Institut für Intermodale Transport- und Logistiksysteme

Technische Universität Braunschweig

Control Engineering 2 (Regelungstechnik 2)

Lecture Notes

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FOREWORD

During summer term 2024 I give the lecture to the module *Control Engineering 2 (Regelungstechnik 2)* at the Technical University of Braunschweig. To structure the lecture and support my students in their learning process, I prepared these lecture notes. The notes are a living document, which I adapt to the needs and requirements of my students. Therefore, I will integrate remarks and corrections throughout the summer term.

The aim of the module is to provide participating students with knowledge of terms of system theory and control engineering. Moreover, students shall be enabled to understand complex control structures, apply control schemes and analyze control systems. After successfully completing the module, students shall additionally be able to apply the discussed methods within real life applications and be able to assess results.

To this end, the module will tackle the subject areas

- System theory and Modeling,
- Methods and Algorithms, and
- Stability and Control Design

for complex and networked linear as well as nonlinear systems. In particular, we discuss the topics

- Frequency domain
  - Modeling of complex control loops
  - Bang-bang and double-setpoint control
  - Multi-input multi-output systems
- Time domain

- Nonlinear control systems
- Backstepping and Sontag's formula
- Digital control systems

within the lecture and support understanding and application within the tutorial and laboratory classes. The module itself is accredited with 5 credits with an add-on of 2 credits if the requirements of the laboratory classes are met.
During the preparation of the lecture, I utilized the books of Jan Lunze [12–14].

Literature for further reading

- Linear part
  - FÖLLINGER, O.: *Regelungstechnik*. 13. überarbeitete Auflage. VDE-Verlag, 2022
  - LUNZE, J.: *Regelungstechnik 1: Systemtheoretische Grundlagen, Analyse und Entwurf einschleifiger Regelungen*. 11. überarbeitete und ergänzte Auflage. Springer, 2016
  - LUNZE, J.: *Automatisierungstechnik*. 5. Auflage. DeGruyter, 2020
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- Nonlinear part
  - ISIDORI, A.: *Nonlinear Control Systems*. 3rd edition. Springer, 1995
  - KHALIL, H.K.: *Nonlinear Systems*. Prentice Hall PTR, 2002. – 750 S. – ISBN 0130673897
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CHAPTER 1
SYSTEM AND MODEL OF A SYSTEM

While on the one hand we want to understand the fundamental limitations that mathematics imposes on what is achievable, irrespective of the precise technology being used, it is also true that technology may well influence the type of question to be asked and the choice of mathematical model.

E.D. Sontag [18]

In Control Engineering 1, basic control structures formed the heart of the lecture. Common ways of describing systems in both frequency and time domain were defined to understand the foundation of control. Moreover, control methods were applied and analyzed.
Within the lecture series Control Engineering 2, we study control structures, which are more complex in both the description and well as in there application and analysis. To have a common basis, this chapter provides terminology and properties of control systems for both frequency and time domain.
For more historic insights, we additionally refer to the books of Cellier [3], Director and Rohrer [6], Ludyk [10], Luenberger [11], Padulo and Arbib [16] and Shearer and Kulakowski [17].

1.1. System

The term of a *system* is used in various different scientific and non-scientific areas. Its meaning, however, is often not defined clearly. Simple formulated, *a system is the connection of different interacting components to realize given tasks*. The interdependence of systems with their environment is given by so called *input and output variables*, cf. Figure 1.1.
More formally, we define the following:

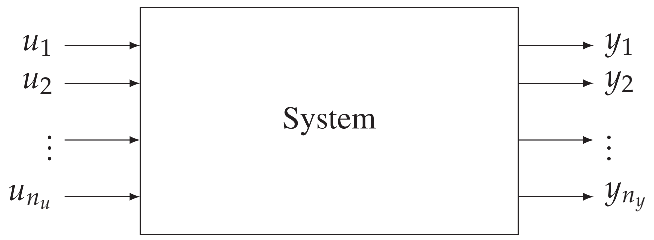


Figure 1.1.: Term of a system

Definition 1.1 (System).
Consider two sets $\mathcal{U}$ and $\mathcal{Y}$. Then a map $f : \mathcal{U} \to \mathcal{Y}$ is called a system.

The inputs $u_1, \ldots, u_{n_u} \in \mathcal{U}$ are variables, which act from the environment to the system and are not dependent on the system itself or its properties. We distinguish between inputs, which are used to specifically manipulate (or control) the system, and inputs, which are not manipulated on purpose. We call the first ones *control or manipulation inputs*, and we refer to the second ones as *disturbance inputs*. The outputs $y_1, \ldots, y_{n_y} \in \mathcal{Y}$ are variables, which are generated by the system and influence the environment. Here, we distinguish output variables depending on whether we measure them or not. We call the measured ones *measurement outputs*.

Remark 1.2
Note that in most cases not all measurable outputs are actually measured. Similarly, in many cases not all manipulable inputs are controlled.

To continue with a system, we need to introduce the concept of time:

Definition 1.3 (Time).
A *time set* $\mathcal{T}$ is a subgroup of $(\mathbb{R}, +)$.

Time allows us to let a system evolve. To see this point, we consider two electical systems illustrated in Figure 1.2 which represent an ideal resistance and an ideal capacitor.
The systems in Figure 1.2 possess the input variable $I(t)$, the output variable $U(t)$ and time t. For the resistance R the output is uniquely defined by the input for every time instant t, i.e. we have

$$y(t) = U(t) = R \cdot I(t) = R \cdot u(t). \tag{1.1}$$

If the outputs depend on the input at the same time instant, we call systems such as this one *static*.

Figure 1.2.: Example of static and dynamic systems

In contrast to this, the computation of the voltage $U(t)$ at the capacitor C at time instant t depends on the entire history $I(\tau)$ for $\tau \leq t$, i.e. we have

$$y(t) = U(t) = \frac{1}{C} \int_{-\infty}^{t} I(\tau) d\tau = \frac{1}{C} \int_{-\infty}^{t} u(\tau) d\tau.$$

If we additionally know the voltage $U(t_0)$ at a time instant $t_0 \leq t$, then only the history $t_0 \leq \tau \leq t$ of the current is required, i.e.

$$y(t) = U(t) = \frac{1}{C} \int_{-\infty}^{t} I(\tau) d\tau = \underbrace{\frac{1}{C} \int_{-\infty}^{t_0} I(\tau) d\tau}_{U(t_0)} + \frac{1}{C} \int_{t_0}^{t} I(\tau) d\tau = U(t_0) + \frac{1}{C} \int_{t_0}^{t} u(\tau) d\tau. \quad (1.2)$$

As we can see from (1.1), the initial value $U(t_0)$ contains all the information on the history $\tau \leq t_0$. For this reason, one typically refers to $U(t_0)$ as the internal *state* of the system capacitor at time instant t_0. If the output of the system depends not only on the input at the time instant but also on the history of the latter, we call these systems *dynamic*.

Remark 1.4
Note that by this definition the set of dynamic systems covers the set of static systems.

If for a system according to Figure 1.1 the outputs $y_1(t), \ldots, y_{n_y}(t)$ depend on the history of the inputs $u_1(\tau), \ldots, u_{n_u}(\tau)$ for $\tau \leq t$ only, then the system is called *causal*. As all technically feasible systems are causal, we will restrict ourselves to this case.

Now, our discussion so far allow us to give the general definition of states of a dynamical system:

Definition 1.5 (State).
Consider a system $f : \mathcal{U} \to \mathcal{Y}$. If the output $\mathbf{y}(t)$ uniquely depends on the history of inputs $\mathbf{u}(\tau)$ for $t_0 \leq \tau \leq t$ and some $\mathbf{x}(t_0)$, then the variable $\mathbf{x}(t)$ is called state of the system.

Task 1.6
Which variable represents a state in case of induction?

Solution to Task 1.6: Current through the inductor

1.2. Continuous time systems

If a dynamical system can be described by a finite number n_x of states, then it is called *system with finite state of order* n_x or concentrated-parametric systems. Such systems with finite state are described by mathematical models featuring differential and algebraic equations. Within this lecture, we restrict ourselves to this class. These systems can be described explicitly via

$$\left.\begin{aligned} \frac{d}{dt}x_1(t) &= f_1(x_1(t),\dots,x_{n_x}(t),u_1(t),\dots,u_{n_u}(t),t) \\ &\vdots \\ \frac{d}{dt}x_{n_x}(t) &= f_{n_x}(x_1(t),\dots,x_{n_x}(t),u_1(t),\dots,u_{n_u}(t),t) \end{aligned}\right\}\text{Differential equations} \tag{1.3a}$$

$$\left.\begin{aligned} x_1(t_0) &= x_{1,0} \\ &\vdots \\ x_{n_x}(t_0) &= x_{n_x,0} \end{aligned}\right\}\text{Initial conditions} \tag{1.3b}$$

$$\left.\begin{aligned} y_1(t) &= h_1(x_1(t),\dots,x_{n_x}(t),u_1(t),\dots,u_{n_u}(t),t) \\ &\vdots \\ y_{n_y}(t) &= h_{n_y}(x_1(t),\dots,x_{n_x}(t),u_1(t),\dots,u_{n_u}(t),t) \end{aligned}\right\}\text{Output equations} \tag{1.3c}$$

We combine the input, output and state variables to (column) vectors

$$\mathbf{u} = [u_1\ u_2\ \dots\ u_{n_u}]^\top \tag{1.4a}$$

$$\mathbf{y} = \left[y_1\ y_2\ \dots\ y_{n_y}\right]^\top \tag{1.4b}$$

$$\mathbf{x} = [x_1\ x_2\ \dots\ x_{n_x}]^\top. \tag{1.4c}$$

Using additionally the short form $\dot{\mathbf{x}}$ for $\frac{d}{dt}\mathbf{x}$ we obtain the compact vector notation

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t),\mathbf{u}(t),t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.5a}$$

$$\mathbf{y}(t) = h(\mathbf{x}(t),\mathbf{u}(t),t). \tag{1.5b}$$

The variables $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{x}$ are called *input*, *output* and *state* of the dynamical system.
If the state $\mathbf{x}$ represents an element of an n_x–dimensional vector space $\mathcal{X}$, then $\mathcal{X}$ is called *state space*. The state of a system at time instant t can then be depicted as a point in the n_x–dimensional state space. The curve of points for variable time t in the state space is called *trajectory* and is denoted by $\mathbf{x}(\cdot)$.

Remark 1.7
Systems with infinite dimensional states are called distributed parametric systems and are described, e.g., via partial differential equations. Examples of such systems are beams, boards, membranes, electromagnetic fields, heat etc..

1.3. Discrete time systems

The setting of continuous time systems as we discussed them so far can be widened to *discrete time systems*. In contrast to continuous time systems where we have $t \in \mathbb{R}$, for discrete time systems time refers to an index $k \in \mathbb{Z}$. Here, the states are denoted by $\mathbf{x}(k)$ and represent a sequence of points in the state space $\mathcal{X}$, which is not a curve. Discrete time systems may arise by *sampling* continuous time systems, e.g. via a A/D and D/A converter. For equidistant sampling with sampling time T, we obtain

$$\mathbb{T} := \{t_k \mid t_k := t_0 + k \cdot T\} \subset \mathbb{R}.$$

where t_0 is some fixed initial time stamp. Apart from equidistant sampling, other types such as event based or sequence based are possible. The equidistant case, however, is important in digital control, which we consider later in the lecture.

Remark 1.8
Note that the class of discrete time systems is larger and contains the class of continuous time systems, i.e. for each continuous time system there exists a discrete time equivalent, but for some discrete time systems no continuous time equivalent exists.

