Observer Design for Nonlinear Systems

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Chapter 1

Context and Motivation

In the following we will use the notation:

x	State variable	time variant, unknown
θ	Parameter	constant, unknown (unlike other system parameters)
d	Distrubance	time variant, unknown (responsible for "strange behaviour" of the system)

The reconstructed variables will be denoted by $\hat{x}$, $\hat{\theta}$, $\hat{d}$ and will in general be used for feedback within the observer system as well as within the real system. The context of these systems and variables is shown in figure 1.1.

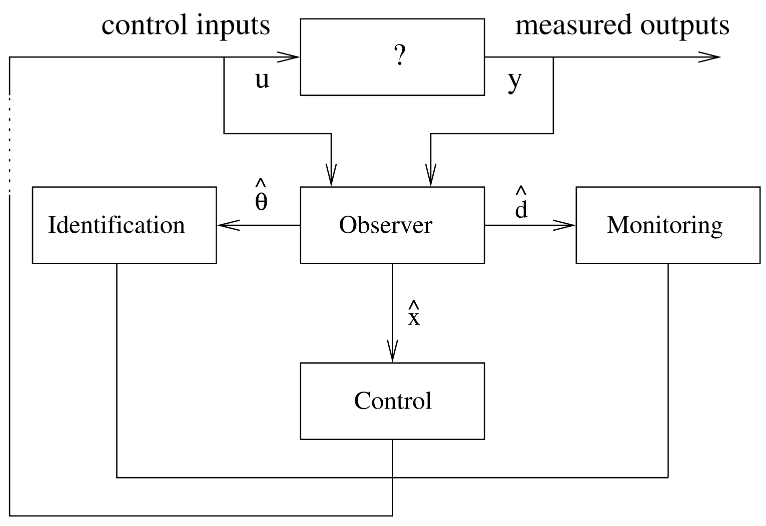


Figure 1.1: Context and relevance of designing observers

Problem and Solution

The overall problem here is to get internal information on a system from external measurements, specifically on can say in general that these measurements are not identical to all signals/components contained within the state variable. To acceive this goal a model– and measurement–based closed–loop information reconstructor (Observer) is used. Hereby model–based means that the general structure of the system is known whereas measurement–based makes clear that current as well as past values of inputs and outputs can be used. Finally a closed–loop reconstructor is used so that one can represent the input as a function of the output which is usually called feedback. The purpose of using a closed–loop reconstructor is to gain the state on–line.

In general one uses the term *model* by means of state–space representation. Therefore distrubances and constant parameters are not included. Here we will assume the state to contain all the variables to be reconstructed. By doing this the parameter identification problem is included within the problem formulation but will not be mentioned explictely.

Therefore the system can be based on equations that are

continuous–time	or	discrete–time,
deterministic	or	stochastic,
finite	or	infinite,
smooth	or	"with singularities".

Chapter 2

Problem Formulation

2.1 Problem

The considered model is of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) && (2.1)\\ y(t) &= h(x(t)) && (2.2) \end{aligned}$$

where the state $x \in \mathbb{X}$ with a $\mathcal{C}^\infty$ connected manifold is assumed to admit some system of coordinate in $\mathbb{R}^n$. The control u takes values in some open set $\mathbb{U} \subset \mathbb{R}^m$ whereas the output $y \in \mathbb{Y} \subset \mathbb{R}^p$ with $\mathbb{Y}$ being an open set as well.
Additionally we assume $u(t)$ to be measurable and bounded, by terms

$$u(t) \in \mathcal{L}^\infty(\mathbb{R}^+, \mathbb{U}). \qquad (2.3)$$

Moreover the functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^\infty$ with respect to all their arguments and the system is complete, meaning the solution of x exists at any time t.
We will denote the solution of x at time t eminating from x_0 at time t_0 under $u(t)$ over the interval $[t_0, t]$ by $\mathcal{X}_u(t, x_0)$

$$\left(\text{namely} \begin{array}{rcl} \frac{d}{dt}\mathcal{X}_u(t, x_0) &=& f(\mathcal{X}_u(t, x_0), u(t)) \\ \mathcal{X}_u(t_0, x_0) &=& x_0) \end{array}\right).$$

More generally one can consider time–variant systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) && (2.4)\\ y(t) &= h(x(t), u(t), t). && (2.5) \end{aligned}$$

Some particular cases are:

- Control affine systems

$$\begin{aligned} f(x, u) &= f_0(x) + g_0(x)u \\ h(x) & \end{aligned}$$

- State affine systems

$$\begin{aligned} f(x,u) &= A(u)x + B(u) \\ h(x) &= Cx \end{aligned}$$

- Bilinear systems

$$\begin{aligned} f(x,u) &= Ax + \sum_i u_i B_i x + Bu \\ h(x) &= Cx \end{aligned}$$

- Linear time–invariant systems

$$\begin{aligned} f(x,u) &= Ax + Bu \\ h(x) &= Cx \end{aligned}$$

- Linear time–variant systems

$$\begin{aligned} f(x,u) &= A(t)x + B(t)u \\ h(x) &= C(t)x \end{aligned}$$

Therefore the problem under consideration can be stated as follows:

> Find some estimate $\hat{x}(t)$ of $x(t)$ from the structural knowledge of the system f, h and the inputs/outputs $u(\tau)$, $y(\tau)$ with $\tau \in [t_0, t]$.

Remark 2.1. *In general one has to face the problem that h is not invertible. This is often the case due to limitations on the number of sensors, for cost reasons or due to technologically reasons (meaning some states can not be measured).*

2.2 Method for generating a solution

We have to design

$$\dot{X}(t) = F(X(t), u(t), y(t)) \tag{2.6}$$

$$\hat{x}(t) = H(X(t), u(t), y(t)) \tag{2.7}$$

meaning we have to find the necessary functions F and H such that

(i) $\hat{\mathbf{x}}(\mathbf{0}) = \mathbf{x}(\mathbf{0}) \quad \Rightarrow \quad \hat{\mathbf{x}}(\mathbf{t}) = \mathbf{x}(\mathbf{t}) \qquad \forall\, \mathbf{t} \geq \mathbf{0}$

(ii) $\hat{\mathbf{x}}(\mathbf{t}) - \mathbf{x}(\mathbf{t}) \to \mathbf{0}, \;\; \mathbf{t} \to \infty \quad$ **if** $\hat{\mathbf{x}}(\mathbf{0}) \neq \mathbf{x}(\mathbf{0})$

If the properties (i) and (ii) are satisfied then the system (2.6), (2.7) is called an observer.

Remark 2.2. *Ideally property (ii) is fulfilled for all $\hat{x}(0)$, $x(0)$. Then the observer is said to be global but this can not be met in general.*

Remark 2.3. *Property (ii) can also be decaying exponentially in some cases. We are talking about an exponential observer in this case.*

If in addition the following property

(iii) with a tunable rate of convergence.

is satisfied, then the system (2.6), (2.7) is called observer with tunable rate of convergence.

Remark 2.4. *Property (iii) can not be satisfied in all cases.*

In practice the following case occurs very often

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + k(y(t) - h(\hat{x}(t)), t) \tag{2.8}$$

with $k(0, t) = 0$ for all $t \geq 0$. Note that by definition this particular case satisfies property (i). Within this notation the correction term k is in general taken to be in the form

$$k(y(t) - h(\hat{x}(t)), t) = k(t) \cdot [y(t) - h(\hat{x}(t))] \tag{2.9}$$

so that k is proportional to the error.

⇒ The observer problem turns out to be the problem of finding k such that property (ii) is satisfied.

Remark 2.5. *Alternatively one can solve (or try to solve) the optimization problem*

$$\min_z \|h(\mathcal{X}_u(t, z)) - y(t)\|^2. \tag{2.10}$$

This has to be valid for all t within the interval and for simplicity reasons one can then take a look at the problem

$$\min_z \int_{t-T}^{T} \|h(\mathcal{X}_u(\tau, z)) - y(\tau)\|^2 d\tau \tag{2.11}$$

and consider a window of size T in which the error within the output is minimized. Since this is a nonlinear optimization problem one faces the usual problems such as

- *computational burden*
- *locally optimal solutions*
- *etc.*

2.3 Conditions for a solution

As a general condition we expect the measured output y to bear the information on x ($\equiv$ "observability condition").

Remark 2.6. *Note the when we are restricting the observer definition to the properties (i) and (ii), then "observability" is not even necessary.*

Example 2.7. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) \quad (2.12)\\ y(t) &= 0 \quad (2.13) \end{aligned}$$

Then y does not contain any information on x, and yet

$$\dot{\hat{x}}(t) = -\hat{x}(t) + u \quad (2.14)$$

is an observer in the sense of (i) and (ii). Indeed, one gets

$$(\dot{\hat{x}} - \dot{x})(t) = -(\hat{x}(t) - x(t)) \quad (2.15)$$

and therefore the error $\hat{x} - x \to 0$ as $t \to \infty$.
This is an observer, but it is not tunable.

When considering observers in the sense of (i), (ii) and (iii), "observability" becomes necessary.

2.3.1 About necessary conditions (except detectability)

Here we will distinguish between

- Formulation and
- Characterization.

Thereby "y should bear the information on x" means that one should be able to distinguish between 2 different initial conditions from the knowledge of $y(\tau)$, $0 \leq \tau \leq t$.

Definition 2.8 (Indistinguishability)**.**
A pair $x_0 \neq x_0'$ is indistinguishable if for all u and for all $t \geq 0$

$$h(\mathcal{X}_u(t, x_0)) \equiv h(\mathcal{X}_u(t, x_0')). \quad (2.16)$$

Then we can say that x is indistinguishable from x_0 if (x, x_0) is indistinguishable.

Definition 2.9 (Observability).
The system (2.1), (2.2) *is observable (O) if it does not admit indistinguishable pairs, respectively if there is no x that is indistinguishable.*

This defintion is quite strong (too "global") to be useful in practice.

Example 2.10. *Consider the system*

$$\begin{aligned}\dot{x}(t) &= u(t)\\ y(t) &= \sin(x(t))\end{aligned}$$

Obviously this system is not observable since x_0 and $x_0 + 2k\pi$ are not distinguishable for any $k \in \mathbb{Z} \setminus 0$, and yet any pair of states can be distinguished on $]-\pi, \pi[$.

Definition 2.11 (Weak Observability).
The system (2.1), (2.2) *is weakly observable (WO) if for all x_0 there exists a neighbourhood U of x_0 such that there is no state $x \in U$ which is indistinguishable from x_0.*

This definition is way better but one can still have to travel for a long time or a long distance to distinguish between two states. Therefore we will cope with a more local definition:

Definition 2.12 (Local Weak Observability).
The system (2.1), (2.2) *is locally weakly observable (LWO) if for all x_0 there exists a neighbourhood U of x_0 such that for all neighbourhoods V of x_0, $V \subset U$, there is no state x which is indistinguishable from x_0 in V as long as trajectories lie in V.*

Remark 2.13. *This roughly means that one can distinguish any state from its neighbours "without going too far".*

Remark 2.14. *Additionally the following conclusions are valid:*

(1) LWO $\Rightarrow$ WO

(2) O $\Rightarrow$ WO

To see that (1) can not be taken the other way round, take the counterexample

$$h(x) = \begin{cases} a & , \quad x \leq 0 \\ a + x & , \quad x > 0 \end{cases}$$

which is illustrated in figure 2.1. Here not for all open neighbourhoods V of x_{01}, x_{02} the trajectories eminating from these initial values are distinguishable, still the system is observable if one waits long enough so that $x > 0$ is fulfilled.

LWO is "characterizable" which will here be presented as kind of *rank condition* based on the observation space. Therfore we define

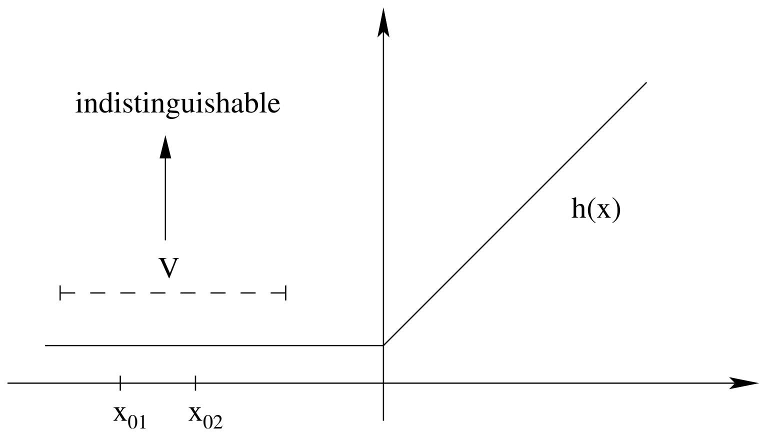


Figure 2.1: Counterexample to WO $\Rightarrow$ LWO

Definition 2.15 (Observation Space).
The smallest real vector space of $\mathcal{C}^\infty$ functions $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ containing the output function h which is invariant under Lie derivation along f for any fixed $u \in \mathbb{R}^m$ is called observation space of a system of the form (2.1), (2.2). *We will denote this space by* $\Theta(h)$. *In particular this means that*

$$\forall \varphi \;\in\; \Theta(h): \quad L_{f_u}\varphi \;\in\; \Theta(h)$$

with $L_{f_u}\varphi = \frac{\partial\varphi}{\partial x} f(x,u)$.

Definition 2.16 (Observability Rank Condition).
The observation space of a system (2.1), (2.2) *can be characterized by the following:*

$$\forall\, x \in \mathbb{R}^n: \quad \dim\left(d\Theta(h)\Big|_x \right) = n \tag{2.17}$$

where $d\Theta(h) := \{ d\varphi \mid \varphi \in \Theta(h)\}$ *and* d *denotes the differential operator.*

Theorem 2.17 (Pointwise Equivalence).
If and only if the observability rank condition is satisfied at x_0 *for a system* (2.1), (2.2), *then this system is LWO at* x_0.

Proof. We will roughly sketch this theorem.

"$\Leftarrow$": By noting that if $x_0 \neq x_0^{'}$ are indistinguishable on some open set V, then $\forall\, \varphi \in \Theta(h)$: $\varphi(x_0) = \varphi(x_0^{'}$ since the output function for x_0, $x_0^{'}$ are the same:

$$\left.\begin{array}{l} \forall\; s_1, \ldots, s_k \\ \forall\; u_1, \ldots, l_k \end{array}\right\} : h(\mathcal{X}_{u_k}^{s_k} \circ \ldots \circ \mathcal{X}_{u_1}^{s_1}(x_0)) = h(\mathcal{X}_{u_k}^{s_k} \circ \ldots \circ \mathcal{X}_{u_1}^{s_1}(x_0^{'}))$$

with $\mathcal{X}_u^s(x) = \mathcal{X}_u(s,x)$ and

$$\frac{d}{ds}\mathcal{X}_u^s = f(\mathcal{X} : u^s(x), u), \quad \mathcal{X}_u^0(x) = x.$$

By derivating by $s_k, \ldots s_1$ one gets

$$L_{f_{u_1}} \ldots L_{f_{u_k}} h(x_0) = L_{f_{u_1}} \ldots L_{f_{u_k}} h(x_0^{'})$$

and by finally evaluating at $s_k = s_{k-1} = \ldots s_1 = 0$ one can see that any $\varphi \in \Theta(h)$ is of the form given above.

