry. Using paper and pencil alone, this procedure takes about 15 min to complete. The accuracy of the method is limited by one's ability to draw equipotential lines. In particular, the spatial resolution is determined by the number of equipotential lines one chooses to draw. Typically, 20–30 lines will lead to a reasonable resolution.

Although students may be more familiar with field lines than equipotential lines, we stress that anybody who is able to draw field lines can construct equipotential lines. In free space, field lines and equipotential lines are orthogonal families, and both contain exactly the same information. In this sense, the determination of surface charges provides a nice opportunity for students to practice the appropriate skill.

ACKNOWLEDGMENTS

The author would like to thank two anonymous referees for their valuable suggestions. They helped to make the presentation of the subject matter much clearer.

[a]Electronic mail: rainer.mueller@tu-bs.de

[1]R. W. Chabay and B. A. Sherwood, *Matter & Interactions*, 3rd ed. (Wiley, New York, 2011).

[2]A. Sommerfeld, *Electrodynamics* (Academic, New York, 1952), pp. 125–130.

[3]J. D. Jackson, "Surface charges on circuit wires and resistors play three roles," Am. J. Phys. **64**, 855–870 (1996).

[4]O. Jefimenko, "Demonstration of the electric fields of current-carrying conductors," Am. J. Phys. **30**, 19–21 (1962).

[5]M. A. Heald, "Electric fields and charges in elementary circuits," Am. J. Phys. **52**, 522–526 (1984).

[6]J. A. Hernandes and A. K. T. Assis, "The potential, electric field and surface charges for a resistive long straight strip carrying a steady current," Am. J. Phys. **71**, 938–942 (2003).

[7]J. A. Hernandes and A. K. T. Assis, "Electric potential for a resistive toroidal conductor carrying a steady azimuthal current," Phys. Rev. E **68**, 046611 (2003).

[8]B. S. Davis and L. Kaplan, "Poynting vector flow in a circular circuit," Am. J. Phys. **79**, 1155–1162 (2011).

[9]H. Haertel, "A qualitative approach to electricity," Report IRL87-0001, Institute for Research on Learning Palo Alto (1987). Available online at <www.astrophysik.uni-kiel.de/hhaertel/PUB/Stromkreis_Lit.htm>

[10]I. Galili and E. Goihbarg, "Energy transfer in electrical circuits: A qualitative account," Am. J. Phys. **73**, 141–144 (2005).

[11]M. K. Harbola, "Energy flow from a battery to other circuit elements: Role of surface charges," Am. J. Phys. **78**, 1203–1206 (2010).

[12]N. W. Preyer, "Surface charges and fields of simple circuits," Am. J. Phys. **68**, 1002–1006 (2000).

[13]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, 1999), Eq. (2.34).

[14]O. D. Jefimenko, *Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields*, 2nd ed. (Electret Scientific, Star City, 1989), Chap. 9.5.

[15]E. Grimsehl and W. Schallreuter, *Lehrbuch der Physik: Bd. II* (B. G. Teubner, Leipzig, 1954).

[16]Computer Program Ansoft Maxwell SV Version 9.0 (Ansoft Corporation, 2003).

[17]R. W. Chabay and B. A. Sherwood, "A unified treatment of eletrostatics and circuits" (2006), WWW Document available online at <www.compadre.org/introphys/items/detail.cfm?ID=10026>.