To mathematically describe discrete time systems so called difference equations and algebraic equations are used. Similar to (1.3) we write

$$\left.\begin{aligned} x_1(k+1) &= f_1(x_1(k), \ldots, x_{n_x}(k), u_1(k), \ldots, u_{n_u}(k), k) \\ &\vdots \\ x_{n_x}(k+1) &= f_{n_x}(x_1(k), \ldots, x_{n_x}(k), u_1(k), \ldots, u_{n_u}(k), k) \end{aligned}\right\} \text{Difference equations} \qquad (1.6a)$$

$$\left.\begin{aligned} x_1(0) &= x_{1,0} \\ &\vdots \\ x_{n_x}(0) &= x_{n_x,0} \end{aligned}\right\} \text{Initial conditions} \tag{1.6b}$$

$$\left.\begin{aligned} y_1(k) &= h_1(x_1(k),\dots,x_{n_x}(k),u_1(k),\dots,u_{n_u}(k),k) \\ &\vdots \\ y_{n_y}(k) &= h_{n_y}(x_1(k),\dots,x_{n_x}(k),u_1(k),\dots,u_{n_u}(k),k) \end{aligned}\right\} \text{Output equations} \tag{1.6c}$$

Again, we combine the input, output and state variables to (column) vectors

$$\mathbf{u} = [u_1\ u_2\ \dots\ u_{n_u}]^\top \tag{1.7a}$$

$$\mathbf{y} = \left[y_1\ y_2\ \dots\ y_{n_y}\right]^\top \tag{1.7b}$$

$$\mathbf{x} = [x_1\ x_2\ \dots\ x_{n_x}]^\top \tag{1.7c}$$

and obtain the compact vector notation

$$\mathbf{x}(k+1) = f(\mathbf{x}(k),\mathbf{u}(k),k), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1.8a}$$

$$\mathbf{y}(k) = h(\mathbf{x}(k),\mathbf{u}(k),k). \tag{1.8b}$$

An example of a discrete time system is the interest rate development of a bank deposit. Consider $\mathbf{x}(k)$ to be the bank deposit in month k and p to be the interest rate in percentage. If we place $\mathbf{u}(k)$ on the deposit at month k, then the model of the bank deposit development reads

$$\mathbf{x}(k+1) = \left(1 + \frac{p}{100}\right) \cdot \mathbf{x}(k) + \mathbf{u}(k). \tag{1.9}$$

Based on this model the bank deposit can be computed for all following months.

Task 1.9 (Gambler's ruin)
Suppose a player is going into a casino to play Roulette. The probability to win is p where $0 < p < 1$, whereas the probability that the casino wins is $q = 1 - p$. Upon start, the player has a chips and the casino has b chips. What is the probability that the player wins all chips? Use the toy data $p = 18/37$, $q = 19/37$, $a = 100$ and $b = 10000$.

Solution to Task :1.9 Suppose the player owns $0 \leq k \leq a + b$ chips, and therefore the casino owns $a + b - k$ chips. Let $\mathbf{x}(k)$ denote the probability that the player with k chips wins. Hence, depending on whether the player wins or not the player will have $k+1$ or $k-1$ chips after this iteration. Therefore, the difference equation is given by

$$\mathbf{x}(k) = p\mathbf{x}(k+1) + q\mathbf{x}(k-1).$$

Additionally, we obtain the boundary conditions $\mathbf{x}(0) = 0$ and $\mathbf{x}(a+b) = 1$. Solving the difference equation (e.g. via `Maple` via `resolve`), we obtain

$$\mathbf{x}(k) = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^{a+b}}.$$

Using the toy data we obtain $\mathbf{x}(a) = 1.538 \cdot 10^{-235}$.

Note that in both discrete and continuous time, the dynamic reveals a *flow* of the system at hand, whereas a *trajectory* is bound to a specific initial value and input sequence. The following Figure 1.3 illustrates the idea of flow and trajectory. In this case, the flow is colored to mark its intensity whereas the arrows point into its direction. The trajectory is evaluated for a specific initial value and „follows“ the flow accordingly.

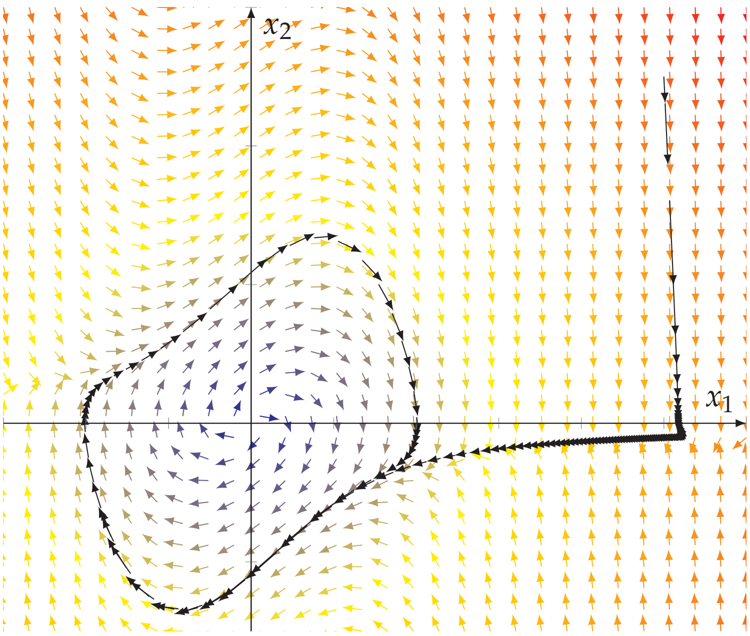
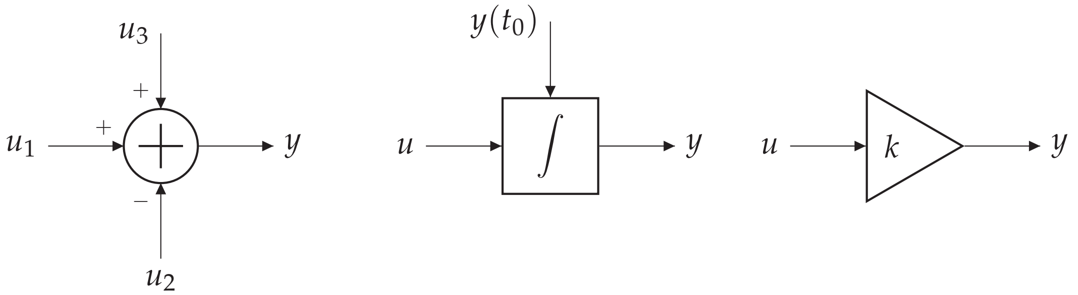


Figure 1.3.: Sketch of a dynamic flow and a trajectory

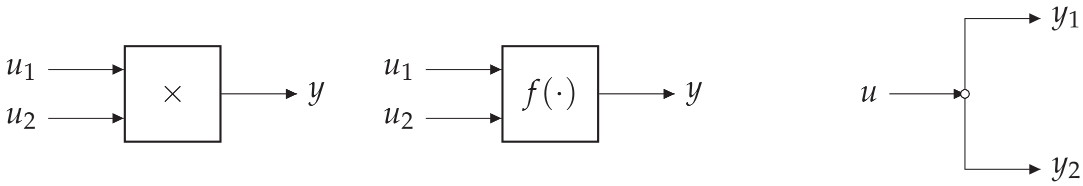
1.4. Block diagram

Although the dynamic behavior of a system is described well by mathematical models, it is convenient to represent models via so called block diagrams. Block diagrams are visual representations using (more or less) standardized symbols. Originally, this graphical representation was developed to simulate differential equations on analog computers. Today, still many simulations programs such as `Matlab/Simulink` as well as automation systems such as `Step7` offer the possibility to enter models via blocks.
Some standard symbols are their meaning are given in Figure 1.4.



$y = u_1 - u_2 + u_3$ $\qquad$ $y(t) = y(t_0) + \int_{t_0}^{t} u(\tau) d\tau$ $\qquad$ $y(t) = ku(t)$

(a) Sum $\qquad$ (b) Integrator $\qquad$ (c) Gain



$y = u_1 \cdot u_2$ $\qquad$ $y = f(u_1, u_2)$ $\qquad$ $y_1(t) = u(t); \quad y_2(t) = u(t)$

(d) Multiplicator $\qquad$ (e) Function $\qquad$ (f) Split

Figure 1.4.: Standard block diagram elements and their meaning

These symbols allow us to visually break down the structure of a system to its elements. For illustration, we consider a separately excited DC machine, which is moving a load via a cable drum, cf. Figure 1.5. Within the example, we denote the armature and excitation currents by I_A and I_F, the armature and excitation voltages by U_A and U_F respectively. Moreover, we denote the winding resistance by R_A and R_F and the magnetic flow by $\Psi_F(I_F)$, where L_A and k represent the armature induction and gain and ω the rotation speed of the motor.

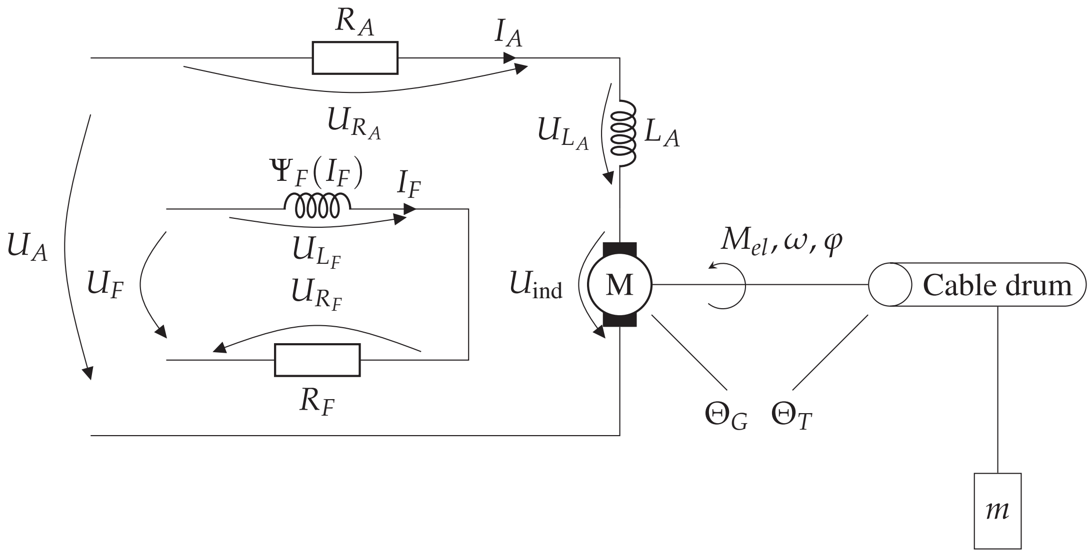


Figure 1.5.: Schematic of separately excited DC machine moving a (hoisting) drum roll

The mathematical model can be derived from the mesh equations of the armature circuit and the excitation circuit

$$\text{Armature circuit} \qquad -U_A + R_A \cdot I_A + U_{L_A} + U_{\text{ind}} = 0 \tag{1.10a}$$

$$\text{Excitation circuit} \qquad -U_F + R_F \cdot I_F + U_{L_F} = 0 \tag{1.10b}$$

where we have

$$U_{L_A} = L_A \cdot \frac{d}{dt} I_A \tag{1.11a}$$

$$U_{L_F} = \frac{d}{dt} \Psi_F(I_F) = \frac{\partial}{\partial I_F} \Psi_F(I_F) \cdot \frac{d}{dt} I_F \tag{1.11b}$$

$$U_{\text{ind}} = k \cdot \Psi_F(I_F) \cdot \omega. \tag{1.11c}$$

Putting (1.11) into (1.10) we obtain

$$L_A \cdot \frac{d}{dt} I_A = U_A - R_A \cdot I_A - k \cdot \Psi_F(I_F) \cdot \omega \tag{1.12a}$$

$$\frac{d}{dI_F} \Psi_F(I_F) \cdot \frac{d}{dt} I_F = U_F - R_F \cdot I_F. \tag{1.12b}$$