"$\Rightarrow$": From the observability rank condition it follows that $\exists\ \varphi_1, \ldots, \varphi_n$ such that

$$\Phi := \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix}$$

is a diffeomorphism on some neighbourhood U of x_0. This is due to the fact that the Jacobian matrix consists of derivatives that are independent according to the definition. Hence for any indistinguishable pair $x_0 \neq x_0^{'}$ on $V \subset U$ we have $\varphi_i(x_0) = \Phi_i(x_0^{'}$ for all $i = 1, \ldots, n$, i.e. $\Phi(x_0) = \Phi(x_0^{'})$ and thus $x_0 = x_0^{'}$ which contradicts our assumption that $x_0 \neq x_0^{'}$ and the result holds.

$\square$

Theorem 2.18 (Uniform Equivalence)**.**
When one is talking about uniform equivalence in x the following statements are valid:

(1) If the observability rank condition is valid for all x, then the system is LWO for all x.

(2) If the system is LWO for all x, then the observability rank condition is generically satisfied, i.e. only at some isolated points the condition is not met.

Example 2.19. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= \sin(x(t)) \end{aligned}$$

Therefore $L_{f_u} h(x(t)) = u\cos(x(t))$. Hence it follows that the observation space is given by

$$d\Theta(h)\Big|_x = span\{\cos(x)\dot{x},\ \sin(x)\dot{x}\}$$

and the dimension of this space is $\dim\left(d\Theta(h)\Big|_x\right) = 1$ for all x. From the definition one can see that the observability rank condition is satisfied and hence the system is LWO.

Example 2.20. *For higher dimension the same example is not necessarily true. Consider the system*

$$\begin{aligned} \dot{x}_1(t) &= u(t) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= \sin(x_1(t)) \end{aligned}$$

Here the observation space is given by $d\Theta(h)\Big|_x = span\{\cos(x_1)\dot{x}_1,\ \sin(x_1)\dot{x}_1\}$*, so its dimension is* $\dim(d\Theta(h)) = 1 < 2$ *and hence the system is not LWO.*

Example 2.21. *Consider the linear n–dimensional system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned}$$

Then the following equivalences are valid:

(1) Observability rank condition $\Leftrightarrow$ *(rank)*$(\Theta(h)) = n$.

(2) Observability rank condition $\Leftrightarrow$ *LWO*

Note that in this case

$$\Theta(h) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is the known observability matrix for linear systems.
This can be proven by considering that $h(x(t)) = Cx(t)$.

(1): It follows that

$$\begin{aligned} L_f h(x(t)) &= CAx(t) \\ &\vdots \\ L_f^k h(x(t)) &= CA^k x(t) \end{aligned}$$

Making use of $\dot{x}(t) = Ax(t)$ *this can be represented in a derivative notation since*

$$\begin{aligned} CAx(t) &= C\dot{x}(t) \\ &\vdots \\ CA^k x(t) &= CA^{k-1}x(t) \end{aligned}$$

Therefore:

$$\dim\left(d\Theta(h)\right) = n \ \Leftrightarrow \ rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

(2) "⇒": If the pair $(x_0, x_0^{'})$ *is indistinguishable then*

$$Ce^{At}x_0 = Ce^{At}x_0^{'} \quad \forall t \geq 0$$

and by derivation it follows for all $k \in \mathbb{N}$

$$CA^k e^{At}x_0 = CA^k e^{At}x_0^{'} \quad \forall t \geq 0.$$

Hence we can conclude

$$\Theta e^{At}x_0 = \Theta e^{At}x_0^{'}.$$

From the observability rank condition it follows that

$$e^{At}x_0 = e^{At}x_0^{'} \quad \Rightarrow \; x_0 = x_0^{'}$$

(2) "⇐": From LWO it follows that there is no indistinguishable pair $(x_0, x_0^{'})$.

$$\Rightarrow \; Kern(\Theta e^{At}) = \{0\}$$

and thus if $x_0 \in Kern(\Theta)$*, i.e.* $e^{-At}x_0 \in Kern(\Theta e^{At})$*, then by LWO it follows that* $e^{-At}x_0 = 0$ *and hence* $x_0 = 0$*. Therefore we get rank*$(\Theta) = n$.

Remark 2.22.
In the linear case we can say the following:

(1) *Observability and the observability rank condition are equivalent for linear systems. This is also true for system of the form*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

since

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau}Bu(\tau)d\tau$$

and therefore the indistinguishability depends only on x_0 *because for identically used inputs u the integral part is identical as well and independent of* x_0*. Hence the integral part is cancelling out itself when one is taking a look at the error.*

(2) *We say that if the observability rank condition is satisfied then* (A, C) *is called observable.*

(3) *The rank condition for the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned}$$

is also sufficient for observer design.

Remark 2.23. *In general this is not true for the nonlinear case since observability does not exclude the possible existence of inputs for which observability is lost.*

Example 2.24. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & u(t) \\ 0 & 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

Clearly for constant $u \not\equiv 0$ it follows that

$$rank(\Theta) = rank \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} = 2$$

i.e. the system is observable. But when $u \equiv 0$ then observability is lost.

We conclude that observability in the sense of LWO is not enough for a possible observer design and that we will have to take a closer look at the inputs u.

2.3.2 About sufficient conditions (up to effective designs)

Here we will take a closer look to sufficient conditions which are related to inputs. As we have seen before there exist cases with "bad inputs" but conversely there can exst inputs which are "always good".

Definition 2.25 (Universal Input)**.**
An input u will be called universal input if

$$\forall\, x_0 \neq x_0^{'} \;\exists\, t \geq 0 : \quad h(\mathcal{X}_u(t, x_0)) \neq h(\mathcal{X}_u(t, x_0^{'})). \tag{2.18}$$

Remark 2.26. *An input u is called local universal input on an interval $[0, \tau]$ if condition* (2.18) *is valid for $t \in [0, \tau]$.*

Definition 2.27 (Singular Input)**.**
An input u is called singular input if it is a non universal input.

Example 2.28. *For the system*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & u(t) \\ 0 & 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

$u(t) \equiv 0$ is a singular input. For any other input $u(t)$ the trajectories will differ at some time t. Therefore $u(t) \equiv 0$ is the only singular input and all $u \not\equiv 0$ are universal inputs.

Remark 2.29. *About singular and universal inputs we can state the following:*

(1) For $\mathcal{C}^\omega$ systems the set of $\mathcal{C}^\omega$ universal inputs is dense in the set of $\mathcal{C}^\omega$ functions.

(2) Characterizing singular inputs is in general not an easy task.

(3) There exist systems without singular inputs, see for instants linear time–variant systems.

Definition 2.30 (Uniformly Observable System)**.**
A system (2.1), (2.2) is uniformly observable if it does not admit any singular input, i.e. any input u is universal.

Remark 2.31. *A local statement can be done here by denoting a system (2.1), (2.2) locally uniformly observable if any input is universal on the interval $[0, t]$ for $t > 0$.*

Example 2.32. *The following systems are uniformly observable:*

(1) Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

with (A, C) observable.

(2) And consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} x(t) + \begin{bmatrix} \varphi_1(x_1(t)) \\ \varphi_2(x_1(t), x_2(t)) \\ \vdots \\ \varphi_n(x(t)) \end{bmatrix} u(t)\\ y(t) &= \begin{bmatrix} 1 & \cdots & 0 \end{bmatrix} x(t)\end{aligned}$$

with $x(t) = (x_1(t), \cdots, x_n(t))^T$.

To see (2) one can choose $x(t) \neq x'(t)$ such that $x_k(t) = x'_k(t)$, $1 \leq k \leq i < n$ and $x_{i+1}(t) \neq x'_{i+1}(t)$. Then

$$\dot{x}_i(t) - \dot{x}'_i(t) = x_{i+1}(t) - x'_{i+1}(t) + \Big[\varphi_i(x_1(t), \ldots, x_i(t)) - \varphi_i(x'_1(t), \ldots, x'_i(t))\Big] u(t) = x_{i+1}(t) - x'_{i+1}(t) \neq 0$$

Therefore a time t_0 exists such that $x_i(t) \neq x_i(t)$ for all $t \in]0, t_0[$. Continuing this it follows that

$$\exists\, t_1 \in]0, t_0[: \quad x_{i-1}(t) \neq x'_{i-1}(t) \quad \forall t \in]0, t_1[.$$

Therefore by iterating this it can be concluded that

$$\exists\, t_{i-1}: \quad x_1(t) \neq x_1^{'}(t) \quad \forall t \in]0, t_{i-1}[,$$

i.e. $y(t) \neq y^{'}(t)$ *for all* u.
Conversely if

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \varphi(x(t))u(t) \\ y(t) &= C_0 x(t) \end{aligned}$$

with

$$A_0 := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad \text{and} \quad C_0 := \begin{bmatrix} 1 & \cdots & 0 \end{bmatrix}$$

is uniformly observable then $\varphi_i(x(t)) = \varphi_i(x_1(t), \ldots, x_i(t))$. *This is true since if we consider* $x(t), x^{'}(t)$: $x_k(t) = x_k^{'}(t)$, $0 \leq k \leq i < n$ *and* $\varphi_i(x(t)) \neq \varphi_i(x^{'}(t))$ *then there exists*

$$u(t) = -\frac{x_{i+1}(t) - x_{i+1}^{'}(t)}{\varphi_i(x(t)) - \varphi_i(x^{'}(t))} \tag{2.19}$$

in some time interval for which the outputs $y(t)$ *and* $y^{'}(t)$ *are identical and thou there exists a singular input* $u(t)$.

Remark 2.33. *For such a system one can hope to design an observer independently of the input* u. *For non uniformly observable systems one will need some additional conditions on the inputs.*

Remark 2.34. *This leads to the question if only universal inputs should be allowed, therefore the admissable set of inputs to be made smaller. In case of disturbances this is not enough since e.g. the system*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & u(t) \\ 0 & 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

with input

$$u(t) = \begin{cases} 1 & , \quad t \in [0, t_1] \\ 0 & , \quad t > t_1 \end{cases}$$

is universal but if some disturbance comes into play at time $t > t_1$, *observability is lost.*

Note that disturbances can only transform a universal input into a singular input if they are related to the state x. Otherwise they cancel out within the error equation and so the observer remains uniform even in the presence of a disturbance.
Because of the case described in this last remark one needs some kind of persistency within the universality property of the input.

Proposition 2.35.
The input u is universal if and only if the inequality

$$\int_0^t \|h(\mathcal{X}_u(\tau, x_0)) - h(\mathcal{X}_u(\tau, x_0))\|^2 d\tau > 0. \tag{2.20}$$

holds for all $x_0 \neq x_0^{'}$.

Definition 2.36 (Persistency)**.**
The input u is persistent if

$$\exists\, T:\ \forall\, x_t \neq x_t^{'}:\quad \int_t^{t+T} \|h(\mathcal{X}_u(\tau, x_t)) - h(\mathcal{X}_u(\tau, x_t))\|^2 d\tau > 0 \quad \forall\, t \geq 0. \tag{2.21}$$

Still this persistency definition within the universal observability is not enough as the following example illustrates:

Example 2.37. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= \frac{1}{(1+t)^2} x(t) \end{aligned}$$

Then the error in the output is given by

$$h(\mathcal{X}_u(\tau, x_t), \tau) - h(\mathcal{X}_u(\tau, x_t^{'}), \tau) = (x_t - x_t^{'}) \frac{1}{(1+\tau)^2}$$

where $(x_t - x_t^{'})$ is independent of τ.

$$\Rightarrow \int_t^{t+T} \|h - h^{'}\| d\tau = \|x_t - x_t^{'}\| \int_t^{t+T} \frac{d\tau}{(1+\tau)^2} = \frac{T \cdot \|x_t - x_t^{'}\|}{(1+t+T)(1+t)} > 0$$

Note that using the norm instead of the squared norm is consistent with our definition of persistency. From the above inequality it follows that

$$\frac{T}{(1+t+T)(1+t)} \to 0, \quad t \to \infty.$$

In particular this means that observability will decay to zero as t tends to infinity and by this disturbances that occur for large t are less observable. To avoid this we need to assure some "regularity" of the persistency:

Definition 2.38 (Regularly Persistent Inputs)**.**
An input u is called regularly persistent if

$$\exists\, t_0 \geq 0,\ T, \alpha > 0 : \quad \int_t^{t+T} \|h(\mathcal{X}_u(\tau, x_t), \tau) - h(\mathcal{X}_u(\tau, x_t^{'}), \tau)\|^2 d\tau \geq \alpha \|x_t - x_t^{'}\|^2. \tag{2.22}$$

for all $x_t \neq x_t^{'}$.

Proposition 2.39.
For state affine systems

$$\begin{aligned} \dot{x}(t) &= A(u(t))x(t) + B(u(t)) \\ y(t) &= Cx(t) \end{aligned}$$

regularly persisten inputs are such that

$$\exists\, t_0 \geq 0,\ T, \alpha > 0 : \quad \int_t^{t+T} \Phi_u(\tau, t)^T C^T C \Phi_u(\tau, t) d\tau \geq \alpha Id \tag{2.23}$$

where $\Phi_u(\tau, t)$ is such that

$$\begin{aligned} \frac{d}{d\tau}\Phi_u(\tau, t) &= A(u(t))x(t) \\ \Phi_u(t, t) &= Id. \end{aligned}$$

Proof. Consider the error

$$\int_t^{t+T} \|h - h^{'}\| d\tau = \int_t^{t+T} (x_t - x_t^{'})^T \Phi_u(\tau, t)^T C^T C \Phi_u(\tau, t)(x_t - x_t^{'}) d\tau$$

where $h(x_t) = \Phi_u(\tau, t)x_t$. Denoting

$$\Gamma = \int_t^{t+T} \Phi_u(\tau, t)^T C^T C \Phi_u(\tau, t) d\tau$$

we can conclude

$$\int_t^{t+T} \|h - h^{'}\| d\tau = (x_t - x_t^{'})^T \cdot \Gamma \cdot (x_t - x_t^{'}) \geq \alpha \|x_t - x_t^{'}\|^2$$

if $\Gamma \geq \alpha \mathrm{Id}$. Conversely we can conclude if

$$\int_{t}^{t+T} \|h - h'\| d\tau \geq \alpha \|x_t - x_t'\|^2$$

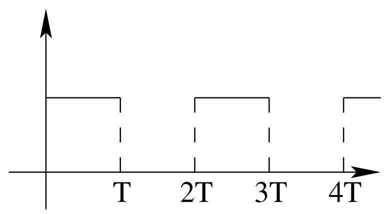
for all $x_t \neq x_t'$ then $\Gamma \geq \alpha \mathrm{Id}$. □

Remark 2.40. *Regular persistency is difficult to be checked in general. As one can see from the previous proposition the regular persistency property becomes independent of the state in the case of state affine systems. To give an idea of the difficulty to check regular persistency consider the system*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & u(t) \\ 0 & 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

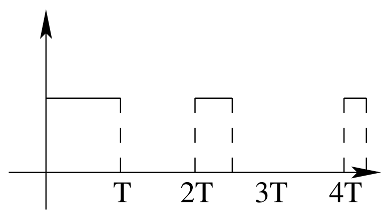
Then the input

$$u(t) = \begin{cases} 1 & , \quad t \in [2kT, (2k+1)T] \\ 0 & , \quad else \end{cases}$$



with $k \in \mathbb{N}_0$ is regularly persistent. But if one uses the input

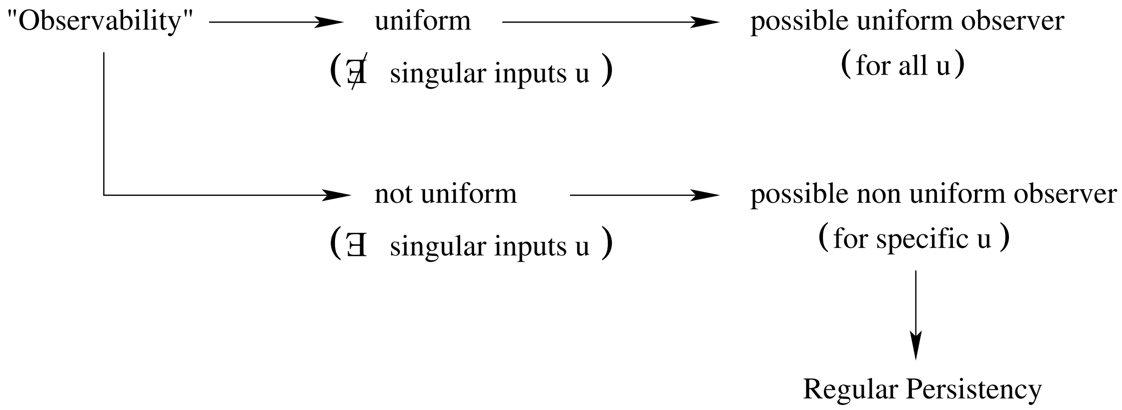
$$u(t) = \begin{cases} 1 & , \quad t \in [2kT, (2k + \frac{1}{k+1})T] \\ 0 & , \quad else \end{cases}$$



with $k \in \mathbb{N}_0$ then the input is clearly not regularly persistent.

Remark 2.41. *Regular persistency for state affine systems reduces to uniform complete observability (Kalman Condition) in the case of linear time–variant systems.*

Remark 2.42. *So far we introduced the following general framework:*



Additionally one has to admit that in special cases one can find a uniformly observer for a system that is not uniform in the input u and vice versa.

Remark 2.43. *Also the following statements are valid:*

(1) *If for systems that do not satisfy the observability rank condition there exists a representation*

$$\begin{aligned} \dot{\zeta}_1(t) &= f_1(\zeta_1(t), \zeta_2(t)) + g_1(\zeta_1(t), \zeta_2(t))u(t) && (2.24)\\ \dot{\zeta}_2(t) &= f_2(\zeta_2(t)) + g_2(\zeta_2(t))u(t) && (2.25)\\ y(t) &= h_2(\zeta_2(t)) && (2.26) \end{aligned}$$

where the $(\dot{\zeta}_2, y)$–subsystem satisfies the observability rank condition, then for the ζ_2–subsystem an observer can be designed.

(2) *For systems that are not observable and if*

$$\forall\ u:\ (x_0, x_0^{'}) \text{ is indistinguishable with } u: \quad \mathcal{X}_u(t, x_0) - \mathcal{X}_u(t, x_0^{'}) \to 0, \quad t \to \infty$$

then one might find some possible observer.
In this case one has to take into account that

- *this is an observer without correction term,*
- *the observer is not tunable,*
- *the observer depends on an internal convergence property of the system itself.*

Chapter 3

"Basic" Designs

Given a system

$$\dot{x}(t) = f(x(t), u(t)) \tag{3.1}$$
$$y(t) = h(x(t)) \tag{3.2}$$

one wants to find

$$\dot{X}(t) = F(X(t), u(t), y(t)) \tag{3.3}$$
$$\hat{x}(t) = H(X(t), u(t), y(t)) \tag{3.4}$$

such that the error $e(t) := \hat{x}(t) - x(t)$ made by this observer tends to zero as $t \to \infty$. In this chapter we will take a closer look at two specific types of function f and h in the linear as well as the nonlinear case.

3.1 Linear Systems

3.1.1 Luenberger Observer

Within this section we will only consider the case of linear time–invariant systems. For this kind of systems a very early result has been presented by Luenberger in the 1960s:

Theorem 3.1 (Luenberger Observer)**.**
Consider a system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.5}$$
$$y(t) = Cx(t) \tag{3.6}$$

If (A, C) is observable then there exists an observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - K\left(C\hat{x}(t) - y(t)\right) \tag{3.7}$$

where the matrix K is such that the matrix $(A - KC)$ is stable and therefore

$$Re(EVal(A - KC)) < 0. \tag{3.8}$$

Proof. To proof this theorem we will first consider a transformation of the system matrizes into a special form and secondly prove that the eigenvalues of $(A - KC)$ can be chosen to lie within the left half of the complex space. We will then conclude the proof by stating that the development of the error tends to zero.

(1) Since (A, C) is observable that there exists a transformation matrix T such that

$$T^{-1}AT = \begin{bmatrix} * & 1 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ * & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad CT = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{3.9}$$

This can be obtained by taking

$$T_1 = \begin{bmatrix} H & AH & \cdots & A^{n-1}H \end{bmatrix}$$

where

$$H = \Theta^{-1} \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}, \quad \text{and the observability matrix } \Theta = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Then for $x(t) = T_1 z(t)$ it follows

$$\dot{x}(t) = \sum_{i=1}^{n} A^{i-1} H \dot{z}_i(t)$$

and

$$\dot{x}(t) = Ax + Bu = \sum_{i=1}^{n} A^{i} H \dot{z}_i(t) + \sum_{i=1}^{n} A^{i-1} H b_i u(t)$$

where $A^n = -a_{n-1}A^{n-1} - \ldots - a_1 A - a_0 \mathrm{Id}$. Then from

$$\begin{aligned} \sum_{i=1}^{n} A^{i-1} H \dot{z}_i(t) \quad &= \quad (b_1 u(t) - a_0 z_n(t))H + (b_2 u(t) - a_1 z_n(t) + z_1(t))AH + \\ &\quad + \ldots + (b_n u(t) - a_{n-1} z_n(t) + z_{n-1}(t))A^{n-1}H \end{aligned}$$

we obtain

$$\begin{aligned}
\dot{z}_i(t) &= b_i u(t) - a_{i-1} z_n(t) + z_{i-1}(t), \\
\text{i.e. } \dot{z}(t) &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_n \end{bmatrix} u
\end{aligned}$$

Additionally if we look at

$$(CT_1)^T = \begin{bmatrix} CH \\ CAH \\ \vdots \\ CA^{n-1}H \end{bmatrix} = \Theta H = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

we see that $y = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix}$ and by reordering the components of z we obtain the canonical form stated above.

(2) For all $\Lambda \in \mathbb{R}^n$ there exists a matrix K_0 such that

$$\mathrm{EVal}(A_0 - K_0 C_0) = \Lambda$$

with $A_0 = T^{-1}AT$, $C_0 = CT$. From this we can set the eigenvalues of $(A_0 - K_0 C_0)$ to any wanted eigenvalues. The einevalues of A_0 are given by the first row of A_0 due to $A^n = -a_{n-1}A^{n-1} - \ldots - a_0 \mathrm{Id}$ which can be transformed into the usual characteristic polynomial of A_0. Therefore

$$\mathrm{EVal}(T^{-1}AT - K_0 CT) = \mathrm{EVal}(A - TK_0 C),$$

i.e. $K = TK_0$ and hence $\mathrm{EVal}(A - KC) = \Lambda$.

(3) Since the error is given by $e(t) = \hat{x}(t) - x(t)$ it follows

$$\dot{e}(t) = (A - KC)e(t)$$

and because of this

$$\Rightarrow \; e(t) \to 0, \;\; t \to \infty$$

if $\Lambda \in (\mathbb{C}^-)^n$.

□

Remark 3.2. *ere we obtain global exponential stability and the observer is tunable via the choice of* Λ.

3.1.2 Kalman Observer

We will now consider the larger class of linear time–variant systems. For this kind of systems the following theorem was obtained by Kalman, also in the 1960's, but here we will give a proof via Lyapunov functions that was developed in the 1980's when one was trying to construct an observer for state–affine systems.

Theorem 3.3 (Kalman Observer).
Consider a system of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad (3.10)\\ y(t) &= C(t)x(t). \qquad (3.11) \end{aligned}$$

If $(A(t), C(t))$ is uniformly completely observalbe and uniformly bounded for all t, then there exists an observer of the form

$$\dot{\hat{x}}(t) = A(t)x(t) + B(t)u(t) - K(t)\left[C(t)\hat{x}(t) - y(t)\right] \qquad (3.12)$$

where $K(t) = M(t)C(t)^T W^{-1}$ with $M(t)$ coming from the Riccati equation

$$\dot{M}(t) = M(t)A(t)^T + A(t)M(t) - M(t)C(t)^T W^{-1}C(t)M(t) + V(t) + \delta M(t), \qquad (3.13)$$

where $M(0)$ and W are positive definite symmetric matrizes and either V is a positive definite symmetric matrix or $\delta > 2\max\|A(t)\|$.

Proof. Set $S(t) := M(t)^{-1}$. Then by derivating this equation we obtain

$$\dot{S}(t) = -A(t)^T S(t) - S(t)A(t) + C(t)^T W^{-1}C(t) - S(t)VS(t) - \delta S(t)$$

and we have:

(1) There exist t_0, α_1 and α_2 such that

$$\alpha_1 \mathrm{Id} \le S(t) \le \alpha_2 \mathrm{Id}$$

.

(2) Given the error $e(t) := \hat{x}(t) - x(t)$ then

$$V(t,e) = e(t)^T S(t)e(t)$$

is a Lyapunov function for the error system.

To show (1) we will first consider the bf case $\delta > \mathbf{2a}$ with $\mathbf{a} := \max\|\mathbf{A(t)}\|$ and $\mathbf{V} = \mathbf{0}$. Then the solution of $S(t)$ is given by

$$S(t) = e^{-\delta t}\Phi(0,t)^T C(t)^T W^{-1}C(t)\Phi(0,t) + \int_0^t e^{-\delta(t-\tau)}\Phi(\tau,t)^T C(t)^T W^{-1}C(t)\Phi(\tau,t)d\tau$$

with Φ as solution of the matrix differential equation

$$\frac{d}{dt}\Phi(\tau,t) = A(\tau)\Phi(\tau,t)$$

using the initial values $\Phi(t,t) = \mathrm{Id}$.
Lower bound: Since the non–integral part is larger than 0 due to $S(0)$ being positiv semidefinit we get

$$\begin{aligned} S(t) &\geq \int_0^t e^{-\delta(t-\tau)}\Phi(\tau,t)^T C(t)^T W^{-1} C(t)\Phi(\tau,t)d\tau \\ &\geq \int_{t-T}^t e^{-\delta(t-\tau)}\Phi(\tau,t)^T C(t)^T W^{-1} C(t)\Phi(\tau,t)d\tau, \quad t \geq T. \end{aligned}$$

Then using uniform complete observability, i.e.

$$\int_t^{t+T} e^{-\delta(t-\tau)}\Phi(\tau,t)^T C(t)^T C(t)\Phi(\tau,t)d\tau \geq \alpha\mathrm{Id},$$

it follows that

$$\int_t^{t+T} e^{-\delta(t-\tau)}\Phi(\tau,t)^T C(t)^T W^{-1} C(t)\Phi(\tau,t)d\tau \geq \overline{\alpha}\mathrm{Id},$$

where $\overline{\alpha}$ and α are related by the eigenvalues of W^{-1}. Using $\Phi(\tau,t) = \Phi(\tau,t-T)\Phi(t-T,t)$ we obtain

$$\begin{aligned} S(t) &\geq \int_{t-T}^t e^{-\delta T}\Phi(t-T,t)^T \left[\Phi(\tau,t)^T C(t)^T W^{-1} C(t)\Phi(\tau,t)\right]\Phi(t-T,t)d\tau \\ &\geq e^{\delta T}\Phi(t-T,t)^T\Phi(t-T,t)\overline{\alpha}. \end{aligned}$$

Since $\Phi(t,t-T) = \mathrm{Id} + \int_{t-T}^{T} A(\tau)\Phi(\tau,t-T)d\tau$ it follows that

$$\|\Phi(t,t-T)\| \geq 1 + \int_{t-T}^t a\|\Phi(\tau,t-T)\|d\tau.$$

Using the Gronwall–Lemma this reveals $\|\Phi(t,t-T)\| \leq e^{aT}$.

$$\Rightarrow \quad \Phi^T(t,t-T)\Phi(t,t-T) \leq e^{2aT}$$

Using $\Phi(t-T,t) = \Phi(t,t-T)^{-1}$ we obtain

$$\Phi^{-T}(t,t-T)\Phi(t,t-T)^T\Phi(t,t-T)\Phi(t,t-T)^{-1} \leq e^{2aT}\Phi(t,t-T)^{-T}\Phi(t,t-T)^{-1}.$$

$$\Rightarrow \quad \Phi(t,t-T)^{-T}\Phi(t,t-T)^{-1} \geq e^{-2aT}\mathrm{Id}$$

and therefore

$$S(t) \geq \overline{\alpha}e^{-\delta T}e^{-2aT}\mathrm{Id}.$$

Upper bound:

$$\begin{aligned} S(t) &\leq e^{-\delta t}e^{2at}\|S(0)\| + \int_0^t e^{-(\delta-2a)(t-\tau)}\|C(t)^T W^{-1}C(t)\|d\tau \\ &\leq \alpha_2 \mathrm{Id} \quad \text{if } \delta > 2a. \end{aligned}$$