Introducing the force F_s of the rope, then by conservation of momentum at the motor we have

$$\frac{d}{dt} \varphi = \omega \tag{1.13a}$$

$$(\Theta_G + \Theta_T) \cdot \frac{d}{dt}\omega = M_{\text{el}} - F_s \cdot r = k\Psi_F(I_F) \cdot I_A - F_s \cdot r \tag{1.13b}$$

where r and φ are the radius and angle of the drum roll and $M_{\text{el}} = k\Psi_F(I_F) \cdot I_A$ is the electric momentum of the motor. Similarly, via conservation of momentum at the rope we have

$$\frac{d}{dt}x = v \tag{1.14a}$$

$$m\frac{d}{dt}v = F_s - m \cdot g \tag{1.14b}$$

where x, v and m are the position, velocity and mass of the load, and g is the acceleration of gravity. Using

$$r \underbrace{\frac{d}{dt}\varphi}_{=\omega} = \frac{d}{dt}x = v$$

in (1.13b) we obtain

$$F_s = m \cdot r \cdot \frac{d}{dt}\omega + m \cdot g.$$

Now, we can combine systems (1.12) and (1.13) to obtain the combined system of differential equations

$$\frac{d}{dt}I_A = \frac{1}{L_A}\left(U_A - R_A \cdot I_A - k \cdot \Psi_F(I_F) \cdot \omega\right) \tag{1.15a}$$

$$\frac{d}{dt}I_F = \frac{1}{\frac{\partial}{\partial I_F}\Psi_F(I_F)}\left(U_F - R_F \cdot I_F\right) \tag{1.15b}$$

$$\frac{d}{dt}\varphi = \omega \tag{1.15c}$$

$$\frac{d}{dt}\omega = \frac{1}{(\Theta_G + \Theta_T + m \cdot r^2)}\left(k\Psi_F(I_F) \cdot I_A - m \cdot g \cdot r\right). \tag{1.15d}$$

Utilizing Definitions 1.1 and 1.5 we identify the inputs $\mathbf{u} = [U_A \; U_F]^\top$, the output $\mathbf{y} = r \cdot \varphi$ and the states $\mathbf{x} = [I_A \; I_F \; \varphi \; \omega]^\top$.

To represent system (1.15) in a block diagram, we first integrate this system of first order differential equations and get

$$I_A(t) = I_A(0) + \frac{1}{L_A}\int_0^t \left(U_A(\tau) - R_A \cdot I_A(\tau) - k \cdot \Psi_F(I_F(\tau)) \cdot \omega(\tau)\right) d\tau \tag{1.16a}$$

$$I_F(t) = I_F(0) + \int_0^t \frac{1}{\frac{\partial}{\partial I_F}\Psi_F(I_F(\tau))}\left(U_F(\tau) - R_F \cdot I_F(\tau)\right) d\tau \tag{1.16b}$$

$$\varphi(t) = \varphi(0) + \int_0^t \omega(\tau) d\tau \tag{1.16c}$$

$$\omega(t) = \omega(0) + \frac{1}{(\Theta_G + \Theta_T + m \cdot r^2)} \int_0^t \left(k\Psi_F(I_F(\tau)) \cdot I_A(\tau) - m \cdot g \cdot r\right) d\tau. \tag{1.16d}$$

In the first step, we consider the simplest equation (1.15c). Utilizing the standard blocks from Figure 1.4 we obtain Figure 1.6.

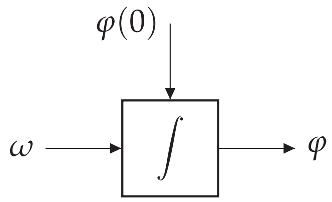


Figure 1.6.: Block diagram of equation (1.16c)

Next, we consider equation (1.15d) and separate it into operating blocks in Figure 1.7.

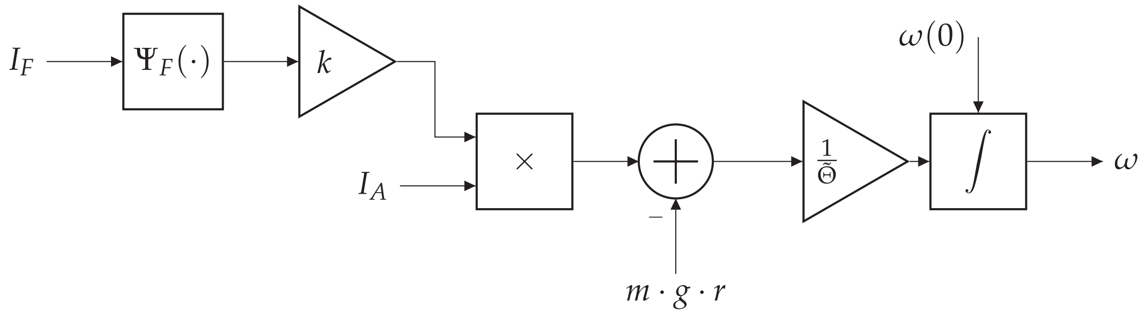


Figure 1.7.: Block diagram of equation (1.16d) with $\tilde{\Theta} = \Theta_G + \Theta_T + m \cdot r^2$

Considering the current in the armature circuit, we obtain the diagram of Figure 1.8.
Last, we get Figure 1.9 for the excitation circuit (1.16a).
Note that now all ingoing values to the block diagrams are either input $\mathbf{u}$ or states $\mathbf{x}$. Hence, we can connect these lines and obtain the overall block diagram.

Task 1.10

Draw the overall block diagram for the separately excited DC machine with drum roll from Figure 1.5.

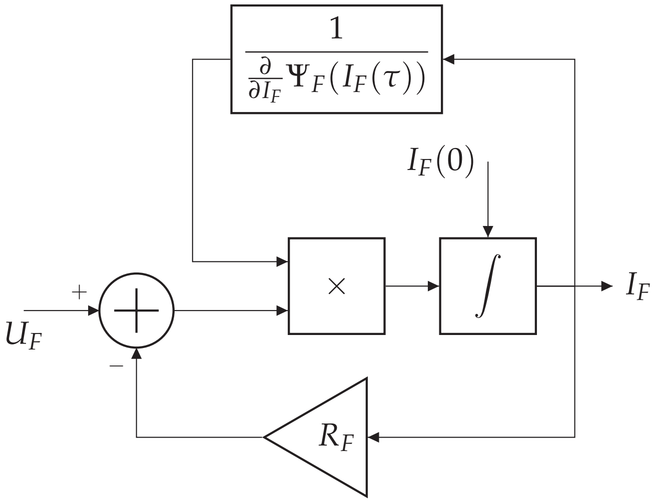


Figure 1.8.: Block diagram of equation (1.16b)

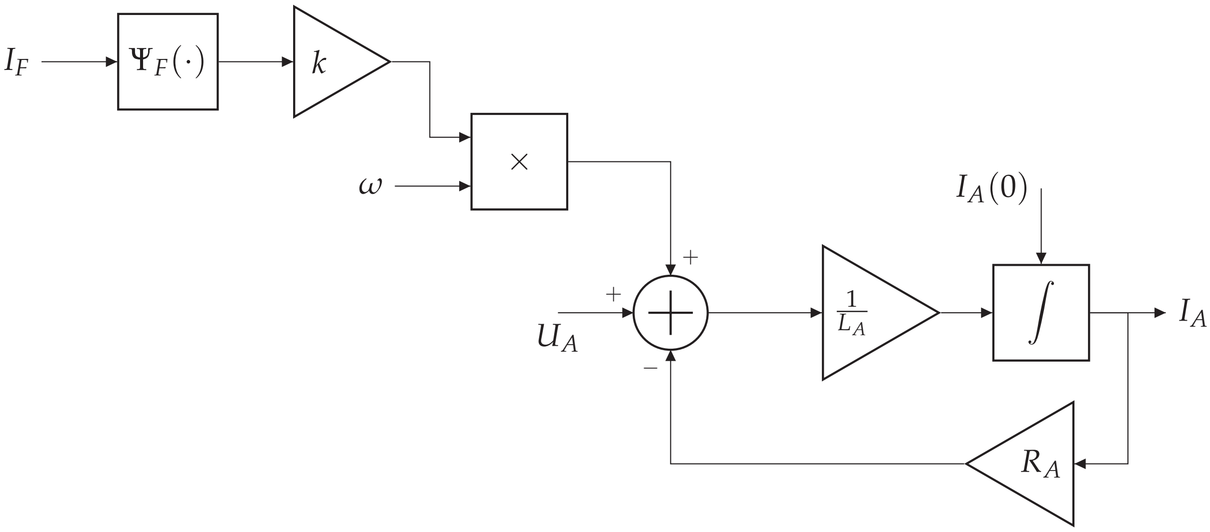


Figure 1.9.: Block diagram of equation (1.16a)

Solution to Task 1.10: The block from Figures 1.6–1.9 are connected via their states, cf. Figure 1.10.

1.5. Properties of systems

In order to analyze, characterize and classify systems, we can use certain properties. These properties not only allow us to get deeper insights into the systems themselves, but also offer possibilities to solve connected problems, design controllers and derive solution and control methods.

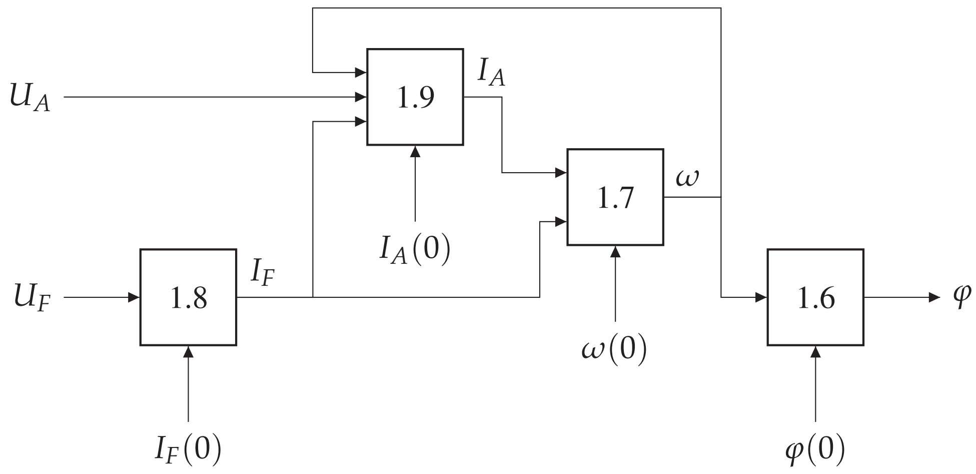
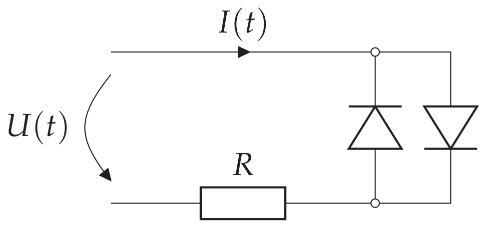
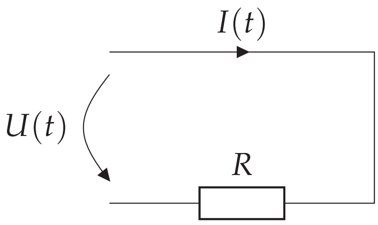


Figure 1.10.: Block diagram for separately excited DC machine with drum roll

The first and possibly most important property used in system theory is *linearity*. Informally, we can say that „a system is linear if all its components and connections are linear“. Note that, unfortunately, we cannot say that „a system is nonlinear if all of its components and connections are nonlinear“. A counterexample is given by the following circuits in Figure 1.11, which are equivalent, yet one system is linear and one contains nonlinear elements.