To show (2) consider the Lyapunov function candidate

$$V(t,e(t)) = e(t)^T S(t)e(t).$$

Then by derivation we obtain

$$\dot{V}(t,e(t)) = \dot{e}(t)^T S(t)e(t) + e(t)^T\dot{S}(t)e(t) + e(t)^T S(t)\dot{e}(t)$$

with

$$\dot{e}(t) = \left(A(t) - S(t)^{-1}C(t)^T W^{-1}\right)e(t)$$

where $S(\cdot)$ is the solution of the Riccati equation

$$\dot{S}(t) = -A(t)^T S(t) - S(t)A(t) + C(t)^T W^{-1}C(t) - \delta S(t).$$

Hence we can simplify to

$$\begin{aligned} \dot{V}(t,e(t)) &= e(t)^T(A(t) - S(t)^{-1}C(t)^T W^{-1})^T S(t)e(t) \\ &\quad + e(t)^T(-A(t)^T S(t) - S(t)A(t) + C(t)^T W^{-1}C(t) - \delta S(t))e(t) \\ &\quad + e(t)^T S(A(t) - S(t)^{-1}C(t)^T W^{-1})e(t) = \\ &= -e(t)^T C(t)^T W^{-1}C(t)e(t) - \delta e(t)^T S(t)e(t) \leq -\delta V(t,e(t)). \end{aligned}$$

To show (1) in the **case $\delta = 0$ and V is a positiv definite symmetric matrix** one instead has to consider

$$\dot{S}(t) = -A(t)^T S(t) - S(t)A(t) + C(t)^T W^{-1}C(t) - S(t)VS(t).$$

Lower bound: It can be shown that for all $\delta > 0$

$$\begin{aligned} S(t) &= e^{-\delta t}\Phi(0,t)^T S(0)\Phi(0,t) + \int_0^t e^{-\delta(t-\tau)}\Phi(\tau,t)^T C(t)^T W^{-1} C(t)\Phi(\tau,t)d\tau \\ &+\delta \int_0^t e^{-\delta(t-\tau)}\Phi(\tau,t)^T \left[S(t) - \frac{S(t)VS(t)}{\delta}\right]\Phi(\tau,t)d\tau. \end{aligned}$$

As before $S(t) \leq \alpha_2\mathrm{Id}$ and $S(t) \geq \alpha_1\mathrm{Id}$. Moreover

$$S(t) - \frac{S(t)VS(t)}{\delta} = \sqrt{S(t)}\left(\mathrm{Id} - \frac{\sqrt{S(t)}V\sqrt{S(t)}}{\delta}\right)\sqrt{S(t)}$$

with $\sqrt{S(t)}V\sqrt{S(t)} \leq \|V\|\,\|S(t)\| \leq \alpha_2\|V\|$. From this it follows that

$$\mathrm{Id} - \frac{\sqrt{S(t)}V\sqrt{S(t)}}{\delta} \geq \mathrm{Id} - \frac{\alpha_2\|V\|}{\delta} > 0$$

if $\delta > \alpha_2\|V\|$. Therefore $S(t) \geq 0$ and hence the previously shown bound applies here as well.
To show (2) in this case consider again the Lyapunov candidate

$$V(t,e(t)) = e(t)^T S(t)e(t).$$

By derivation we get

$$\begin{aligned} \dot V(t,e(t)) &= -e(t)^T C(t)^T W^{-1}C(t)e(t) - e(t)^T S(t)VS(t)e(t) \\ &\leq -e(t)^T S(t)VS(t)e(t) = -e(t)^T\sqrt{S(t)}S\sqrt{S(t)}e(t) \\ &\leq -v\alpha_1 e(t)^T S(t)e(t) = -v\alpha_1 V(t,e(t)) \end{aligned}$$

where v denotes the minimum eigenvalue of V. □

Remark 3.4. *Properties of the Kalman observer:*

(1) *From Lypunov stability theory we obtain global exponential stability. Moreover the observer is tunable via either the parameter δ or the matrix V.*

(2) *If $\delta = 0$ and $V \neq 0$ we get the classical Kalman observer. Additionally in considering the dual system we get that the usual condition of uniform complete controllability is satisfied by $V = V^T > 0$.*

(3) *The Kalman observer is optimal in the sense of minimizing the expression*

$$\int_0^t \left[(y(\tau)-C(\tau)z(\tau))^T W^{-1}(y(\tau)-C(\tau)z(\tau)) + (\dot z(\tau)-A(\tau)z(\tau))^T V^{-1}(\dot z(\tau)-A(\tau)z(\tau))\right]d\tau \tag{3.14}$$

where one is also free to add the initial values $\|z(0) - \hat x_0\|^2_{M(0)}$.

(4) The Kalman observer is also optimal with respect to minimization of

$$E\big[(\hat{x}(t) - x(t))^T(\hat{x}(t) - x(t))\big] \tag{3.15}$$

when

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + v(t)\\ y(t) &= Cx(t) + w(t)\end{aligned}$$

with

$$\begin{aligned}E\big[v(\tau)v(t)^T\big] &= V\delta(t-\tau)\\ E\big[w(\tau)w(t)^T\big] &= W\delta(t-\tau)\\ E\big[w(\tau)v(t)^T\big] &= 0.\end{aligned}$$

In this case one an notice, for $e(t) = \hat{x}(t) - x(t)$*, that*

$$\begin{aligned}\dot{e}(t) &= (A - KC)e(t) - v(t) + Kw(t)\\ ST \quad & \dot{x}(t) = Ax(t) + Bu(t) + v(t)\\ & y(t) = Cx(t) + w(t)\\ & \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - K(C\hat{x} - y)\end{aligned}$$

If we extend this design to the nonlinear case we get

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t))\\ y(t) &= h(x(t))\\ \text{and } \dot{\hat{x}}(t) &= f(\hat{x}(t), u(t)) + k(h(\hat{x}(t) - y(t)))\end{aligned}$$

and for the error $e(t) = \hat{x}(t) - x(t)$ it follows

$$\dot{e}(t) = \big[f(\hat{x}(t), u(t)) - f(x(t), u(t))\big] - K\big[h(\hat{x}(t)) - h(x(t))\big]$$

and we have to find a k such that $e(t) \to 0$ as $t \to \infty$. Using Taylor expansion we get

$$\begin{aligned}\dot{e}(t) &= \frac{\partial f}{\partial x}\Big|_{\hat{x}(t)} \cdot (\hat{x}(t) - x(t)) + O(\|\hat{x}(t) - x(t)\|)\\ &\qquad -K(t)\left[\frac{\partial h}{\partial x}\Big|_{\hat{x}(t)} \cdot (\hat{x}(t) - x(t)) + O(\|\hat{x}(t) - x(t)\|)\right]\\ &= \left[\frac{\partial f}{\partial x}\Big|_{\hat{x}(t)} - K(t)\frac{\partial h}{\partial x}\Big|_{\hat{x}(t)}\right] e(t) + O(\|\hat{x}(t) - x(t)\|) + K(t)O(\|\hat{x}(t) - x(t)\|).\end{aligned}$$

Using the shorthand notation $A(t) \approx \frac{\partial f}{\partial x}\Big|_{\hat{x}(t)}$, $C(t) \approx \frac{\partial h}{\partial x}\Big|_{\hat{x}(t)}$, $v(t) = O(\|\hat{x}(t) - x(t)\|)$ and $w(t) = O(\|\hat{x}(t) - x(t)\|)$ it follows

$$\dot{e}(t) = \big(A(t) - K(t)C(t)\big)e(t) + v(t) + K(t)w(t)$$

Now take $K(t) = M(t)C^T W^{-1}$ where $M(t)$ is given by the matrix Riccati differential equation

$$\dot{M}(t) = M(t)A(t)^T + A(t)M(t) - M(t)C(t)^T W^{-1} C(t)M(t) + V + \delta M(t)$$

and $A(t)$ and $C(t)$ are given as before. This is also called "Extended Kalman Filter".
Here v and w are not uncorrelated and not white noise. Therefore the conditions for the Kalman observer to be optimal are not satisfied. In general there is no guarantee that $e(t) \to 0$, $t \to \infty$ as it can be seen from the following example.

Example 3.5. *Consider the system*

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_1^2(t) \\
y(t) &= x_1(t) \\
\dot{\hat{x}}_1(t) &= \hat{x}_2(t) - k_1(t)e_1(t) \\
\dot{\hat{x}}_2(t) &= \hat{x}_1^2(t) - k_2(t)e_1(t)
\end{aligned}$$

with $e_1(t) = \hat{x}_1(t) - x_1(t)$ *and* $K(t) = [k_1(t),\ k_2(t)]^T$. *Then the error is given by*

$$\dot{e}(t) = \begin{bmatrix} -k_1(t) & 0 \\ -k_2(t) + 2\hat{x}_1(t) & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ e_1^2(t) \end{bmatrix}.$$

Hence the usual Lyapunov candidate $V(t, e(t)) = e^T(t)S(t)e(t)$ *yields*

$$\dot{V}(t, e(t)) \leq \lambda V(t, e(t)) - 2e^T(t)S(t) \begin{bmatrix} 0 \\ e_1^2(t) \end{bmatrix}.$$

If e_1 *is too large the* $e(t)$ *might grow since the last term might become dominant.*

3.2 Nonlinear Systems

Here we will focus on and restrict us to specific structures of f and h and extend the previously mentioned linear structures.

3.2.1 Luenberger–like Observer

Here we handle uniformly observable systems.

Output Additive Nonlinearities

Example 3.6. *In the previous example 3.5 the only nonlinearity is* x_1^2 *and the output is measuring* x_1. *Therefore we will use this in the observer.*

$$\begin{aligned}
\dot{\hat{x}}_1(t) &= \hat{x}_2(t) - k_1 e_1(t) \\
\dot{\hat{x}}_2(t) &= y^2(t) - k_2 e_1(t)
\end{aligned}$$

with $e(t) := \hat{x}_1(t) - x_1(t)$. *Then we obtain*

$$\begin{aligned}\dot{e}_1(t) &= e_2(t) - k_1 e_1(t)\\ \dot{e}_2(t) &= -k_2 e_2(t),\end{aligned}$$

which can be written as

$$\dot{e}(t) = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} e(t)$$

which is an exact linear observer since $e(t) \to 0$, $t \to \infty$ *globally, exponentially an exactly for appropriate* k_1, k_2 *and the oberserver is tunalbe by choice of* k_1 *and* k_2.

Theorem 3.7.
For any system of the form

$$\dot{x}(t) = Ax(t) + \varphi(Cx(t), u(t)) \tag{3.16}$$

$$y(t) = Cx(t) \tag{3.17}$$

where the pair (A, C) *is observable, there exists a* $\dot{\hat{x}}$ *satisfying*

$$\dot{\hat{x}}(t) = A\hat{x}(t) - K\Big(C\hat{x}(t) - y(t)\Big) + \varphi(y(t), u(t)) \tag{3.18}$$

with $(A - KC)$ *stabil such that* $\hat{x}$ *is an globally exponential and exact observer.*

Triangular Additive Nonlinearities

Example 3.8. *Consider the system*

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \frac{1}{5}\sin(x_2(t))\\ y(t) &= x_1.\end{aligned}$$

Note that $\dot{x}_2(t) = \frac{1}{5}\sin(x_2(t)) \neq \varphi(x_1(t))$. *The observer is then given by*

$$\begin{aligned}\dot{\hat{x}}_1(t) &= \hat{x}_2(t) - k_1 e_1(t)\\ \dot{\hat{x}}_2(t) &= \frac{1}{5}\sin(\hat{x}_2(t)) - k_2 e_1(t).\end{aligned}$$

Take for instance $k_1 = 2$, $k_2 = 1$.

$$\Rightarrow A - KC = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$

which is stable. Then take P such that

$$\begin{aligned} P(A-KC)+(A-KC)^T P &= -Id \\ \Rightarrow P = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} &= P^T > 0. \end{aligned}$$

So choosing for $e(t)=\hat{x}(t)-x(t)$ an Lyapunov function $V(e(t)) = e(t)^T P e(t)$ it follows that

$$\dot{V}(e(t)) = -e(t)^T e(t) + 2e(t)^T P \begin{bmatrix} 0 \\ \frac{\sin(\hat{x}_2(t))-\sin(x_2(t))}{5} \end{bmatrix}.$$

Since sin *is Lipschitz we have*

$$|\sin(\hat{x}_2(t)) - \sin(x_2(t))| \leq |\hat{x}_2(t) - x_2(t)| = \|e(t)\|$$

and the eigenvalues of P are given by $\lambda_{1,2} = 2 \pm \sqrt(2)$. Therefore

$$\begin{aligned} \dot{V}(e(t)) &= -e(t)^T e(t) + 2e(t)^T P \begin{bmatrix} 0 \\ \frac{\sin(\hat{x}_2(t))-\sin(x_2(t))}{5} \end{bmatrix} \quad \text{with } P \leq 2 \\ &\leq -\|e(t)\|^2 + \frac{4}{5}\|e(t)\|^2 = -\frac{1}{5}\|e(t)\|^2 \end{aligned}$$

and hence $e(t) \to 0$ as $t \to \infty$.

Theorem 3.9.
Consider a system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \varphi(Cx(t), u(t)) && (3.19) \\ y(t) &= Cx(t) && (3.20) \end{aligned}$$

and the pair (A, C) shall be observable. Then there exists an $\dot{\hat{x}}$ satisfying

$$\dot{\hat{x}}(t) = A\hat{x}(t) - K\Big(C\hat{x}(t) - y(t)\Big) + \varphi(y(t), u(t)) \tag{3.21}$$

with $(A-KC)$ stable such that $\hat{x}$ is a globally exponential and exact observer.