(a) Circuit with two diodes and one resistor



(b) Circuit with two diodes and one resistor

Figure 1.11.: Counterexample for linearity of systems

As outlined before, we focus on systems of the form (1.3), or for short

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.17a}$$

$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.17b}$$

with state $\mathbf{x} \in \mathbb{R}^{n_x}$, input $\mathbf{u} \in \mathbb{R}^{n_u}$ and output $\mathbf{y} \in \mathbb{R}^{n_y}$.
Suppose that $\varphi(t; \mathbf{x}_0, t_0, \mathbf{u})$ denotes the solution of the system (1.17) with initial value $\mathbf{x}(t_0) = \mathbf{x}_0$ and input $\mathbf{u}(\tau)$, $t_0 \leq \tau \leq t$. Then the linearity definition reads as follows:

Definition 1.11 (Linearity).
Consider a system of the form (1.17). If for all (feasible) inputs $\mathbf{u}$ and all initial times $t_0 \geq 0$ we have that the output $\mathbf{y}(\mathbf{x}_0, \mathbf{u}, t) = h(\varphi(t; \mathbf{x}_0, t_0, \mathbf{u}), \mathbf{u}(t), t)$ satisfies

$$\mathbf{y}(\alpha_1 \mathbf{x}_{0,1} + \alpha_2 \mathbf{x}_{0,2}, 0, t) = \alpha_1 \mathbf{y}(\mathbf{x}_{0,1}, 0, t) + \alpha_2 \mathbf{y}(\mathbf{x}_{0,2}, 0, t) \tag{1.18a}$$

$$\mathbf{y}(0, \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2, t) = \beta_1 \mathbf{y}(0, \mathbf{u}_1, t) + \beta_2 \mathbf{y}(0, \mathbf{u}_2, t) \tag{1.18b}$$

$$\mathbf{y}(\mathbf{x}_0, \mathbf{u}, t) = \mathbf{y}(\mathbf{x}_0, 0, t) + \mathbf{y}(0, \mathbf{u}, t) \tag{1.18c}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ for all $t \geq t_0$, then we call system (1.17) linear.

Within Definition 1.11 we call equation (1.18a) *superposition principle*, equation (1.18b) *zero-input-linearity* and equation (1.18c) *zero-state-linearity*. Given Definition 1.11, we get the following:

Theorem 1.12 (Linear system).
System (1.17) *is linear if and only if it can be transformed into*

$$\dot{\mathbf{x}}(t) = A(t) \cdot \mathbf{x}(t) + B(t) \cdot \mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.19a}$$

$$\mathbf{y}(t) = C(t) \cdot \mathbf{x}(t) + D(t) \cdot \mathbf{u}(t). \tag{1.19b}$$

Here, the matrices $A(t) \in \mathbb{R}^{n_x \times n_x}$, $B(t) \in \mathbb{R}^{n_x \times n_u}$, $C(t) \in \mathbb{R}^{n_y \times n_x}$ and $D(t) \in \mathbb{R}^{n_y \times n_u}$ depend on time $t \in \mathbb{R}$ only.

Task 1.13
Is system

$$\dot{\mathbf{x}}(t) = 2 + \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{x}(t)$$

linear?

Solution to Task 1.13: No, it isn't.

Apart from linearity, the dependence on time is a key element for systems (1.17). In particular, if a system is independent from time, the starting point may be shifted freely without change on the behavior of the system and its output. Note that the dependence of time of a model may be different for the dependence of time of a system. For example, a model of a rocket without its environment does not depend on weather or the orbital mechanics including the moon etc., yet the system itself clearly depends on these aspects which are varying over time.

Definition 1.14 (Time invariance).
Consider a system of the form (1.17). Suppose $\mathbf{y}(t)$ is the output at time $t \geq t_0$ for initial value $\mathbf{x}(t_0) = \mathbf{x}_0$ and input $\mathbf{u}(\tau)$, $t_0 \leq \tau \leq t$. If for all (feasible) inputs $\mathbf{u}$ and all initial times $t_0 \geq 0$ we have

$$\mathbf{y}(t - T) = h(\varphi(t; \mathbf{x}_0, t_0 + T, \mathbf{u}(\cdot - T)), \mathbf{u}(t - T)),$$

then (1.17) is time invariant.

Remark 1.15
Note that by considering a function evaluation of any function f at time instant $t - T$ with $T > 0$, the function is „shifted to the right by T“.

Typically, the time invariant version of system (1.17) is written

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.20a}$$

$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t)) \tag{1.20b}$$

Task 1.16
Consider the example from Task 1.13. Is the system time invariant?

Solution to Task 1.16: Yes, it is.

Regarding time invariance, the following necessary and sufficient conditions hold:

Theorem 1.17 (Linear time invariant system).
System (1.17) *is linear time invariant if and only if it can be transformed into*

$$\dot{\mathbf{x}}(t) = A \cdot \mathbf{x}(t) + B \cdot \mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1.21a}$$

$$\mathbf{y}(t) = C \cdot \mathbf{x}(t) + D \cdot \mathbf{u}(t). \tag{1.21b}$$

We like to note that linearity and time invariance is not limited to systems of the form (1.17). To see this, consider the following:

Task 1.18
Consider a conveyor belt and let $\mathbf{u}(t)$ *denote the amount of material put on the lower end of the belt and let* $\mathbf{x}(t)$ *denote the amount of material issued at the upper end of the belt at time* t*. For transportation from lower to upper end, the time* t_T *(dead time) is required, i.e. we have* $\mathbf{x}(t) = \mathbf{u}(t - t_T)$*. Is the system linear and time invariant? Can the system be formulated in the form* (1.17)*?*

Solution to Task 1.18: The system is linear and time invariant, yet no description of form (1.17) exists.

If we focus further on the so called linear autonomous time invariant system

$$\dot{\mathbf{x}}(t) = A \cdot \mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1.22}$$

then we know by *Lipschitz continuity* of A that a unique solution exists and can derive the corresponding solution by applying *Picard's method of successive approximation*. The latter reveals

$$\mathbf{x}(t) = \lim_{j \to \infty} \mathbf{x}_j(t) = \lim_{j \to \infty} \left(\mathrm{Id} + A \cdot t + A^2 \frac{t^2}{2} + \ldots + A^j \frac{t^j}{j!} \right) = \left(\sum_{j=0}^{\infty} A^j \frac{t^j}{j!} \right) \mathbf{x}_0,$$

which allows us to define the following solution operator:

Definition 1.19 (Transition matrix).
Consider system (1.22). Then we call

$$\Phi(t) := \exp(A \cdot t) = \left(\sum_{j=0}^{\infty} A^j \frac{t^j}{j!} \right) \tag{1.23}$$

transition matrix of the system.

In particular, the following holds:

Theorem 1.20 (Solution of autonomous linear time invariant systems).
Consider system (1.22). *Then we obtain the solution*

$$\mathbf{x}(t) = \Phi(t) \cdot \mathbf{x}_0. \tag{1.24}$$

Task 1.21

Compute the transition matrix of the system

$$\begin{pmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{pmatrix}.$$

Solution to Task 1.21:

$$\Phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot t + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2}_{=0} \cdot \frac{t^2}{2} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

For the linear time invariant system (1.21), we can utilize the transition matrix and the method of variation of constants to see the following:

Theorem 1.22 (Solution for linear time invariant systems).
Consider system (1.21) *and let* Φ *denote the transition matrix of system* (1.21) *with* $\mathbf{u} \equiv 0$. *Then the solution of system* (1.21) *is given by*

$$\mathbf{x}(t) = \Phi(t) \cdot \mathbf{x}_0 + \int_0^t \Phi(t-\tau) \cdot B \cdot \mathbf{u}(\tau) d\tau \tag{1.25a}$$

$$\mathbf{y}(t) = C \cdot \mathbf{x}(t) + D \cdot \mathbf{u}(t). \tag{1.25b}$$

Task 1.23

Consider the PI controller given by Figure 1.12 with corresponding equations

$$\dot{U}_C(t) = \frac{1}{R_1 C} \mathbf{u}(t)$$

$$\mathbf{y}(t) = -U_C(t) - \frac{R_2}{R_1} \cdot \mathbf{u}(t).$$

Use Theorem 1.22 to compute the output $\mathbf{y}(t)$ *for any feasible input* $\mathbf{u}(t)$.

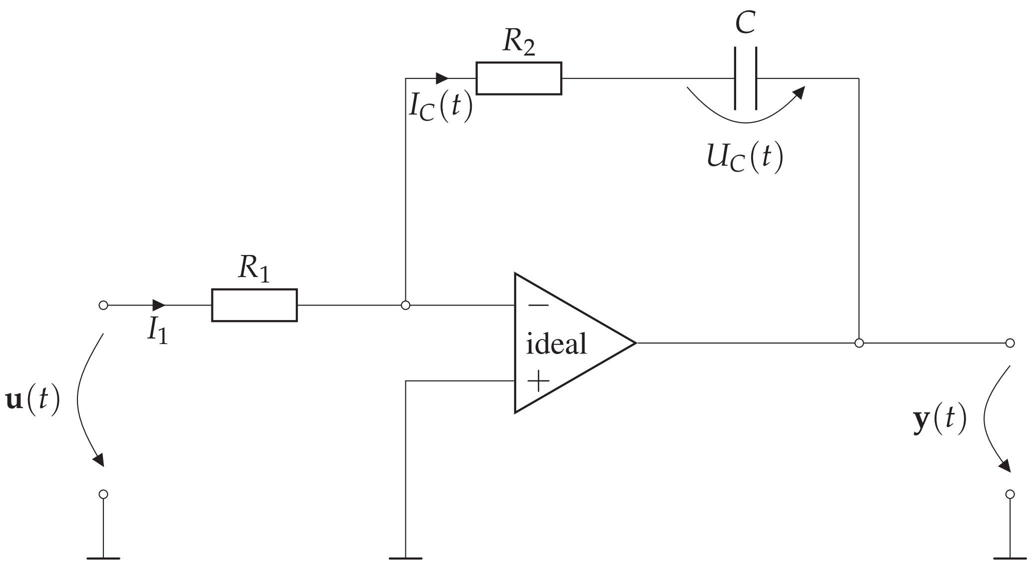


Figure 1.12.: Electronic circuit of a PI controller

Solution to Task 1.23: We directly obtain

$$\mathbf{y}(t) = -\frac{1}{R_1 C} \int_0^t \mathbf{u}(\tau) d\tau - \frac{R_2}{R_1} \mathbf{u}(t),$$

which gives us the proportional parameter $K_P = -R_2/R_1$ and the integral parameter $K_I = -1/(R_1 C)$ of the controller.