In general it is difficult to get

$$\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} > \gamma \tag{3.22}$$

to be satisfied. A systematic solution is obtained for uniformly observable systems by the following:

Theorem 3.10 (High–Gain Observer Design)**.**
Consider a system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \varphi(x(t), u(t)) && (3.23) \\ y(t) &= C_0 x(t) && (3.24) \end{aligned}$$

with

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \quad \text{and} \quad C_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}.$$

If $\frac{\partial \varphi_i}{\partial x_j} = 0$, $n \geq j > i \geq 1$ *and* φ *is globally Lipschitz in* x *with Lipschitz constant* γ *and uniformly in* u *then*

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) - \Lambda(\lambda) K_0 (C_0 \hat{x}(t) - y(t)) + \varphi(\hat{x}(t), u(t)) \tag{3.25}$$

where $K_0 := (A_0 - K_0 C_0)$ *is stable and*

$$\Lambda(\lambda) = \begin{bmatrix} \lambda & & & 0 \\ & \lambda^2 & & \\ & & \ddots & \\ 0 & & & \lambda^n \end{bmatrix} \tag{3.26}$$

is an observer for the system for λ *being sufficiently large.*

Proof. Define the error $e(t) := \hat{x}(t) - x(t)$. Then the derivative is given by

$$\dot{e}(t) = (A_0 - \Lambda K_0 C_0) e(t) + \varphi(\hat{x}(t), u(t)) - \varphi(x(t), u(t)).$$

(1) By inspecting the matrices one can check that

$$\Lambda^{-1}(A_0 - \Lambda K_0 C_0)\Lambda = \lambda(A_0 - K_0 C_0)$$

(2) It also follows for λ large enough, $\lambda > 1$, that

$$\|\Lambda^{-1}\Big(\varphi(\hat{x}(t), u(t)) - \varphi(x(t), u(t))\Big)\| \leq \gamma \|\Lambda^{-1}(\hat{x}(t) - x(t))\| = \gamma \|\Lambda^{-1} e(t)\|$$

since

$$\begin{aligned} \frac{1}{\lambda^i}\|(\varphi_i(\hat{x}(t), u(t)) - \varphi_i(x(t), u(t)))\| &\leq \frac{1}{\lambda^i}\left\| \begin{pmatrix} \hat{x}_1(t) \\ \vdots \\ \hat{x}_i(t) \end{pmatrix} - \begin{pmatrix} x_1(t) \\ \vdots \\ x_i(t) \end{pmatrix} \right\| \\ &\leq \left\| \begin{bmatrix} \frac{1}{\lambda} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda^i} \end{bmatrix} \left[\begin{pmatrix} \hat{x}_1(t) \\ \vdots \\ \hat{x}_i(t) \end{pmatrix} - \begin{pmatrix} x_1(t) \\ \vdots \\ x_i(t) \end{pmatrix} \right] \right\| \\ &\leq \|\Lambda^{-1}(\hat{x}(t) - x(t))\| \end{aligned}$$

Then we use the Lyapunov candidate $V(e(t)) = e(t)^T \Lambda^{-1} P \Lambda^{-1} e(t)$ with

$$P(A_0 - K_0C_0) + (A_0 - K_0C_0)^T P = -\mathrm{Id}, \quad P = P^T > 0.$$

Then it follows from the derivative

$$\begin{aligned}
\dot{V}(e(t)) &= 2e(t)^T \Lambda^{-1} P \Lambda^{-1}(A_0 - \Lambda K_0 C_0)e(t) \\
&\quad +2e(t)^T \Lambda^{-1} P \Lambda^{-1}(\varphi(\hat{x}(t), u(t)) - \varphi(x(t), u(t))) \\
&\leq -\lambda \|\Lambda^{-1}e(t)\|^2 + 2\|\Lambda^{-1}e(t)\|\lambda_{\max}(P)\gamma\|\Lambda^{-1}e(t)\| \\
&\leq -\|\Lambda^{-1}e(t)\|^2(\lambda - 2\lambda_{\max}(P)\gamma) \\
&\leq -\delta\|\Lambda^{-1}e(t)\|^2
\end{aligned}$$

for $\lambda > 2\lambda_{\max}(P)\gamma + \delta$ that the error $e(t) \to 0$ exponentially by Lyapunov arguments. □

Remark 3.11. *The choice of λ is crucial:*

(1) The rate of convergence is tunable by λ. So not only convergene by itself is guaranteed by it.

(2) Notice the "peaking phenomenon". The faster the convergence is, the larger the overshoot within the transient behavior may be.

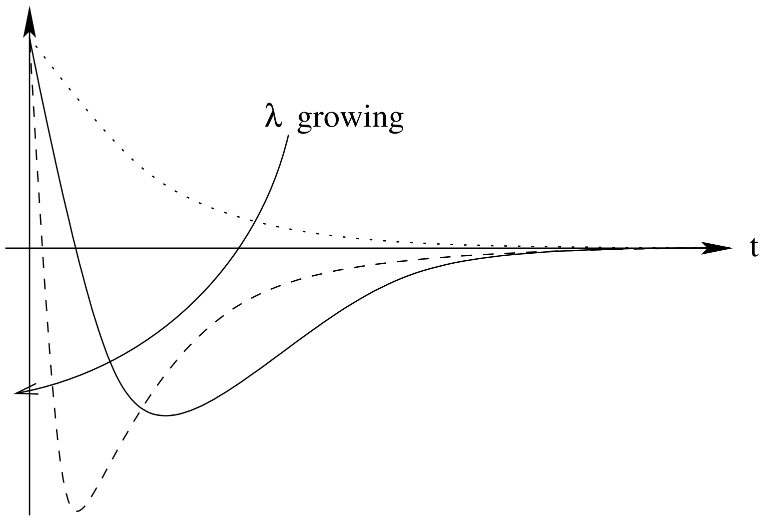


Figure 3.1: Peaking phenomenon for growing λ

(3) The larger a disturbance is then for large λ its effects will be larger.

(4) This result can be extended to systems of the form

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), u(t)) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), x_3(t), u(t)) \\
&\vdots \\
\dot{x}_{n-1}(t) &= f_{n-1}(x_1(t), \dots, x_n(t), u(t)) \\
\dot{x}_n(t) &= f_n(x(t), u(t)) \\
y(t) &= x_1(t)
\end{aligned}$$

where $\frac{\partial f_i}{\partial x_{i+1}} \geq \alpha_i > 0 \; \forall x, u$ *and* f *is Lipschitz with Lipschitz constant* γ *globally in* x*, uniformly in* u*.*

3.2.2 Kalman–like Observer

Here we will only consider the case of state–affine systems. For this special kind of systems an observer can be constructed as follows.

Theorem 3.12.
Consider a system of the form

$$\begin{aligned}
\dot{x}(t) &= A(u(t))x(t) + B(u(t)) && (3.27) \\
y(t) &= Cx(t). && (3.28)
\end{aligned}$$

If the input $u(t)$ *is regularly persistent and such that* $A(u(t))$ *is bounded, then the Kalman observer stated in Theorem 3.3 is an observer for the system.*

Remark 3.13. *This result can be extended to systems that are affine in the unmeasured states*

$$\begin{aligned}
\dot{x}(t) &= A(u(t), y(t))x(t) + B(u(t), y(t)) \\
y(t) &= Cx(t)
\end{aligned}$$

if $u(t)$ *makes*

$$v(t) := \begin{pmatrix} u(t) \\ C\mathcal{X}_u(t, x_0) \end{pmatrix}$$

regularly persistent for the system

$$\begin{aligned}
\dot{x}(t) &= A(v)x(t) \\
y(t) &= Cx(t)
\end{aligned}$$

for all x_0*.*

Combining

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \varphi(x(t), u(t)) \\ y(t) &= C_0 x(t)\end{aligned}$$

and

$$\begin{aligned}\dot{x}(t) &= A(u(t), y(t))x(t) + B(u(t), y(t)) \\ y(t) &= Cx(t)\end{aligned}$$

to

$$\dot{x}(t) = A_0(u(t), y(t))x(t) + \varphi(x(t), u(t)) \quad (3.29)$$

$$y(t) = C_0 x(t) \quad (3.30)$$

where A_0, φ and C_0 satisfy the properties of Theorem 3.10, then some kind of High–Gain Kalman–like Observer can be derived under appropriate observability properties which can be characterized as regular observability for arbitrarily short times.

Chapter 4

“Advanced” Designs

In this chapter we will treat the case when a system does not exhibit the form of one of the previously mentioned systems.

4.1 Interconnection

The basic scheme of interconnected systems can be shown graphically as follows:

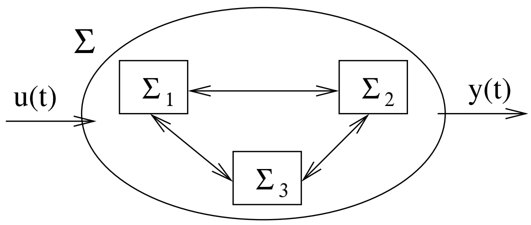


Figure 4.1: System of interconnected subsystems

For this kind of system Σ we are not only looking for (sub-)observers Θ_i for each of the subsystems Σ_i as shown in 4.2, but for an observer Θ for the interconnection of the subsystems, see 4.3.
Therefore we would like to obtain conditions on the subobservers itself as well as on the interconnection term that will allow us to construct such an observer for the overall system. We will see that up till now this is only possible for special kind of systems, meaning the subsystems have to fulfill certain structural criteria.

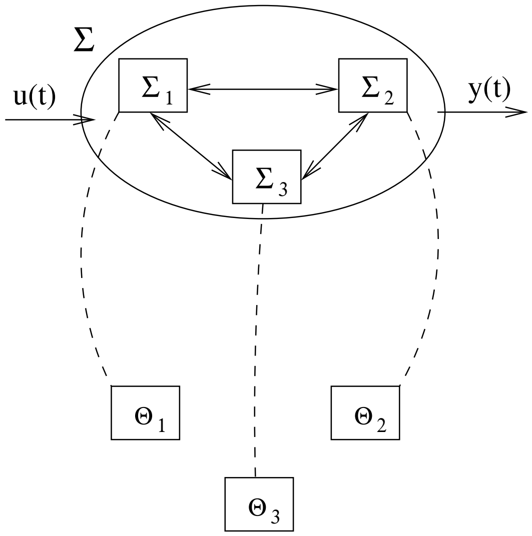


Figure 4.2: Observer for every single system within a system of interconnected systems

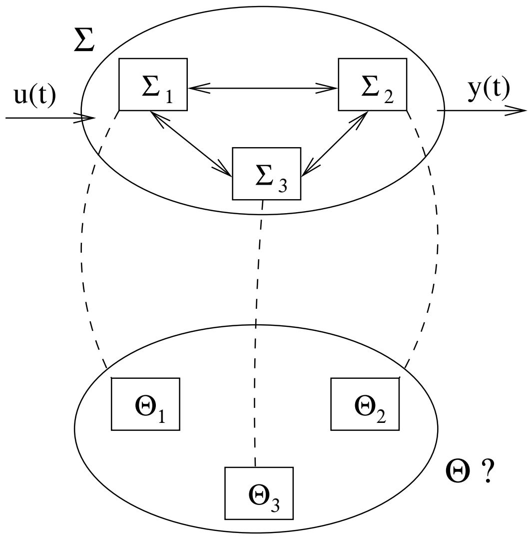


Figure 4.3: Observer for a system of interconnected systems

Example 4.1. *Consider the system*

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u_1(t)\\ \dot{x}_3(t) &= x_4(t) + \varphi(x_2(t))\\ \dot{x}_4(t) &= u_2(t)\\ y(t) &= \begin{pmatrix} x_1(t)\\ x_3(t)\end{pmatrix}\end{aligned}$$

This system can be divided into two subsystems Σ_1, Σ_2 *and one can construct observers* Θ_1 *and* $\Theta_2(\hat{x}_2(t))$ *for each of these subsystems.*

$$\Sigma_1 \quad \begin{cases}\dot{x}_1(t) = x_2(t)\\ \dot{x}_2(t) = u_1(t)\\ y_1(t) = x_1(t)\end{cases} \qquad \longleftarrow \Theta_1$$

$$\Sigma_2 \quad \begin{cases}\dot{x}_3(t) = x_4(t) + \varphi(x_2(t))\\ \dot{x}_4(t) = u_2(t)\\ y_2(t) = x_3(t)\end{cases} \qquad \longleftarrow \Theta_2(\hat{x}_2(t))$$

The obvious question is if

$$\Theta_1 + \Theta_2(\hat{x}_2(t))$$

is an observer for the system Σ, which can actually be shown if φ is Lipschitz.
If instead

$$\Sigma_2 \begin{cases} \dot{x}_3(t) = \varphi(x_2(t))x_4(t) \\ \dot{x}_4(t) = u_2(t) \\ y(t) = x_3(t) \end{cases}$$

we need $x_2(t)$ to be regularly persistent for Σ_1 and φ needs to be Lipschitz to get a solution.

One can give conditions on subobservers (exponential convergence) and interconnection terms (Lipschitz, observability) to make it possible to get an observer by interconnecting subobservers either for

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), u(t)) \\ y_1(t) &= h_1(x_1(t)) \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t), u(t)) \\ y_2(t) &= h_2(x_1(t), x_2(t)) \\ &\vdots \end{aligned}$$

or for

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t), u(t)) \\ \dot{x}_2(t) &= f_2(x_2(t), x_1(t), u(t)) \\ y(t) &= h(x) \end{aligned}$$

where in the second case $(x_2(t), u(t))$ is considered as the input of the first differential equation f_1 and $(x_1(t), u(t))$ the input of the second differential equation f_2.

4.2 Transformation

If we consider a system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned}$$

the idea is to use a transformation

$$z(t) = \Phi(x(t)) \quad \text{such that} \quad x(t) = \Psi(z(t)) \tag{4.1}$$

to get a system of the form

$$\dot{z}(t) = F(z(t), u(t)) \tag{4.2}$$
$$y(t) = H(z(t)) \tag{4.3}$$

for which one knows how to design an observer, i.e.

$$\dot{\hat{z}} = F(\hat{z}(t), u(t)) + K(H(\hat{z}(t)) - y(t)) \tag{4.4}$$

and hence we can take

$$\hat{x}(t) = \Psi(\hat{z}(t)) \tag{4.5}$$

as an observer of the untransformed system.
The problem here is to find appropriate transformations Φ and Ψ.
There exist conditions to transform a system into one of the following forms:

$$\Sigma_1 \qquad \begin{cases} \dot{z}(t) = Az(t) + \varphi(y(t), u(t)) \\ y(t) = Cz(t) \end{cases} \tag{4.6}$$

$$\Sigma_2 \qquad \begin{cases} \dot{z}(t) = A(u(t), y(t))z(t) + \varphi(y(t), u(t)) \\ y(t) = Cz(t) \end{cases} \tag{4.7}$$

$$\Sigma_3 \qquad \begin{cases} \dot{z}(t) = A(u(t))z(t) + \varphi(y(t), u(t)) \\ y(t) = Cz(t) \end{cases} \tag{4.8}$$

$$\Sigma_4 \qquad \begin{cases} \dot{z}(t) = A_0 z(t) + \varphi(z(t), u(t)) \\ y(t) = C_0 z(t) \end{cases} \tag{4.9}$$

$$\Sigma_5 \qquad \begin{cases} \dot{z}(t) = A_0(u(t), y(t))z(t) + \varphi(z(t), u(t)) \\ y(t) = C_0 z(t) \end{cases} \tag{4.10}$$

Chapter 5

From Theory to Practice

5.1 Infinitessimal Observability

Here we will be dealing with systems of the form

$$\Sigma \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \tag{5.1}$$

where again x is the state, y is the observation and u is the control. Furthermore we will suppose that f is a smooth vectorfield and h a smooth function. Here $x \in X \subset \mathbb{R}^n$ is a connected manifold, $y \in \mathbb{R}^p$ and $u \in L^\infty(U)$ where $U \subset \mathbb{R}^m$.

Definition 5.1 (Input/Output Function)**.**
The input/output function is given by

$$\begin{array}{rcccccc} P: & L^\infty(U) & \times & X & \to & L(\mathbb{R}^p) \\ & \Big((u_t)_{0 \le t \le t_u} & , & x_0 \Big) & \rightsquigarrow & (y_t)_{0 \le t \le e(u,x_0)} \end{array} \tag{5.2}$$

where $e(u, x_0)$ is the minimum of t_u and the escape time if any exists. The system is observable if the set

$$\left\{ t \mid P_t(u, x_0) \neq P_t(u, x_0^{'}) \right\} \tag{5.3}$$

has positive Lebesque measure.

Remark 5.2. *This is observability for any input u.*

We will define the first variation of the system Σ denoted by $T\Sigma$ as follows:

Definition 5.3 (First Variation)**.**
Consider a system Σ according to (5.1) *with*

$$f \;:\; X \times U \to TX \tag{5.4}$$

$$T_x f \;:\; TX \times U \to TTX \tag{5.5}$$

$$h \;:\; X \times U \to \mathbb{R}^p \tag{5.6}$$

$$d_x h \;:\; TX \times U \to \mathbb{R}^p. \tag{5.7}$$

Then the first variation is given by

$$T\Sigma \begin{cases} \frac{d\xi}{dt} = T_x f(\xi(t), u(t)) \\ \eta(t) = d_x h(\xi(t), u(t)) \end{cases} \tag{5.8}$$

which is the well known linearization of Σ along a state trajectory.
Additionally the input/output function of $T\Sigma$ between 0 and $e(u, x_0)$ is defined by

$$dP_t(u, \xi_0) = \eta_t, \quad 0 \leq t < e(u, x_0). \tag{5.9}$$

Remark 5.4. *dP_t is also the Frechet derivative of the input/output function P of Σ.*

Definition 5.5 (Infinitesimal Observability)**.**
A system σ is infinitesimally observable at u if $dP_{u,x_0}(.)$ is injective (where $dP_{u,x_0}(.)\xi_0 \rightsquigarrow \left(dP_{u,x_0}(\xi_0)\right)_{0\leq t\leq e(u,x_0)}$). Σ is uniformly infinitesimally observable if it is infinitesimally observable for any input u and any x_0.

This definition splits the classes of systems Σ into two cases:

(1) If $p \leq m$, then infinitesimal observability is a very strong property. Only a few systems verify this hypothesis. So systems that are uniformly infinitesimally observable are "very special".

(2) If $p > m$, then uniform infinitesimal observability is a generic property, that is to say it is verified by almost all systems.

5.1.1 Uniformly infinitessimally observable systems if $p \leq m$

For the first case we will suppose that $U = I^p \subset \mathbb{R}^m$ and also that $p = 1$. There is an analytic set M in X of codimension at least 1 such that ${}^{\Sigma}/_{X\times\bar{M}}$ can (locally) be put in the form

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), u(t)) && (5.10)\\
\dot{x}_2(t) &= f_1(x_1(t), x_2(t), x_3(t), u(t)) && (5.11)\\
&\vdots \\
\dot{x}_n(t) &= f_n(x(t), u(t)) && (5.12)\\
y(t) &= h(x_1(t), u(t)) && (5.13)
\end{aligned}$$

where

$$\frac{\partial f_1}{\partial x_2} \neq 0, \quad \frac{\partial f_2}{\partial x_3} \neq 0, \quad \dots \quad \frac{\partial h}{\partial x_1} \neq 0. \tag{5.14}$$

Let us verify that this system is infinitesimally observable:

$$\begin{aligned}
\dot{\xi}_1(t) &= \frac{\partial f_1}{\partial y_1}(x_1(t), x_2(t), u(t))\xi_1(t) + \frac{\partial f_1}{\partial x_2}(x_1(t), x_2(t), u(t))\xi_2(t) && (5.15)\\
\dot{\xi}_1(t) &= \frac{\partial f_2}{\partial y_1}(x_1(t), x_2(t), x_3(t), u(t))\xi_1(t) + \frac{\partial f_2}{\partial x_2}(x_1(t), x_2(t), x_3(t), u(t))\xi_2(t) && (5.16)\\
&\quad + \frac{\partial f_2}{\partial x_3}(x_1(t), x_2(t), x_3(t), u(t))\xi_3(t) && (5.17)\\
&\vdots \\
\eta(t) &= \frac{\partial h}{\partial x_1}(x_1(t), u(t))\xi_1(t) && (5.18)
\end{aligned}$$

There is a special result for control affine systems

$$\begin{aligned}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t)\\
y(t) &= h(x(t))
\end{aligned}$$

with $u \in \mathbb{R}^m$, $y \in \mathbb{R}$. If this system is uniformly infinitesimally observable, then it can locally be written as

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + g_1(x_1(t))u(t) && (5.19)\\
\dot{x}_2(t) &= x_3(t) + g_2(x_1(t), x_2(t))u(t) && (5.20)\\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t) + g_{n-1}(x_1(t), \ldots, x_{n-1}(t))u(t) && (5.21)\\
\dot{x}_n(t) &= \varphi(x(t)) + g_n(x(t))u(t) && (5.22)\\
y(t) &= x_1(t) && (5.23)
\end{aligned}$$

up to change of coordinates.

Remark 5.6. *If g_1 was a function of $x_1(t)$ and $x_2(t)$*

$$\begin{aligned}
\dot{\xi}_1(t) &= \xi_2(t) + u\left[\frac{\partial g_1}{\partial x_1}(x_1(t), x_2(t))\xi_1(t) + \frac{\partial g_1}{\partial x_2}(x_1(t), x_2(t))\xi_2(t)\right]\\
&= \left[1 + u\frac{\partial g_1}{\partial x_2}(x_1(t), x_2(t))\right]\xi_2(t) + u\frac{\partial g_1}{\partial x_1}(x_1(t), x_2(t))\\
&\Rightarrow u = \frac{-1}{\frac{\partial g_1}{\partial x_2}(x_1(t), x_2(t))}
\end{aligned}$$

This would give us no information on $x_2(t)$ and hence the system is not observable.

Continuing, this is equivalent to a system of the form

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + g(x(t))u(t) + \tilde{\varphi}(x(t)) && (5.24)\\
y(t) &= C_0 x(t) && (5.25)
\end{aligned}$$

where the linear components are given by the matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \quad \text{and} \quad C_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}.$$

and the nonlinear components are of the form

$$g(x(t)) = \begin{pmatrix} g_1(x_1(t)) \\ g_2(x_1(t), x_2(t)) \\ \vdots \\ g_n(x(t)) \end{pmatrix}, \qquad \tilde{\varphi}(x(t)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varphi(x(t)) \end{pmatrix}$$

Here the set

$$M = \left\{ x \in X \mid d_x h(x(t)) = 0,\ d_x L_f h(x(t)) = 0, \dots,\ d_x L_f^{n-1} h(x(t)) = 0 \right\} \tag{5.26}$$

and the change of coordinates

$$\Phi(x) = \begin{pmatrix} h(x(t)) \\ L_f h(x(t)) \\ \vdots \\ L_f^{n-1} h(x(t)) \end{pmatrix} \tag{5.27}$$

are given explicitly.

5.1.2 Uniformly infinitessimally observable systems if $p > m$

Definition 5.7 (k-jet input/k-jet output function)**.**
A function

$$\begin{aligned} \Phi_k^{\Sigma} : X \times U \times \mathbb{R}^{m\cdot(k-1)} &\to \mathbb{R}^{p\cdot k} & (5.28)\\ \big(x_0, u(t), \dot{u}(t), \ddot{u}(t), \dots, u^{(k-1)}(t)\big) &\rightsquigarrow \big(y(t), \dot{y}(t), \ddot{y}(t), \dots, y^{(k-1)}(t)\big) & (5.29) \end{aligned}$$

is called k-jet input/k-jet output function for $k \in \mathbb{N}$.

Remark 5.8. *At time $t = 0$ we got the following:*

$$\begin{aligned} y(t) &= h(x_1(t), u(t)) \\ \dot{y}(t) &= \frac{\partial h}{\partial x}(x_0, u(t)) f(x_0, u(t)) + \frac{\partial h}{\partial u}(x_0, u(t)) \dot{u}(t) \\ &\ \vdots \\ y^{k-1}(t) &= \textit{a function of}\, x_0,\ u(t),\ \dot{u}(t), \dots,\ u^{(k-1)}(t) \end{aligned}$$

Definition 5.9 (Differentially Observable)**.**
The system Σ is differentially observable of order k if the function

$$\begin{aligned} S\Phi_k^\Sigma : X \times U \times \mathbb{R}^{m\cdot(k-1)} &\rightarrow \mathbb{R}^{k\cdot p} \times U \times \mathbb{R}^{(k-1)m} &\quad (5.30)\\ \big(x_0, u(t), \dot{u}(t), \ldots, u^{(k-1)}(t)\big) &\rightsquigarrow \big(y(t), \dot{y}(t), \ldots, y^{(k-1)}(t), u(t), \dot{u}(t), \ldots, u^{(k-1)}(t)\big) &\quad (5.31) \end{aligned}$$

is injective. The system Σ is strongly differentially observable of order k if $S\Phi_k^\Sigma$ is an injective immersion (i.e. an embedding).

Theorem 5.10.
The set of systems Σ such that $S\Phi_k^\Sigma$ is an immersion contains an open dense subset of

$$\Omega = \{(f, h) \mid f, h \in C^\infty\} \tag{5.32}$$

for the topology of uniform convergence for $k \geq 2n$.

Theorem 5.11.
The set of systems Σ which are strongly differentially observable is a residual set of Ω, where a residual set is a countable intersection of open dense sets.

Theorem 5.12.
If X is analytic then the set of analytic systems Σ such that $S\Phi_k^\Sigma$ is an injective immersion is dense in Ω.

Remark 5.13. *These theorems are true for systems of the form*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

as well as for systems

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned}$$

Lemma 5.14.
Let Σ be an analytical system. Then Σ is observable for all L^∞–inputs u if and only if Σ is observable for all C^ω–inputs.

Therefore if a $L^\infty(U)$–input is a “bad” input in the sense that it makes the system unobservable, then there exists an analytic “bad” input.

5.2 Observer Construction

We will consider systems that are in the canonical form of observability for a nonlinear system that is uniformly infinitesimally observable:

$$\begin{aligned}
\dot{x}_1(t) &= f(x_1(t), x_2(t), u(t)) && (5.33)\\
\dot{x}_1(t) &= f(x_1(t), x_2(t), x_3(t), u(t)) && (5.34)\\
&\vdots && (5.35)\\
\dot{x}_1(t) &= f(x(t), u(t)) && (5.36)\\
y(t) &= h(x(t), u(t)) && (5.37)
\end{aligned}$$

Let us suppose that

$$\begin{aligned}
0 < \alpha_1 &\leq \frac{\partial h}{\partial x_1} \leq \beta_1 && (5.38)\\
0 < \alpha_{i+1} &\leq \frac{\partial f}{\partial x_{i+1}} \leq \beta_{i+1} && (5.39)\\
& && (5.40)
\end{aligned}$$

Additionally we assume that h and f_i, $i \in \{1, \ldots, n\}$ are globally Lipschitz in x and uniformly in u.

5.2.1 Luenberger Observer

Chapter 6

Identification

6.1 Theoretical part

We will consider systems of the form

$$\Sigma : \begin{cases} \dot{x}(t) &= f\left(x(t), u(t), \varphi \circ \pi(x(t))\right) \\ y(t) &= h\left(x(t), u(t), \varphi \circ \pi(x(t))\right) \end{cases} \tag{6.1}$$

where

$$\begin{array}{llllll} \varphi \circ \pi : & X & \to & Z & \to & I \subset \mathbb{R} \\ & x & \to & z = \pi(x) & \to & \varphi(\pi(x)) \end{array} \tag{6.2}$$

and

$$\begin{array}{lccccccc} P_\Sigma : & X & \times & \mathcal{L}^\infty[U] & \times & \mathcal{L}^\infty[I] & \to & \mathcal{L}^\infty[\mathbb{R}^{d_y}] \ . \\ & \Big(x_0 & , & u(\cdot) & , & \hat{\varphi}(\cdot)\Big) & \to & y(\cdot) \end{array} \tag{6.3}$$

Within this system φ is an unknown function of $\pi(x)$, $\hat{\varphi}$ is a function of time and P_Σ is the input/output function of Σ. To shorten notation I denotes the unit interval $[0, 1]$.

Definition 6.1 (Identifiability)**.**
A system Σ is called identifiable at

$$\left(\left(u(t)\right)_{0 \le t \le t_u}, \left(y(t)\right)_{0 \le t \le l(u,x_0)}\right) \in \mathcal{L}^\infty[U] \times \mathcal{L}^\infty[\mathbb{R}^p] \tag{6.4}$$

where $e(u, x_0)$ is the minimum of t_u and , if it exists, the smallest possible finite escape time, if there is at most a single pair

$$(x_0, \hat{\varphi}) \in X \times \mathcal{L}^\infty(I) \tag{6.5}$$

such that for almost all times t the input/output function P_Σ satisfies the equality

$$P_\Sigma(x_0, u, \hat{\varphi})(t) = y(t) \tag{6.6}$$

and $\hat{\varphi}(t) = \varphi \circ \pi(x(t))$ for some smooth function $\varphi : Z \to I$.
The system Σ is identifiable if it is identifiable at any admissible pair $(u(.), y(.))$.

Definition 2:

$$T\Sigma : \begin{cases} \frac{d\xi}{dt} & = & T_{x,\varphi} f(x, u, \varphi; \xi, \eta) \\ \hat{y} & = & d_{x,\varphi} h(x, u, \varphi; \xi, \eta) \end{cases}$$

where $(\xi, \eta) \in T_x X \times T_\varphi I$, we set

$$\begin{aligned} P^t_{T\Sigma}(\xi_0, \eta) &= d_{x,\varphi} h(x, u, \hat{\varphi}; T_{x,\varphi}\phi_t(x, u, \hat{\varphi}; \xi_0, \eta), \eta) \\ &= T_{x,\varphi} P^t_\Sigma(\xi_0, \eta) \end{aligned}$$

Σ is infinitesimally identifiable at $(x_{0,}\ u,\ \hat{\varphi}) \in X \times L^\infty[U] \times L^\infty[I]$ if $P^t_{T\Sigma}$ is injective $\forall t > 0$
Σ is uniformly infinitesimally identifiable if this is true at all $(x_{0,}\ u,\ \hat{\varphi})$
Let $D_k\Phi = X \times (U \times \mathbf{R}^{(k-1)d_u}) \times (I \times \mathbf{R}^{k-1})$ be the space of k-jets of the system Σ
$(j^k(u) = \left(u(0), u'(0), \ldots, u^{(k-1)}(0)\right))$, we set

$$\begin{array}{rccc} \Phi^\Sigma_k : & D_k\Phi & \to & \mathbf{R}^{kd_y} \\ & (x_0, j^k(u), j^k(\hat{\varphi})) & \to & j^k(y) \end{array}$$

$$\begin{array}{rccc} \Phi^{\Sigma,*}_{k,2} : & D_k\Phi \times D_k\Phi & \to & \mathbf{R}^{kd_y} \times \mathbf{R}^{kd_y} \\ & (z_1, z_2) & \to & (\Phi^\Sigma_k(z_1), \Phi^\Sigma_k(z_2)) \end{array}$$