The last task is an example of the core of this lecture, the systematic manipulation of systems to fulfill tasks or force behavior. As we will see later in the lecture, the systematic manipulation of nonlinear systems is in general more complicated as compared to linear systems. Yet, for sufficiently small neighborhoods of points in the operating range of a system, results for linear systems apply also to nonlinear systems. This is particularly useful if these points are equilibria (constant operating points) or reference trajectories. To this end, we consider autonomous nonlinear systems of the form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \tag{1.26}$$

and define:

Definition 1.24 (Equilibrium).
Given a system of form (1.26), we call a point $\mathbf{x}^\star \in \mathbb{R}^{n_x}$ an equilibrium if

$$f(\mathbf{x}^\star) = 0 \qquad \forall t \geq 0. \tag{1.27}$$

Task 1.25
Compute the equilibria for the systems

$$\dot{\mathbf{x}}(t) = (\mathbf{x} - 1) \cdot (\mathbf{x} - 2) \cdot (\mathbf{x} - 3), \tag{1.28a}$$

$$\begin{pmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_2(t) \cdot \exp -\mathbf{x}_1(t) \\ \sin(\mathbf{x}_2(t)) \end{pmatrix}, \tag{1.28b}$$

$$\dot{\mathbf{x}}(t) = \mathbf{x}^2(t) + 1. \tag{1.28c}$$

Solution to Task 1.25: For system (1.28a) we have three equilibria $\mathbf{x}_1^\star = 1$, $\mathbf{x}_2^\star = 2$ and $\mathbf{x}_3^\star = 3$.
For system (1.28b) we have infinitely many equilibria $\mathbf{x}^\star = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid x_2 = 0 \wedge x_1 \in \mathbb{R} \right\}$.
For system (1.28c) there exists no equilibrium.

Remark 1.26
For autonomous linear time invariant systems (1.22) *we have*

- $\mathbf{x}^\star = 0$ *if and only if A is regular, i.e. $\det(A) \neq 0$, or*
- *there exist infinitely many equilibria if and only if A is singular, i.e. $\det(A) = 0$.*

If the nonlinear system is not autonomous, i.e.

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \tag{1.29}$$

then the input $\mathbf{u} \in \mathbb{R}^{n_u}$ needs to be constant and fixed to $\mathbf{u} = \mathbf{u}^\star$ in order to compute the equilibria.

Definition 1.27 (Operating point).
Consider system (1.29). Then the pairs $(\mathbf{x}^\star, \mathbf{u}^\star)$ satisfying

$$f(\mathbf{x}^\star, \mathbf{u}^\star) = 0 \tag{1.30}$$

are called *operating points* of the system. If (1.30) holds true for any $\mathbf{u}^\star$, then the operating point is called strong or robust operating point.

Remark 1.28
For linear time invariant systems (1.21a) *we have*

- $\mathbf{x}^\star = -A^{-1} \cdot B \cdot \mathbf{u}^\star$ *iff* $det(A) \neq 0$,
- *infinitely many operating points iff* $det(A) = 0$ *and* $rank(A) = rank([A, B \cdot \mathbf{u}^\star])$,
- *no operating points iff* $det(A) = 0$ *and* $rank(A) \neq rank([A, B \cdot \mathbf{u}^\star])$

The result/property we are most interested in control theory is stability. Utilizing Definition 1.27 we can introduce two concepts of stability and asymptotic stability, robustness and controllability. These concepts depend on the interpretation of $\mathbf{u}$ as an external control or a disturbance.

Definition 1.29 (Stability and Controllability).
For a system (1.29) we call $\mathbf{x}^\star$

- *strongly* or *robustly stable* operating point if, for each $\varepsilon > 0$, there exists a real number $\delta = \delta(\varepsilon) > 0$ such that for all $\mathbf{u}$ we have

$$\|\mathbf{x}_0 - \mathbf{x}^\star\| \leq \delta \implies \|\mathbf{x}(t) - \mathbf{x}^\star\| \leq \varepsilon \qquad \forall t \geq 0 \tag{1.31}$$

- *strongly* or *robustly asymptotically stable* operating point if it is stable and there exists a positive real constant r such that for all $\mathbf{u}$

$$\lim_{t \to \infty} \|\mathbf{x}(t) - \mathbf{x}^\star\| = 0 \tag{1.32}$$

  holds for all $\mathbf{x}_0$ satisfying $\|\mathbf{x}_0 - \mathbf{x}^\star\| \leq r$. If additionally r can be chosen arbitrary large, then $\mathbf{x}^\star$ is called *globally strongly* or *robustly asymptotically stable*.

- *weakly stable* or *controllable* operating point if, for each $\varepsilon > 0$, there exists a real number

$\delta = \delta(\varepsilon) > 0$ such that for each $\mathbf{x}_0$ there exists a control $\mathbf{u}$ guaranteeing

$$\|\mathbf{x}_0 - \mathbf{x}^\star\| \leq \delta \implies \|\mathbf{x}(t) - \mathbf{x}^\star\| \leq \varepsilon \qquad \forall t \geq 0. \tag{1.33}$$

- *weakly asymptotically stable* or *asymptotically controllable* operating point if there exists a control $\mathbf{u}$ depending on $\mathbf{x}_0$ such that (1.33) holds and there exists a positive constant r such that

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}^\star\| = 0 \qquad \forall \|\mathbf{x}_0 - \mathbf{x}^\star\| \leq r. \tag{1.34}$$

If additionally r can be chosen arbitrary large, then $\mathbf{x}^\star$ is called *globally asymptotically stable*.

Task 1.30
Draw solutions of systems for each of the cases in Definition 1.29.

Remark 1.31 (BIBO stability)
In some books, the concept of strong/robust stability is also termed BIBO (bounded input bounded output) stability.

Note that strongly asymptotically stable control systems are boring from a control point of view since the chosen control does not affect the stability property of the system. Still, it can be used to improve the performance of the system. Moreover, strong asymptotic stability is interesting in the presence of significant measurement or discretization errors. Its most interesting application is in the analysis of robustness of a system, i.e. whether or not there exists an external input (in that case a disturbance) which can destabilize the system.
The concept of weak stability, on the other hand, naturally leads to the question how to compute a control law such that $\mathbf{x}^\star$ is weakly stable, and, in particular, how to characterize the quality of a control law.
In the following chapter, our focus will be to design a control law such that the stability property can be forced to hold for a given sytem.

Part I.

Frequence Domain

CHAPTER 2

MODELING OF COMPLEX CONTROL LOOPS

In modeling of control systems, we used a white box idea in Chapter 1 and introduced the state of a system. In practice, however, deriving such a white box model is neither always necessary nor productive. In many (especially in simple) cases, a black box approach allows us to derive a control with required properties much more easily. This approach utilizes the so called *frequency domain*. In that case, the map between input and output is not defined via a state dependent dynamic, but via a direct map from input to output, the so called *transfer function*. As we learned in *Control Engineering 1*, there exists a linear and invertible transformation between the time domain which we used in Chapter 1 and the frequency domain, the so called *Laplace transform* (or z transform in the discrete time case).

Within this chapter, we first recall the connection of frequency and time domain for simple systems before moving to more complex control loops.

For further details we additionally refer to the DIN 19226 [4, 5]

2.1. Laplace transform

The Laplace transform is a one to one map of functions of time t to functions of complex variables s. For causal functions $f(t)$, that is function satisfying $f(t) = 0$ for $t < 0$, the (one sided) map itself is defined as follows:

Definition 2.1 (Laplace transform function).
Consider a function $f(t)$ which is

- causal, i.e. $f(t) = 0$ for $t < 0$,
- piecewise constant for every finite time interval $t \geq 0$, and

- bounded by $|f(t)| \leq M\exp(\gamma t)$ for suitable constants $\gamma, M > 0$.

Then we call the integral

$$\hat{f}(s) = \mathcal{L}(f(t)) = \int_0^\infty \exp(-st) \cdot f(t)dt, \qquad s = \alpha + i\omega \tag{2.1}$$

Laplace transform of the time function $f(t)$ and the set $\mathbb{C}_\gamma = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \gamma\}$ is called *area of existence* of $\hat{f}(s)$.

Remark 2.2
Note that the integral (2.1) *converges absolutely if* $Re(s) > \gamma$.

Task 2.3
Compute the Laplace transform and its area of existence for $f(t) = \exp(at)$.

Solution to Task 2.3: We obtain

$$\hat{f}(t) = \int_0^\infty \exp(-st) \cdot \exp(at)dt = \left.\frac{\exp(-(s-a)t)}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}$$

for $\mathrm{Re}(s) > a$, i.e. area of existence $\mathbb{C}_a = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > a\}$.

Two special cases of Laplace transformed functions are the so called *Heaviside* and *Dirac delta* function. The Heaviside function represents a unit jump, which is not differentiable but integrable via the Laplace function, and also not defined at the jump point.

Task 2.4
Compute the Laplace transform of the Heaviside function

$$\eta(t) = \begin{cases} 0, & t < 0 \\ \textit{undefined}, & t = 0 \\ 1, & t > 0 \end{cases}.$$

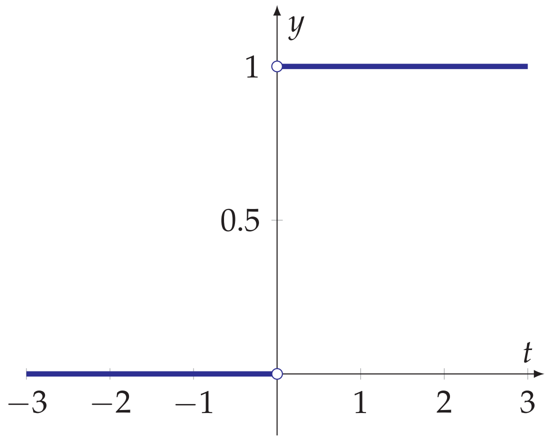


Figure 2.1.: Sketch of the Heaviside function

Solution to Task 2.4:

$$\mathcal{L}(\eta(t)) = \int_0^\infty \exp(-st)dt = \left.\frac{\exp(-st)}{-s}\right|_0^\infty = \frac{1}{s} \qquad \text{for } \mathrm{Re}(s) > 0$$

The Dirac delta function is the left sided jump height of the Heaviside function, or may be interpreted as functional to align a n-times continuously differentiable function to an initial value.

Task 2.5

Compute the Laplace transform of the Dirac delta function

$$\int_{-\infty}^{\infty} \delta(t) \cdot g(t)dt = g(0)$$

$$\int_{-\infty}^{\infty} \left(\frac{d^n}{dt^n} \cdot \delta(t)\right) g(t)dt = (-1)^n \cdot \left(\frac{d^n}{dt^n} g\right)(0)$$

or (via the Heaviside function)

$$\delta(t) = \lim_{\tau \to 0} \frac{\eta(t) - \eta(t-\tau)}{\tau}.$$

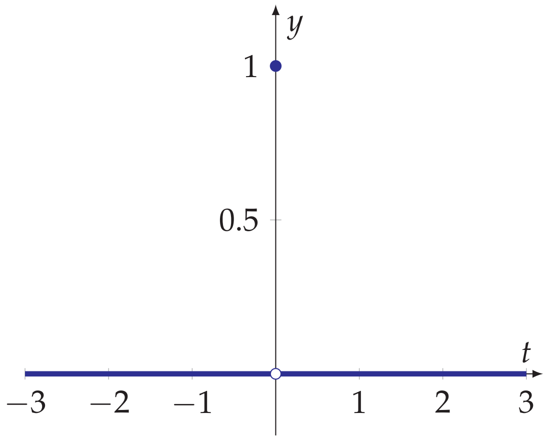


Figure 2.2.: Sketch of the Dirac delta function

Solution to Task 2.5:

$$\begin{aligned}\mathcal{L}(\delta(t)) &= \lim_{\tau\to 0} \mathcal{L}\left(\frac{\eta(t)-\eta(t-\tau)}{\tau}\right)\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\cdot\int\limits_0^\infty \eta(t)\cdot\exp(-st)dt - \int\limits_0^\infty \eta(t-\tau)\cdot\exp(-st)dt\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\cdot\int\limits_0^\tau \exp(-st)dt = \lim_{\tau\to 0}\frac{1-\exp(-s\tau)}{\tau s} \underbrace{=}_{\text{l'Hospital}} \lim_{\tau\to 0}\frac{s\cdot\exp(\tau s)}{s} = 1\end{aligned}$$

Since the Laplace transform is one to one, it can be inverted:

Definition 2.6 (Inverse Laplace transform function).
Consider a function $\hat{f}(s) : \mathbb{C}_\gamma \to \mathbb{C}$, which is well defined on $\mathbb{C}_\gamma$. Then we call the integral

$$f(t) = \mathcal{L}(\hat{f}(s)) = \frac{1}{2\pi i}\cdot\int\limits_{r-i\infty}^{r+i\infty} \hat{f}(s)\cdot\exp(st)ds, \qquad t\geq 0,\ r\in\mathbb{R} \tag{2.2}$$

the inverse Laplace transform.

The reason why the Laplace transform or Laplace transformed functions are used quite often to solve dynamic problems is due to its properties: While in time domain, the solution of a dynamics requires the computation of integral, derivatives, time delay/advance, convolution etc., in frequency domain these problems can be solved using algebraic equations only. To recall the main properties and laws of computation, we refer to Table A.1.

Note that typically the computation of a Laplace transform and of its inverse is not done via equations (2.1) or (2.2) but via equivalence tables. Table B.1 summarizes a few of these equivalencies. The main mathematical tool used to apply the equivalences from Table B.1 is the partial fraction decomposition. Since the entire fraction is typically not contained in the table, this method allows us to split fraction into components, which are available in the table and therefore can be transformed.

Theorem 2.7 (Partial fraction decomposition).
Consider a function

$$\hat{f}(s) = \frac{\hat{p}(s)}{\hat{q}(s)} \tag{2.3}$$

where $\hat{p}(s)$ and $\hat{q}(s)$ are coprime real polynoms satisfying $grad(\hat{p}(s)) \leq grad(\hat{q}(s)) = n$. Furthermore, suppose that $\hat{q}(s)$ can be transformed into the form

$$\hat{q}(s) = \prod_{j=1}^{h} (s - \lambda_j)^{k_j} \prod_{j=1}^{m} \left((s - \alpha_j)^2 + \beta_j^2 \right)^{l_j} \tag{2.4}$$

with $grad(\hat{q}(s)) = n = \sum_{j=1}^{h} k_j + 2 \sum_{j=1}^{m} l_j$. Then, function $\hat{f}(s)$ can be uniquely reformulated to to the partial fraction decomposition

$$\hat{f}(s) = c_0 + \sum_{j=1}^{h} \sum_{i=1}^{k_j} \frac{c_{ji}}{\left(s - \lambda_j\right)^i} + \sum_{j=1}^{m} \sum_{i=1}^{l_j} \frac{d_{ji} + e_{ji}s}{\left((s - \alpha_j)^2 + \beta_j^2 \right)^i} \tag{2.5}$$

with $c_0 = \lim_{s \to \infty} \frac{\hat{p}(s)}{\hat{q}(s)}$ and real coefficients c_{ji}, d_{ji} and e_{ji}.

While the Laplace transform is not necessary to derive a description of a black box model, it is very useful to see the interconnection between the white box and black box description.

2.2. Transfer matrix

Within the lecture, we aim to extend the concept of control, stability and controllability to more complex systems. To this end, we abstract from the transfer function treated in *Control Engineering 1* for the one dimensional case and introduce the transfer matrix.

A transfer matrix in general is a map of a system in the sense of Definition 1.1. If we reconsider the linear time invariant system from equation (1.21)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A \cdot \mathbf{x}(t) + B \cdot \mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= C \cdot \mathbf{x}(t) + D \cdot \mathbf{u}(t),\end{aligned}$$

then we obtain

$$\begin{aligned}s \cdot \hat{\mathbf{x}}(s) - \mathbf{x}_0 &= A \cdot \hat{\mathbf{x}}(s) + B \cdot \hat{\mathbf{u}}(s) \\ \hat{\mathbf{y}}(s) &= C \cdot \hat{\mathbf{x}}(s) + D \cdot \hat{\mathbf{u}}(s)\end{aligned}$$

which gives us

$$\hat{\mathbf{y}}(s) = C \cdot (s \cdot \mathrm{Id} - A)^{-1} \cdot \mathbf{x}_0 + \Big(C \cdot (s \cdot \mathrm{Id} - A)^{-1} \cdot B + D\Big) \cdot \hat{\mathbf{u}}(s) \tag{2.7}$$

Technically, we are not restricted to the linear case of system (1.21). For this reason, we can directly define the following:

Definition 2.8 (Transfer matrix).
Consider a time invariant system (1.20) and a function $G \in \mathbb{C}^{n_y \times n_u}$. We call $G(s)$ transfer matrix of system (1.20) if it satisfies

$$\hat{\mathbf{y}}(s) = G(s) \cdot \hat{\mathbf{u}}(s). \tag{2.8}$$

Remark 2.9 (SISO)
Note that in the one dimensional case, that is one input u and one output y, we call G transfer function. This case is also termed SISO – single input single output. Unless termed otherwise, we directly state all results for the multi input multi output (MIMO) case.

In the linear case, we know from Theorem 1.22 that the general solution to system (1.21) reads

$$\begin{aligned}\mathbf{x}(t) &= \Phi(t) \cdot \mathbf{x}_0 + \int_0^t \Phi(t - \tau) \cdot B \cdot \mathbf{u}(\tau) d\tau \\ \mathbf{y}(t) &= C \cdot \mathbf{x}(t) + D \cdot \mathbf{u}(t)\end{aligned}$$

with transition matrix $\Phi(t)$. Applying the Laplace transform to the state dynamics and the convolution rule from Table A.1 we have

$$\hat{\mathbf{x}}(s) = \hat{\Phi}(s) \cdot \mathbf{x}_0 + \hat{\Phi}(s) \cdot B \cdot \hat{\mathbf{u}}(s) \tag{2.9}$$

and thereby

$$\hat{\Phi}(s) = (s \cdot \mathrm{Id} - A)^{-1}. \tag{2.10}$$

Remark 2.10

The two parts of (2.9) *can be interpreted physically.*

- *The first part* $\hat{\Phi}(s) \cdot \mathbf{x}_0$ *represents the response of the system if no input is applied. For this reason, it is called zero input response.*
- *The second part* $\hat{\Phi}(s) \cdot B \cdot \hat{\mathbf{u}}(s)$ *represent the response to an input if the system state is zero. Similarly, it is termed zero state response.*

For the zero state response of the system, i.e. $\mathbf{x}_0 = 0$, we can conclude

Theorem 2.11 (Transfer matrix).

Consider a linear time invariant system (1.21) *with* $\mathbf{x}_0 = 0$*. Then the transfer matrix is given by*

$$G(s) = C \cdot (s \cdot Id - A)^{-1} \cdot B + D = C \cdot \hat{\Phi}(s) \cdot B + D. \tag{2.11}$$

Remark 2.12

Note that due to non-commutativity of matrix multiplication, the sequence of transfer matrices is important and may not be switched as in the one dimensional case of transfer functions.

The inverted way, i.e. to derive a state description from a transfer matrix/function, is called *realization problem*. For a system, the related property is called *properness*.

Definition 2.13 (Properness).

Consider a transfer matrix $G(s)$ defining the system

$$\hat{\mathbf{y}}(s) = G(s) \cdot \hat{\mathbf{u}}(s).$$

If there exists a (possibly nonlinear) system (1.20) such that $G(s)$ is the transfer function of the system, then the transfer matrix $G(s)$ is called *proper*.

For the latter, if a solution exists then the solution is not unique and one typically addresses a *minimal realization* only. The main result is the following, which we state for the one dimensional case:

Theorem 2.14 (Properness).
Consider a transfer function

$$G(s) = \frac{z(s)}{n(s)} \tag{2.12}$$

with polynoms $z(s)$ *and* $n(s)$*. The transfer function is proper if and only if*

$$grad(z(s)) \leq grad(n(s)) \qquad \text{or equivalently} \qquad \lim_{s\to\infty} |G(s)| < \infty. \tag{2.13}$$

Remark 2.15
Properness of a system can be extended to the MIMO case. In order to compute the latter, one typically applies a parameter transformation first (similar to the computation of the Jordan matrix) to separate the connections between inputs and outputs. For further details, we refer to the book of Isidori [8, Chapter 5] for the general nonlinear case or the article of Müller [15] for the linear case.

In the literature, there are two canonical minimal realizations, which can be obtained via partial fraction decomposition, cf. Theorem 2.7. The canonical minimal realizations are called *controllable normal form* and *observable normal form* and are given in the appendix, cf. Definitions B.1 and B.1 respectively. As the MIMO case (many) introduces further zeros and ones, we restrict ourselves to the SISO case here. A full description can be found in Müller [15].

Theorem 2.16 (Poles and zeros of transfer function).
Consider a transfer function

$$G(s) = \frac{z(s)}{n(s)} \tag{2.14}$$

with coprime polynoms $z(s)$ *and* $n(s)$*. Then we have* $grad(z(s)) \leq grad(n(s)) \leq n_x$ *and the zeros of* $n(s)$ *are called poles of the transfer function* $G(s)$ *and equal to the Eigenvalues of* A*.*

Theorem 2.17 (Poles and zeros of transfer matrix).
For a transfer matrix $G(s)$, the poles and zeros are given by the interconnection of the transfer functions by applying the computing rules for Laplace transformed functions.

Utilizing Definition 1.29 on stability in the frequency domain, we can show that the following holds:

Theorem 2.18 (Strong/robust/BIBO stability).
Consider a transfer matrix $G(s)$ defining the system

$$\hat{\mathbf{y}}(s) = G(s) \cdot \hat{\mathbf{u}}(s).$$

Then the system is strong/robust/BIBO stable if and only if for the impulse response

$$g(t) = \mathcal{L}^{-1}\left(G(s) \cdot Id\right) \tag{2.15}$$

the following inequality holds

$$\int_0^\infty \|g(t)\| dt < \infty. \tag{2.16}$$

If we know, that the transfer matrix corresponds to a linear time invariant system, then Theorem 2.18 simplifies to

Theorem 2.19 (Strong/robust/BIBO stability for linear time invariant systems).
A linear time invariant system (1.21) *is strong/robust/BIBO stable if and only if inequality* (2.16) *holds for the impulse response*

$$g(t) = \mathcal{L}^{-1}\left(G(s) \cdot Id\right) = C \cdot \left(s \cdot Id - A\right)^{-1} \cdot B. \tag{2.17}$$

The latter can be verified by checking the poles of the transfer function.

Theorem 2.20 (Strong/robust/BIBO stability for linear time invariant systems via transfer function).
A linear time invariant system (1.21) *is strong/robust/BIBO stable if and only if all poles*

$s_j = \alpha_j + i\omega_j$ of the transfer function $G(s)$ satisfy

$$Re(s_j) = \alpha_j < 0. \tag{2.18}$$

Before coming to structures with multiple inputs and multiple outputs, we consider two intermediate cases where we use multiple outputs to improve the controller for a single input.

2.3. Cascade control

The concept of a cascade control can be applied to systems (1.20) if more than one output but only one input is available. The idea is to distinguish between fast and slow processes using fast and slow outputs/sensors. In an inner control loop, disturbances on the system are included in a fast manner, while on the outer loop low frequent changes are tackled. The block diagram of the cascade control is sketched in Figure 2.3.