Definition 3: Σ is differentially identifiable of order k if
$\Phi^{\Sigma,*}_{k,2}(z_1, z_2) \in \Delta_k \Rightarrow (x_1, \hat{\varphi}_1(0)) = (x_2, \hat{\varphi}_2(0))$
Proposition. Differential Identifiability $\Rightarrow$ Identifiability
Theorem 1.
• If $d_y \geq 3$, differential identifiability of order $2n + 1$ is a **generic property** in the class of C^∞ systems.
• If $d_y < 3$, differential identifiability **is not** a generic property.
$Z_i = \left(x_i, \varphi_i, \varphi'_i, \ldots, \varphi^k_i, j^k_\Sigma(x_i, \varphi_i)\right)$, $i = 1, 2$
$Z = (Z_1, Z_2)$
$\Phi(Z) = \Phi^\Sigma_k(Z_1) - \Phi^\Sigma_k(Z_2) \in R^{k\,d_y}$,
$k = 2n + 1$, $d_y \geq 3$
Let us suppose that Φ is a submersion
$\mathrm{codim}\Phi^{-1}(0) = k\,d_y$
Let $\Pi\Phi^{-1}(0) = \left(x_i, \varphi_i, j^k_\Sigma(x_i, \varphi_i)\right)_{i=1,2}$

$$\begin{aligned} \mathrm{codim}\Pi\Phi^{-1}(0) &\geq k\,d_y - 2(k-1) = k(d_y - 2) + 2 \\ &\geq k + 2 \geq 2n + 3 \end{aligned}$$

$$\begin{array}{rccc} \rho_\Sigma : & (X \times I)^2 \setminus \Delta & \to & \left(J^k_\Sigma\right)^2 \\ & (x_1, \varphi_1, x_2, \varphi_2) & \to & \left(x_i, \varphi_i, j^k_\Sigma(x_i, \varphi_i)\right)_{i=1,2} \end{array}$$

Multijet transversality theorem: the set of Σ such that ρ_Σ is transversal to $\Pi\Phi^{-1}(0)$
is residual.

$$\dim\left(X\times I\right)^2\setminus\Delta=2n+2$$
$$\Downarrow$$
$$\text{generically, } \rho_\Sigma \text{ avoids } \Pi\Phi^{-1}\left(0\right)$$

Theorem 2. If Σ is uniformly infinitesimally identifiable then
i) $\frac{\partial}{\partial\varphi}\left\{h, L_{f_\varphi}h, \ldots, (L_{f_\varphi})^{n-1}h\right\}\equiv 0$
ii) $\frac{\partial}{\partial\varphi}L^n_{f_\varphi}h\neq 0$
iii) $d_xh\wedge\ldots\wedge d_xL^{n-1}_{f_\varphi}h\neq 0$,
Therefore, locally, the system can be written

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & x_2\\ & \vdots & \\ \dot{x}_{n-1} & = & x_n\\ \dot{x}_n & = & \psi(x,\varphi)\\ y & = & x_1\end{array}\right. \quad \text{and } \frac{\partial}{\partial\varphi}\psi(x,\varphi)\neq 0$$

Theorem 3. If Σ meets the following conditions,
i) $\frac{\partial}{\partial\varphi}\left\{h, L_{f_\varphi}h, \ldots, (L_{f_\varphi})^{n-1}h\right\}\equiv 0$
ii) $\frac{\partial}{\partial\varphi}L^n_{f_\varphi}h\neq 0$
iii) $d_xh\wedge\ldots\wedge d_xL^{n-1}_{f_\varphi}h\neq 0$,
then Σ is
1) locally identifiable,
2) loc. unif. infinitesimally identifiable,
3) loc. diff. identifiable of order $n+1$.
Let $k<n$ be the first k such that $d_\varphi L^k_f h\not\equiv 0$:

$$\Sigma\quad\left\{\begin{array}{lcl}y & = & x_1\\ \dot{x}_1 & = & x_2\cdots\\ \dot{x}_{k-1} & = & x_k\\ \dot{x}_k & = & L^k_f\left(x,\varphi\right)=f_k(x,\varphi)\cdots\\ \dot{x}_n & = & f_n(x,\varphi)\end{array}\right.$$

$$T\Sigma\quad\left\{\begin{array}{lcl}\dot{x} & = & f\left(x,\varphi\right)\\ \widehat{y} & = & \xi_1\\ \dot{\xi}_1 & = & \xi_2\cdots\\ \dot{\xi}_{k-1} & = & \xi_k\\ \dot{\xi}_k & = & d_xf_k\left(x,\varphi\right)\xi+d_\varphi f_k\left(x,\varphi\right)\eta\end{array}\right.$$

A feedback $\eta=-\dfrac{d_xf_k\left(x,\varphi_0\right)\xi}{d_\varphi f_k\left(x,\varphi_0\right)}$ in φ_0 s.t. $d_\varphi f_k\left(x,\varphi_0\right)\neq 0$ gives $\frac{d\xi_k}{dt}=0$ which contradict observability.

If $\frac{\partial}{\partial \varphi} L^n_{f_\varphi} h = 0$ at (x, φ)

$$\begin{array}{ccl} X \times I & \supset & E = \left\{ (x,\varphi)\,;\, d_\varphi L^n_f h = 0 \right\} \\ & & \downarrow \Pi \\ X & \supset & \Pi E \end{array}$$

Hardt's theorem $\Rightarrow \exists \widehat{\varphi}$

$$\left\{ \begin{array}{llllllll} y & = & x_1, & \dot{x}_1 & = & x_2, & \ldots & \dot{x}_n = \psi\left(x, \widehat{\varphi}\left(x\right)\right) \\ \widehat{y} & = & \xi_1, & \dot{\xi}_1 & = & \xi_2, & \ldots & \dot{\xi}_n = d_x \psi\left(x, \widehat{\varphi}\left(x\right)\right) + 0 \end{array} \right.$$

Define $E_l = \left\{ d_x h_i, d_x L_{f_\varphi} h_i, \ldots, d_x L^{l-1}_{f_\varphi} h_i \,,\, i = 1, 2 \right\}$ and $N\left(l\right) = \operatorname{rank}\left(E_l\right)$ at a generic point: k is defined by

$N(0)$	$N(1)$	$\cdots$	$N(k-1)$	$N(k)$	$N(k+1)$	$\cdots$	$N(k+m)$
0	2		$2k-2$	$2k$	$2k+1$		$2k+m$

$(2k + m \leq n)$

The **order** of the system is the first integer r such that $d_\varphi L^r_{f_\varphi}\left(h_1, h_2\right) \not\equiv 0$.

Lemma: If Σ is uniformly infinitesimally identifiable then (1) $2k + m = n$
(2) $r \leq k + m$

Proof:

(1) $\varphi = \varphi_0 =$cte $\left\{ \begin{array}{lcl} \dot{x} & = & f\left(x, \varphi_0\right) \\ \dot{\xi} & = & g\left(x, \xi, \varphi_0\right) \\ y & = & h\left(x, \varphi_0\right) \end{array} \right.$ contradict observability

(2) $\left\{ \begin{array}{lcl} \dot{x} & = & f\left(x\right) \\ y & = & h\left(x\right) \end{array} \right.$ contradict identifiability

Définition 5. A system Σ is regular if (1) and (2) holds.

$$\left\{ \begin{array}{lcllcl} y_1 & = & x_1 & y_2 & = & x_2 \\ \dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\ & \vdots & & & \vdots & \\ \dot{x}_{n-3} & = & x_{n-1} & \dot{x}_{n-2} & = & x_n \\ \dot{x}_{n-1} & = & f_{n-1}(x,\varphi) & \dot{x}_n & = & f_n(x,\varphi) \end{array} \right.$$

with $\frac{\partial}{\partial \varphi}(f_{n-1}, f_n) \neq 0$

$N\left(l\right)$ increases by steps of 2 until the last derivative and apparition of φ.

$$\begin{cases} y_1 & = & x_1 & y_2 & = & x_2 \\ \dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\ & \vdots & & & \vdots & \\ \dot{x}_{2k-3} & = & x_{2k-1} & \dot{x}_{2k-2} & = & x_{2k} \\ \dot{x}_{2k-1} & = & \multicolumn{4}{l}{f_{2k-1}(x_1, ..., x_{2k+1})} \\ \dot{x}_{2k} & = & x_{2k+1} & & & \\ & \vdots & & & & \\ \dot{x}_{n-1} & = & x_n & & & \\ \dot{x}_n & = & f_n(x, \varphi) & & & \end{cases}$$

with $\frac{\partial f_n}{\partial \varphi} \neq 0$.
$N\left(l\right)$ increases by steps of 1 when φ appears for the first time, $\simeq$ single-output case.

$$\begin{array}{llllll} y_1 & = & x_1 & y_2 & = & x_2 \\ \dot{x}_1 & = & x_3 & \dot{x}_2 & = & x_4 \\ & \vdots & & & \vdots & \\ \dot{x}_{2r-3} & = & x_{2r-1} & \dot{x}_{2r-2} & = & x_{2r} \\ \dot{x}_{2r-1} & = & \psi(x,\varphi) & \dot{x}_{2r} & = & F_{2r}(x_1, \ldots, x_{2r+1}, \psi(x,\varphi)) \\ & & & \dot{x}_{2r+1} & = & F_{2r+1}(x_1, \ldots, x_{2r+2}, \psi(x,\varphi)) \\ & & & & \vdots & \\ & & & \dot{x}_{n-1} & = & F_{n-1}(x, \psi(x,\varphi)) \\ & & & \dot{x}_n & = & F_n(x,\varphi) \end{array}$$

with $\frac{\partial \psi}{\partial \varphi} \neq 0, \frac{\partial F_{2r}}{\partial x_{2r+1}} \neq 0, ..., \frac{\partial F_{n-1}}{\partial x_n} \neq 0$
φ appears when $N\left(l\right)$ increases by steps of 2.
If $r = k$ before the last derivative:
$d_x h_1 \wedge \cdots \wedge d_x L_{f_\varphi}^{k-1} h_1 \wedge d_x L_{f_\varphi}^{k-1} h_2 \wedge d_x L_{f_\varphi}^{k} h_2 \not\equiv 0$
If $d_\varphi L_{f_\varphi}^k h_1 \neq 0$, we obtain φ using y_1 and x_{2k},. . . ,x_n using y_2
If $d_\varphi L_{f_\varphi}^k h_1 \equiv 0$, we obtain φ using y_2

$$\begin{cases} \dot{x}_1 = F_1\left(x_1, x_2, u\right) & \frac{\partial F_1}{\partial x_2} \neq 0 \\ \dot{x}_2 = F_2\left(x_1, x_2, x_3, u\right) & \frac{\partial F_2}{\partial x_3} \neq 0 \\ \quad \vdots & \\ \dot{x}_n = F_n\left(x, u\right) & \end{cases}$$

$\xi_1 = y = x_1$, $\xi_2 = F_1(x_1, x_2, u)$
$\xi_3 = \frac{\partial F_1}{\partial x_2} F_2\left(x_1, x_2, u\right), \cdots$
$\xi_{i+1} = \frac{\partial F_1}{\partial x_2} \cdots \frac{\partial F_{i-1}}{\partial x_i} F_i\left(x_1, \ldots, x_{i+1}, u\right)$

$$\begin{cases} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 + \frac{\partial F_1}{\partial x_1}\dot{x}_1 + \frac{\partial F_1}{\partial u}\dot{u} \\ & \vdots \\ \dot{\xi}_n &= G\left(x, u, \dot{u}\right) \end{cases}$$

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$$\begin{cases} \dfrac{dx}{dt} &= A(t)x + b(x,u) \\ y &= C(t)x \end{cases}$$

$$A(t) = \begin{pmatrix} 0 & a_2\left(t\right) & 0 & \cdots & 0 \\ & & a_3\left(t\right) & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & a_n\left(t\right) \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$C(t) = \begin{pmatrix} a_1(t) & 0 & \cdots & 0 \end{pmatrix}$$

$$0 < a_m \leq a_i(t) \leq a_M$$

$$b\left(x,u\right) = b_1(x_1,u)\frac{\partial}{\partial x_1} + b_2(x_1,x_2,u)\frac{\partial}{\partial x_2} + b_n(x_1,...,x_n,u)\frac{\partial}{\partial x_n}$$

$$\begin{array}{rcl} \frac{dz}{dt} &=& A(t)z + b(z,u) - S(t)^{-1}C\left(t\right)' r^{-1}(C\left(t\right)z - y(t)) \\ \frac{dS}{dt} &=& -(A(t) + b^*(z,u))'S - S(A(t) + b^*(z,u)) \\ && +C\left(t\right)' r^{-1} C\left(t\right) - SQ_\theta S \end{array}$$

$\frac{d\theta}{dt} = \lambda(1-\theta)$

$$\Delta = \begin{pmatrix} 1 & & & \\ & \frac{1}{\theta} & & \\ & & \ddots & \\ & & & (\frac{1}{\theta})^{n-1} \end{pmatrix}$$

$$Q_\theta = \theta^2 \Delta^{-1} Q \Delta^{-1}$$

If θ is large, high-gain observer (HGEKF)
If $\theta \approx 1$, Classical Extended Kalman filter (EKF)
There exist $\lambda_0 > 0$ such that for any $0 \leq \lambda \leq \lambda_0$, there exist θ_0 such that for any $\theta\left(0\right) > \theta_0$, for any $S\left(0\right) \geq c\, Id$, for any compact $K \subset \mathbf{R}^n$, for any $z\left(0\right) \in K$ then if we set $\varepsilon\left(t\right) = z\left(t\right) - x\left(t\right)$ for any $t \geq 0$

$$||\varepsilon(t)||^2 \leq R(\lambda, c)e^{-at}\Lambda(\theta\left(0\right), t, \lambda)||\varepsilon\left(0\right)||^2$$

where

$$\Lambda(\theta(0), t, \lambda) = \theta(0)^{2(n-1)+\frac{a}{\lambda}} e^{-\frac{a}{\lambda}\theta(0)(1-e^{-\lambda t})}$$

and a is a positive constant and $R(\lambda, c)$ is a decreasing function of c.

Change of variables $\begin{cases} \widetilde{x} & = \Delta x \\ \widetilde{P} & = \frac{1}{\theta}\Delta P \Delta \qquad (P = S^{-1}) \end{cases}$

+ time change $d\tau = \theta(t)\, dt$

We set $\varepsilon = z - x = error$ then we calculate ${}^{T}\varepsilon(\tau)\, S(\tau)\, \varepsilon(\tau)$.

Observability give us $\alpha I \leq S(\tau) \leq \beta I$ then

$${}^{T}\varepsilon(\tau)\, S(\tau)\, \varepsilon(\tau) \longrightarrow 0 \Longleftrightarrow \varepsilon(\tau) \longrightarrow 0$$

When $\tau \leq T$

$$\|\varepsilon(\tau)\|^2 \leq \theta(\tau)^{2(n-1)} H(c)\, e^{-(a_1\theta(T)-a_2)\tau} \|\varepsilon(0)\|^2$$

We use N observers in parallel. At times kT :

• a new observer is initialized with $\theta(kT) = \theta_0$,

• the older observer is killed.

Therefore, at any time t, we have N observers initialized at times kT, $(k-1)\,T \ldots (k-N+1)\,T$ where $k = \left\lfloor \frac{t}{T} \right\rfloor$.