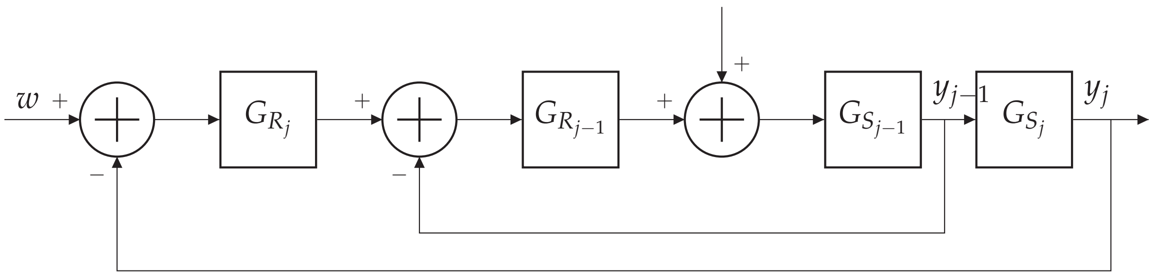


Figure 2.3.: Block diagram of a cascade control

The reason for utilizing such a control structure is that, in order for a controller to react, disturbances must directly affect the state and the output. As a consequence, the output will suffer and diverge from the target point, whereas the speed of recovery is determined by the response speed of the control loop. Unfortunately, plants which are difficult to control often exhibit low gains and long integral times for stability, hence have a slow response. Such plants are prone to error from disturbances. Examples of such applications can, e.g., be found in motor control, cf. Example 2.21.

Example 2.21 (Cascaded motor control)
Consider the control structure in Figure 2.4 of an electric engine where y_M denotes the drive torque, y_ω denotes the rotating speed of the engine, and y_φ denotes the position angle of the engine. The controller has the three levels

1. *current / torque control u_M,*
2. *speed control u_ω, and*
3. *position control u_φ.*

Here, the disturbance is modeled as a load moment d_M.

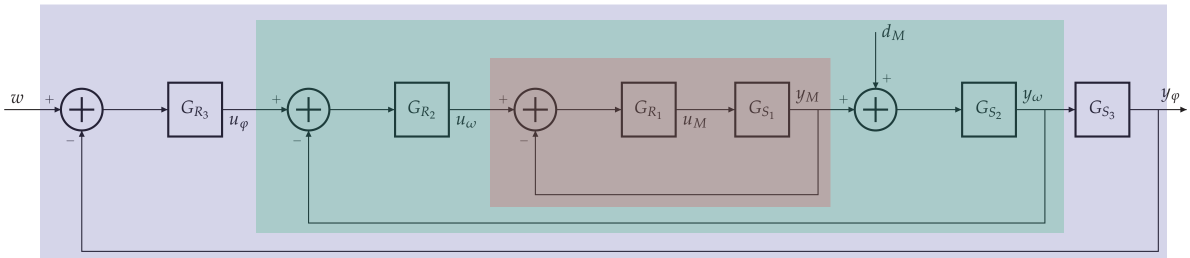


Figure 2.4.: Block diagram of a cascade motor control

Remark 2.22
Closed loop systems can in general be considered to show the behavior of a second order system with a natural rotating frequency and a damping factor. At frequencies faster than the natural frequency, one can see that the closed loop gain decreases rapidly (at 12_{dB} per octave). Consequently, disturbances faster than double the natural frequency will remain more or less uncorrected. In case of underdamped systems, i.e. damping factor less then unity, the disturbances with frequency around the natural frequency may be magnified.

If the disturbance can be measured, and its effect known, (even approximately), the idea of a cascade controller can be imposed. As such, a correcting signal can be added either on the inner or outer loop to compensate the issues described above.
While typically a two loop structure as shown in Figure 2.3, a cascade controller technically distinguishes between outer and inner loops only. Hence, also concatenated loops are possible. More formally, extending on Figure 2.3, the transfer function of the cascade control is given by the following.

Definition 2.23 (Transfer function cascade control).
Consider a system (1.20) with one input and at least $j \geq 2$ outputs and tracking reference $\mathbf{w}(t)$. Let $G_{R_j}(s)$ and $G_{S_j}(s)$ denote the control and system transfer functions for $j = 1, \ldots, j_{\max}$.

Suppose the structure of Figure 2.3 holds iteratively for these controllers and systems. Then we call

$$\hat{\mathbf{y}}(s) = G_{\text{cascade}_{j\text{max}}}(s) \cdot \hat{\mathbf{w}}(s) \tag{2.19}$$

transfer function of a cascade controlled system where the transfer functions $G_{\text{cascade}_j}(s)$ are defined via

$$G_{\text{cascade}_j}(s) := \frac{G_{R_j}(s) \cdot G_{\text{cascade}_{j-1}}(s) \cdot G_{S_j}(s)}{1 + G_{R_j}(s) \cdot G_{\text{cascade}_{j-1}}(s) \cdot G_{S_j}(s)} \qquad \forall j = 1, \ldots, j_{\max} \tag{2.20}$$

with $G_{\text{cascade}_0}(s) = \text{Id}(s)$.

In that sense, an outer loop control can therefore be interpreted as feed forward for an inner loop. It supplies a reference trajectory which shall be followed by the (typically faster) inner controller. On the downside, the usage of cascade control increases the complexity of the control structure. Additionally, it requires additional measurement devices and controllers.

Remark 2.24
In general, if the inner loop is at least three times faster than the outer loop, then the improved performance justifies the investment of a cascade controller.

The nice property of the cascade control is that the problems to be tackled by the concatenated loops can be separated and therefore treated subsequently.

Remark 2.25
Note that the problems are not treatable independently but require subsequent steps.

The steps to be taken can be combined into the following method:

Algorithm 2.26 (Design of a cascade control)
Consider the setting of Definition 2.23. Suppose performance criteria and input/output bounds to be given for each loop.
For each $j = 1, \ldots, j_{\max}$ do

(1) Disregard all loops $k > j$ and consider the open loop $G_{R_j}(s) \cdot G_{\text{cascade}_{j-1}}(s) \cdot G_{S_j}(s)$. Design the controller $G_{R_j}(s)$ such that the closed loop satisfies the respective performance criteria and input/output bounds.

(2) Compute and simplify the closed loop $G_{\text{cascade}_j}(s)$.

As noted before, the problems are nested and tackled in a subsequent manner. From Algorithm 2.26 we observe, that the closed loop behavior of the inner loop is the open loop behavior of the subsequent outer loop. Hence, the design of the inner loop control heavily influences the difficulty of designing the subsequent outer loop. I.e., if a fast behavior is desired, then the inner control needs to be aggressive.

In most cases, only the performance of the most outer loop is of interest to the user, whereas all inner loops are only a means to an end. Therefore, criteria such as „no constant deviation“ or „minimal overshoot“ are not of interest for the inner loops. For this reason, on the inner loops typically P or PDT_1 controllers are used, which are also faster, whereas the much slower PI controller are applied on the most outer loop.

Task 2.27

Consider the cascade control from Figure 2.5. What is the advantage of using a cascade controller utilizing $y_1(s)$ and $y_2(s)$ as compared to a SISO controller based on $y_2(s)$ only?

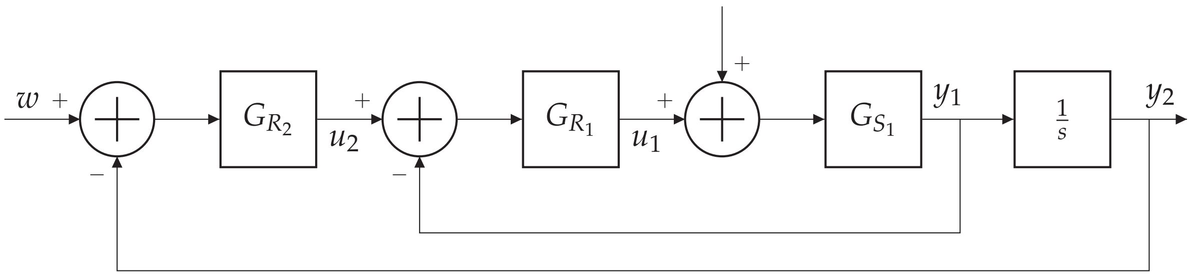


Figure 2.5.: Block diagram of a cascade control for integral outer loop

Solution to Task 2.27: Designing the inner loop as a P controller is equivalent of using a D controller on the outer loop. This configuration allows to get rid of the disadvantages of the D controller such as noise sensitivity and the rank problem in the transfer function.

Task 2.28

Consider a robot arm with variables as shown in Figure 2.6. For robots, three typical control types exist:

- *torque control, i.e. to apply a defined torque / moment within a working area,*
- *position control, i.e. to guarantee sufficiently accurate movement of the arm independent from torques / moments, and*

- *hybrid control, i.e. an application dependent switching between torque and position control.*

Consider the cascade control from Figure 2.7 to be applied for position control of one of the angles φ_j, $j = 1, 2, 3$. *Suppose the transfer function block coefficients of drive j to read*

$$
\begin{aligned}
K_{P\varphi} &= 0.2, & K_{DT\varphi} &= 0.009s \\
K_{I\omega} &= 0.9s^{-1} & & \\
K_{Pu} &= 2.8, & K_{DTu} &= 0.073s \\
K_{P\varphi} &= 3.5, & K_{DT\varphi} &= 0.069s
\end{aligned}
$$

For inner loop control coefficients $K_{P1} = 25.5$ *and* $K_{T1} = 0.073s$ *compute the optimal coefficients* K_{P2}, K_{I2} *of the external PI controller.*

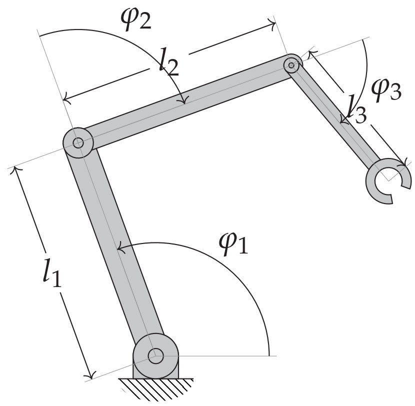


Figure 2.6.: Sketch of a 3DOF robot arm

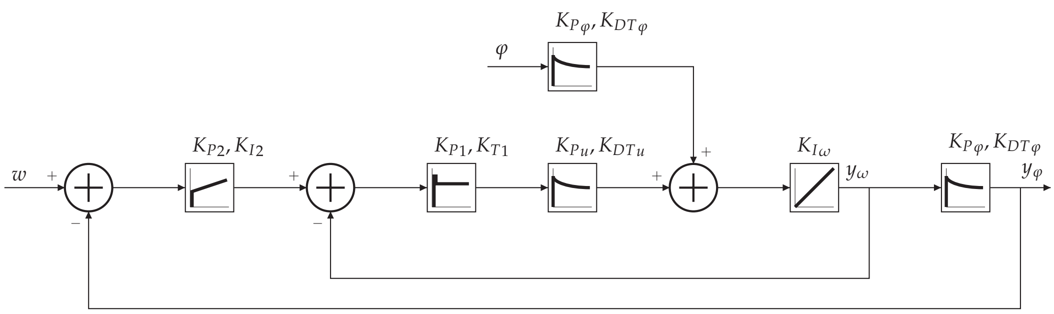


Figure 2.7.: Block diagram of a cascade robot control for one joint drive

Solution to Task 2.28: The inner open loop transfer function of the drive reads

$$G_{01}(s) = K_{P1} \cdot (1 + s \cdot K_{T1}) \cdot \frac{K_{Pu}}{1 + s \cdot K_{DTu}} \cdot \frac{K_{I\omega}}{s}.$$