State estimation: the estimation given by the observer with smallest innovation $\|y - C\widehat{x}\|$.

6.2 Biological reactor

$$\begin{cases} \dfrac{ds(t)}{dt} & = & -\mu(s(t))\, x(t) + D(t)(S_{in} - s(t)) \\ \dfrac{dx(t)}{dt} & = & (\mu(s(t)) - D(t))x(t) \end{cases}$$

$s(t)$: substrates
$x(t)$: biomass
$D(t)$: influent flow rate
S_{in} : substrate concentration in the influent

$\mu(s) > 0$, $\mu(0) = 0$ specific growth rate,

$$\begin{array}{ll} \text{Monod} & \mu(s) = \frac{\mu_0 s}{k_m + s} or \\ \text{Haldane} & \mu(s) = \frac{\mu_0 s}{k_m + s + \frac{s^2}{k_i}} \qquad \cdots \end{array}$$

Only $s(t)$ is measured

$$\frac{ds(t)}{dt} = -\mu(s(t))\, x(t) + D(S_{in} - s(t))$$

The system is observable.
Only $x(t)$ is measured

$$\begin{aligned}\frac{dx(t)}{dt} &= (\mu(s(t)) - D(t))x(t) \\ \frac{d\mu(s(t))}{dt} &= \mu'(s(t))\left(-\mu(s(t))\,x(t) + D(t)\,(S_{in} - s(t))\right)\end{aligned}$$

The system is observable (depending on μ).
Same questions but $\mu(s)$ is unknown

$$\begin{cases} \dfrac{ds(t)}{dt} &= -\mu(s(t))\,x(t) + D(S_{in} - s(t)) \\ \dfrac{dx(t)}{dt} &= (\mu(s(t)) - D)x(t) \end{cases}$$

Only $x(t)$ is measured,
can we reconstruct $\mu(s)$ and $s(t)$? no
Only $s(t)$ is measured,
can we reconstruct $\mu(s)$ and $x(t)$? yes if...
Both $x(t)$ and $s(t)$ are measured,
can we reconstruct $\mu(s)$? yes

$$\begin{cases} \dfrac{dx(t)}{dt} &= (\mu(s) - D)x \\ \dfrac{ds(t)}{dt} &= -\mu(s)\;x + D(S_{in} - s) \\ y &= x \end{cases}$$

$$\begin{aligned} s(t) &= e^{-Dt}s(0) + \int_0^t e^{-D(t-\tau)}(-(\mu x)(\tau) + DS_{in})d\tau \\ s(0) &= s_0 \end{aligned}$$

$$\begin{aligned} \tilde{s}(t) &= e^{-Dt}\tilde{s}_0 + \int_0^t e^{-D(t-\tau)}(-(\mu x)(\tau) + DS_{in})d\tau \\ \tilde{s}(0) &= \tilde{s}_0 \approx s_0 \end{aligned}$$

$\widetilde{\mu}(\widetilde{s}) = \mu(s) \Rightarrow \dfrac{dx(t)}{dt} = (\widetilde{\mu}(\widetilde{s}(t)) - D)x(t)$

Let us denote $z(t) = \mu(s(t))\,x(t)$, and assume that $\dfrac{d^k z}{dt^k} = 0$

$$\left\{\begin{array}{rcl} \dot{s} & = & -z + D(t)(S_{in} - s) \\ \dot{x} & = & z - D(t)\ x \\ \dot{z} & = & z_1 \cdots \\ \dot{z}_{k-2} & = & z_{k-1} \\ \dot{z}_{k-1} & = & 0 \\ y & = & (s, x) \end{array}\right.$$

where $\frac{ds}{dt} = \dot{s}$

Linear (optimal) Kalman observer

$$\left\{\begin{array}{rcl} X & = & x + s \\ \widetilde{D}\,(t) & = & \int_0^t D\,(\tau)\,d\tau \\ \Lambda\,(t) & = & e^{\widetilde{D}(t)}(s - S_{in}) + S_{in} \end{array}\right.$$

$$\begin{aligned} \dot{\Lambda} &= -e^{\tilde{D}(t)}(X - s)\mu\,(s) \\ &= (\Lambda - X_0)\mu\,(s) \\ \text{with } \Lambda\,(0) &= s\,(0) \end{aligned}$$

If $s\,(t_0) = s\,(t_1)$, $t_0 < t_1$ then

$$\frac{\dot{\Lambda}(t_0)}{\Lambda(t_0) - X_0} = \mu(s(t_0)) = \mu(s(t_1)) = \frac{\dot{\Lambda}(t_1)}{\Lambda(t_1) - X_0}$$

gives X_0 hence $\mu(s(t)) = \frac{\dot{\Lambda}(t)}{\Lambda(t) - X_0}$

$\mu\,(s)$ is identifiable $\Longleftrightarrow$ $s(t)$ visits twice the same value

1. μ is identified at sample values $k\Delta s$, at time t, giving $\widehat{\mu}_t\,(h\Delta s)$;

2. $x\,(t)$ is estimated using a linear Kalman filter and $\widehat{\mu}_t\,(h\Delta s)$

Simulation:

$\mu\,(s)$ is the Haldane law, $\widehat{\mu}_0\,(s)$ is the Monod law,

6.3 FCC process

Reactor model

Temperature in the reactor:

$$
\begin{aligned}
S_c H_{ra} \dot{T}_{ra} &= S_c R_c \left(T_{rg} - T_{ra}\right) + S_{tf} R_{tf} \left(T_{tf} - T_{ra}\right) \\
&\quad - \Delta H_{fv} R_{tf} - \Delta H_{cr} R_{tf} C_{tf} \\
C_{tf} &= \frac{R_{cr}}{R_{cr} + R_{tf}} \\
R_{cr} &= \frac{\sqrt{k_{cr} R_c P_{ra} H_{ra}}}{10\, C_{rc}^{0.12}} \exp\left(-\frac{1}{2} \frac{A_{cr}}{R T_{ra}}\right),
\end{aligned}
$$

with:

- Reactor operating conditions $T_{ra}|_{t=0} = 775\,\mathrm{K}$, $H_{ra} = 1.85\,10^{-4}\,\mathrm{kg}$, $P_{ra} = 211.7\,\mathrm{kPa}$,
- Feed properties $R_{tf} = 41\,\mathrm{kg}\,/\,\mathrm{s}$, $T_{tf} = 492.8\,\mathrm{K}$, $S_{tf} = 3140\,\mathrm{J}\,/\,(\mathrm{kg}\,.\,\mathrm{K})$,
- Catalyst recirculation $R_c|_{t=0} = 290\,\mathrm{kg}\,/\,\mathrm{s}$, $0 < R_c^{\min} \leq R_c \leq R_c^{\max}$, $S_c = 1047\,\mathrm{J}\,/\,(\mathrm{kg}\,.\,\mathrm{K})$,
- Heat constants $\Delta H_{cr} = 4.65\,10^5\,\mathrm{J}\,/\,\mathrm{kg}$, $\Delta H_{fv} = 1.74\,10^5\,\mathrm{J}\,/\,\mathrm{kg}$, $\Delta H_{rg} = 3.02\,10^7\,\mathrm{J}\,/\,\mathrm{kg}$,
- $k_{cr} = 25.96\ \mathrm{kPa}^{-1}\,\mathrm{s}^{-1}$, $A_{cr} = 83.8\,10^3\,\mathrm{J}\,/\,\mathrm{mol}$
- $R = 8.314\,\mathrm{J}\,/\,(\mathrm{mol}\,.\,\mathrm{K})$

Carbon concentration on spent catalyst in the reactor:

$$
\begin{aligned}
H_{ra} \dot{C}_{sc} &= R_c \left(C_{rc} - C_{sc}\right) + 100\, R_{cf} \\
R_{cf} &= R_{cc} + R_{ad} \\
R_{cc} &= \frac{\sqrt{k_{cc} R_c P_{ra} H_{ra}}}{10\, C_{rc}^{0.03}} \exp\left(-\frac{1}{2} \frac{A_{cc}}{R T_{ra}}\right) \\
R_{ad} &= F_{cf} R_{tf}
\end{aligned}
$$

with:

- $C_{sc}|_{t=0} = 1.2$
- $k_{cc} = 2.66\,10^{-4}\,\mathrm{kPa}^{-1}\,\mathrm{s}^{-1}$, $A_{cc} = 4.18\,10^4\,\mathrm{J}\,/\,\mathrm{mol}$

Regenerator model

Temperature in the regenerator:

$$
\begin{aligned}
S_c H_{rg} \dot{T}_{rg} &= S_c R_c \left(T_{ra} - T_{rg}\right) + S_{ai} R_{ai} \left(T_{ai} - T_{rg}\right) + \Delta H_{rg} R_{cb} \\
R_{cb} &= \frac{R_{ai}}{242} \left(21 - O_{fg}\right) \\
O_{fg} &= 21 \exp\left(\frac{-\frac{P_{rg} H_{rg}}{R_{ai}}}{\frac{1}{K_{od}} + \frac{1}{K_{or} C_{rc}}}\right) \\
K_{od} &= 6.34\,10^{-9} R_{ai}^2 \\
K_{or} &= 1.16\,10^{-5} \exp\left(\frac{A_{or}}{R\left(\frac{1}{866.7} - \frac{1}{T_{rg}}\right)}\right)
\end{aligned}
$$

with:

- Regenerator operating conditions $T_{rg}|_{t=0} = 943\,\mathrm{K}$, $H_{rg} = 1.53\,10^5\,\mathrm{kg}$, $P_{rg} = 254.4\,\mathrm{kPa}$,
- Air properties $R_{ai}|_{t=0} = 26\,\mathrm{kg}\,/\,\mathrm{s}$, $0 < R_{ai}^{\min} \leq R_{ai} \leq R_{ai}^{\max}$, $T_{ai} = 394\,\mathrm{K}$, $S_{ai} = 1130\,\mathrm{J}\,/\,(\mathrm{kg}\,.\,\mathrm{K})$
- $A_{or} = 1.47\,10^5\,\mathrm{J}\,/\,\mathrm{mol}$

Carbon concentration on regenerated catalyst in the regenerator:

$$
H_{rg} \dot{C}_{rc} = R_c \left(C_{sc} - C_{rc}\right) - 100\, R_{cb}
$$

with $C_{rc}|_{t=0} = 0.3$

$$
\begin{aligned}
\varphi\left(x; u\right) &= \varphi\left(T_{rg}, T_{ra}, C_{rc}, C_{sc}, F_{cf}; R_{ai}\right) \\
&= \left(T_{rg}, T_{ra}, C_{tf}\left(C_{rc}, T_{ra}\right), \tfrac{C_{sc}}{C_{rc}}, \tfrac{F_{cf}}{C_{rc}}\right) = \xi
\end{aligned}
$$

$$
\left\{
\begin{array}{llcl}
\dot{x}_1 & = \dot{T}_{rg} & = & \psi\left(x, \varphi\left(x_1\right), u\right) \\
\dot{x}_2 & = \dot{T}_{ra} & = & a_3(t) x_3 + f_2\left(x_1, x_2\right) \\
\dot{x}_3 & \simeq \dot{C}_{rc} & = & a_4(t) x_4 \\
 & & & + f_3\left(x_1, x_2, x_3, \psi\left(x, \varphi\left(x_1\right), u\right), u, \dot{u}\right) \\
\dot{x}_4 & \simeq \dot{C}_{sc} & = & a_5(t) x_5 + f_4\left(x_1, x_2, x_3, x_4\right) \\
\dot{x}_5 & \simeq \dot{F}_{cf} & = & F\left(x\right)
\end{array}
\right.
$$

Here, $\psi = R_{cb}$, $\varphi = K_{or}$ and $\pi\left(x\right) = T_{rg} = x_1$. $u = \left(R_{ai}, P_{ra}\right)$

$$
\frac{dx}{dt} = F\left(x, u\right)
$$

Our diffeomorphism $\xi = \varphi(x,u)$ depend on u supposed to be smooth, hence:

$$\begin{aligned} \tfrac{d\xi}{dt} &= D_\varphi\left(\varphi_u^{-1}(\xi)\right) f\left(\varphi_u^{-1}(\xi), u\right) + \tfrac{\partial\varphi\left(\varphi_u^{-1}(\xi),u\right)}{\partial u}\dot{u} \\ &= F(\xi, u, \dot{u}) \end{aligned}$$

$$\left\{ \begin{aligned} \tfrac{d\hat{\xi}}{dt} &= F\left(\hat{\xi}, u, \dot{u}\right) + PC^T R^{-1}\left(y - C\hat{\xi}\right) \\ \tfrac{dP}{dt} &= F^*\left(\hat{\xi}, u, \dot{u}\right) P + PF^*\left(\hat{\xi}, u, \dot{u}\right) + Q_\theta - PC^T R^{-1} CP \end{aligned} \right.$$

Since $C\varphi_u(x) = Cx$, equations are those of a modified extended Kalman filter

$$\left\{ \begin{aligned} \tfrac{d\hat{x}}{dt} &= f(\hat{x}, u) + pC^T R^{-1}(y - C\hat{x}) \\ \tfrac{dp}{dt} &= f^*(\hat{x}, u)\, p + pf^*(\hat{x}, u)^T + q_\theta(\hat{x}) \\ &-ph^*(\hat{x}, u)^T R^{-1} h^*(\hat{x}, u)\, p \\ &+D_{\psi_u}^{-1}(\hat{x})\, D_{\psi_u}^2 \cdot \left(ph^*(\hat{x}, u)^T R^{-1}(h(\hat{x}, u) - y)\right) p \\ &+pD_{\psi_u}^2 \cdot \left(ph^*(\hat{x}, u)^T R^{-1}(h(\hat{x}, u) - y)\right) D_{\psi_u}^{-1}(\hat{x})^T \end{aligned} \right.$$

where $q_\theta(\hat{x}) = D_{\varphi_u}(\hat{x})^{-1} Q_\theta \left(D_{\varphi_u}(\hat{x})^{-1}\right)^T$
The two last lines (transposed) correspond to the change of coordinate.
We use a second order system to estimate K_{or} i.e. $\dfrac{d^3 K_{or}}{dt^3} = 0$
We use three parallel extended Kalman filters such that
• $\theta_0 = 3$ (starting value for each observers)
• $\theta_{HG} = 2$ (minimal value of θ ensuring high-gain)
• Time between two consecutive initializations: 2 hours
At last,

$$\xi = \left(T_{rg}, R_{cb}, \dot{K}_{or}, \ddot{K}_{or}, T_{ra}, C_{tf}, \frac{C_{sc}}{C_{rc}}, \frac{F_{cf}}{C_{rc}}\right)$$

and

$$\Delta^{-1} = \operatorname{diag}\left(1, \theta, \theta^2, \theta^3, 1, \theta, \theta^2, \theta^3\right)$$

with $Q_\theta = \theta^2 \Delta^{-1} Q \Delta^{-1}$
and $R_\theta = (C\Delta^{-1}C')\, R\, (C\Delta^{-1}C')$

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