Since $K_{T1} \equiv K_{DTu} = 0.073s$ we obtain

$$G_{01}(s) = \frac{K_{P1} \cdot K_{Pu} \cdot K_{I\omega}}{s}$$

and obtain the inner closed loop transfer function

$$G_{w1}(s) = \frac{1}{1 + \frac{1}{G_{01}(s)}} = \frac{1}{1 + \frac{s}{K_{P1} \cdot K_{Pu} \cdot K_{I\omega}}} = \frac{1}{1 + s \cdot K_{w1}}$$

with

$$K_{w1} := 1/(K_{P1} \cdot K_{Pu} \cdot K_{I\omega}) = \frac{1}{25.5 \cdot 2.8 \cdot 0.9s^{-1}} = 0.016s.$$

Using the latter, the outer open loop transfer function reads

$$\begin{aligned} G_{02}(s) &= \frac{K_{P2} \cdot (1 + sK_{I2})}{s \cdot K_{I2}} \cdot G_{w1}(s) \cdot \frac{K_{P\varphi}}{1 + s \cdot K_{DT\varphi}} \\ &= \frac{K_{P2} \cdot (1 + sK_{I2})}{s \cdot K_{I2}} \cdot \frac{1}{1 + s \cdot K_{w1}} \cdot \frac{K_{P\varphi}}{1 + s \cdot K_{DT\varphi}} \end{aligned}$$

Hence, we can already fix the integrator constant and set

$$K_{I2} := K_{w1} = 0.016s$$

and obtain the outer open loop transfer function

$$G_{02}(s) = \frac{K_{P2} \cdot K_{P\varphi}}{s \cdot K_{I2} \cdot \left(1 + s \cdot K_{DT\varphi}\right)}.$$

Therefore, the outer closed loop transfer function reads

$$G_{w2}(s) = \frac{1}{1 + \frac{1}{G_{02}(s)}} = \frac{1}{1 + \frac{s \cdot K_{I2} \cdot \left(1 + s \cdot K_{DT\varphi}\right)}{K_{P2} \cdot K_{P\varphi}}} = \frac{K_{P2} \cdot K_{P\varphi}}{K_{P2} \cdot K_{P\varphi} + s \cdot K_{I2} \cdot \left(1 + s \cdot K_{DT\varphi}\right)}.$$

For optimality, we require $|G_{w2}(i\omega)|^2 \approx 1$ and see

$$G_{w2}(i\omega) = \frac{K_{P2} \cdot K_{P\varphi}}{K_{P2} \cdot K_{P\varphi} + i\omega \cdot K_{I2} \cdot \left(1 + i\omega \cdot K_{DT\varphi}\right)}$$
$$= \frac{K_{P2} \cdot K_{P\varphi}}{K_{P2} \cdot K_{P\varphi} - \omega^2 \cdot K_{I2} \cdot K_{DT\varphi} + i\omega \cdot K_{I2}}$$

Hence, we have

$$|G_{w2}(i\omega)|^2 = \frac{K_{P2}^2 \cdot K_{P\varphi}^2}{\left(K_{P2} \cdot K_{P\varphi} - \omega^2 \cdot K_{I2} \cdot K_{DT\varphi}\right)^2 + (\omega \cdot K_{I2})^2}$$
$$= \frac{K_{P2}^2 \cdot K_{P\varphi}^2}{K_{P2}^2 \cdot K_{P\varphi}^2 - 2 \cdot K_{P2} \cdot K_{P\varphi} \cdot \omega^2 \cdot K_{I2} \cdot K_{DT\varphi} + \omega^4 \cdot K_{I2}^2 \cdot K_{DT\varphi}^2 + \omega^2 \cdot K_{I2}^2}$$
$$= \frac{K_{P2}^2 \cdot K_{P\varphi}^2}{K_{P2}^2 \cdot K_{P\varphi}^2 - \omega^2 \cdot \left(2 \cdot K_{P2} \cdot K_{P\varphi} \cdot K_{I2} \cdot K_{DT\varphi} - K_{I2}^2\right) + \omega^4 \cdot K_{I2}^2 \cdot K_{DT\varphi}^2}$$

Since we cannot influence $\omega^4 \cdot K_{I2}^2 \cdot K_{DT\varphi}^2$, we obtain

$$2 \cdot K_{P2} \cdot K_{P\varphi} \cdot K_{I2} \cdot K_{DT\varphi} - K_{I2}^2 \approx 0$$
$$\iff \quad K_{P\varphi} \approx \frac{K_{I2}}{2 \cdot K_{P2} \cdot K_{DT\varphi}} = \frac{0.016s}{2 \cdot 0.2 \cdot 0.009s} = 4.44.$$

To summarize, a cascade control shows the following advantages/disadvantages given in Table 2.1 if compared to SISO.

Table 2.1.: Advantages and disadvantages of cascade control

Advantage	Disadvantage
✓ Identical design of control per loops as for SISO	✗ More controllers required
✓ Possible integration of multiple outputs	✗ More sensors required
✓ Improved disturbance rejection	✗ Slower response due to higher order

Continued on next page

Table 2.1 – continued from previous page

Advantage	Disadvantage
✓ Improved controllability of local non-linearities	
✓ Simplified integration of input/output bounds per loop	✗ Possible wind-up for integral control caused by local bound

2.4. Disturbance control

An alternative way of utilizing inner knowledge of the system to improve its performance is the so called *disturbance control*. Disturbance control is closely connected to *feed forward control*, which we analyze first. As we will see, unstable zeros cannot be treated via feed forward but require a feedback control structure. At the same time, by design of a feedback control, a disturbance $\mathbf{d}(t)$ can only be suppressed if the reference input $\mathbf{w}(t)$ and the disturbance $\mathbf{d}(t)$ are within the same frequency range. Hence, if they exhibit different frequency ranges, then an additional component must be added to the loop. In that case, the advantages of both feed forward and feedback can be utilized to design *disturbance suppression* and *reference tracking*. Last, if the disturbance can additionally be measured, we extend the concept to disturbance control.
A feed forward control $G_F(s)$ aims to annihilate the dynamic of the system, i.e. to push the combined dynamic of the feed forward and the system to identity $G_F(s) \cdot G_S(s) \stackrel{!}{=} \mathrm{Id}(s)$. A feed forward is sketched in Figure 2.8.

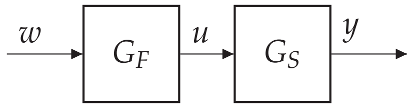


Figure 2.8.: Simple feed forward

If such an identity can be reached, that is $G_F(s) = 1/G_S(s)$, the control would truly be optimal. However, the main reasons why such a behavior is not realizable are:

1. Degree of nominator is larger than degree of denominator. This is necessary (but not realizable) since the system $G_S(s)$ exhibits degree of nominator to be smaller than degree of denominator, only in exceptional cases both are identical.

   → Remedy: Add time constants to denominator to equalize degree of nominator and denominator, for example:

$$G_S(s) = \frac{K}{(1+T_1 s)\cdot(1+T_2 s)} \longrightarrow G_F(s) = \frac{(1+T_1 s)\cdot(1+T_2 s)}{K}\cdot\frac{1}{(1+Ts)^2}$$
$$\Longrightarrow G_w(s) = \frac{1}{(1+Ts)^2}.$$

2. Negative lag time. If the system exhibits a lag time, then its inverse must have a negative lag time, i.e. a predictor. Hence, the controller is not causal and can only be applied if the output is known in advance.

   → Remedy: Ignore negative lag time, for example

$$G_S(s) = \frac{K}{(1+T_1 s)}\cdot\exp(-T_2 s) \longrightarrow G_F(s) = \frac{(1+T_1 s)}{K}\cdot\frac{1}{(1+Ts)}$$
$$\Longrightarrow G_w(s) = \frac{1}{(1+Ts)}\cdot\exp(-T_2 s).$$

3. Unstable behavior. If the system $G_S(s)$ shows unstable zeros, i.e. it is not minimal phase, then the feed forward $G_F(s)$ must have unstable poles. The latter may lead to unbounded controls and if the poles and zeros do not cancel out exactly, then the loop will be unstable.

   → Remedy: Ignore unstable zeros, for example

$$G_S(s) = \frac{K\cdot(1-T_1 s)}{(1+T_2 s)\cdot(1+T_3 s)} \longrightarrow G_F(s) = \frac{(1+T_2 s)\cdot(1+T_3 s)}{K}\cdot\frac{1}{(1+Ts)^2}$$
$$\Longrightarrow G_w(s) = \frac{(1-T_1 s)}{(1+Ts)^2}.$$

   → Remedy: Set the amplitude response $|G_w(i\omega)| = 1$. To this end, any unstable zero of the system is compensated by its complex transposed in the denominator of the control, for example

$$G_S(s) = \frac{K\cdot(1-T_1 s)}{(1+T_2 s)\cdot(1+T_3 s)} \longrightarrow G_F(s) = \frac{(1+T_2 s)\cdot(1+T_3 s)}{K\cdot(1+T_1 s)}\cdot\frac{1}{(1+Ts)}$$
$$\Longrightarrow G_w(s) = \frac{(1-T_1 s)}{(1+T_1 s)\cdot(1+Ts)}.$$

   This transfer function is also called *all pass*.

   → Remedy: Set the phase response $\angle G_w(i\omega) = 0$. To this end, any unstable zero of the system is compensated by its complex transposed in the nominator of the control, for example

$$G_S(s) = \frac{K \cdot (1 - T_1 s)}{(1 + T_2 s) \cdot (1 + T_3 s)} \longrightarrow G_F(s) = \frac{(1 + T_2 s) \cdot (1 + T_3 s) \cdot (1 + T_1 s)}{K \cdot (1 + Ts)^3}$$

$$\Longrightarrow G_w(s) = \frac{(1 - T_1 s) \cdot (1 + T_1 s)}{(1 + Ts)^3} = \frac{\left(1 - T_1^2 s^2\right)}{(1 + Ts)^3}.$$

Note that in all cases, the unstable zeros remain uncompensated. Hence, considering realizability of a control, the choice of T is always a compromise between sensitivity wrt. noise and speed of the control.

As a consequence, we know that unstable systems cannot be stabilized via feed forward, but require a feedback structure to enforce stable behavior.

Remark 2.29

In fact, unstable zeros (and weakly damped one, that is zeros close to the stability boundary) should never be canceled out to avoid instability of the control (or strong oscillations in case of weakly damped zeros).

Yet, a disturbance can only be suppressed by design of a feedback control, if the reference input and the disturbance are within the same frequency range. To tackle the case where these variables exhibit different frequency ranges, we introduce the precontrol and the equivalent prefilter concept.

To integrate a feed forward and a feedback into one loop, two structures are possible, cf. Figure 2.9 and 2.10 respectively.

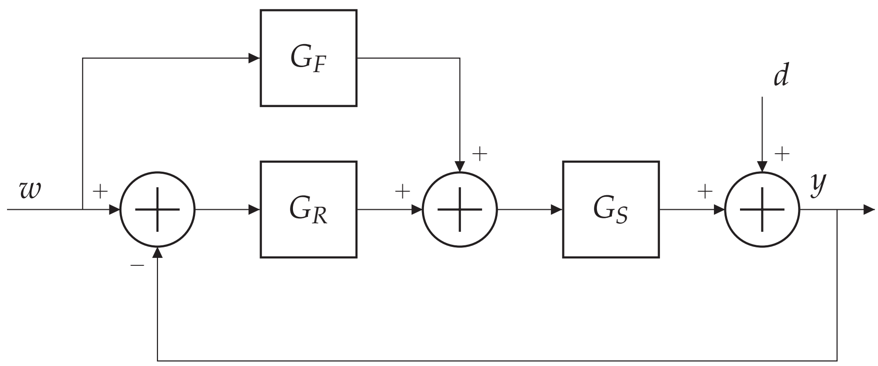


Figure 2.9.: Structure of a precontrol

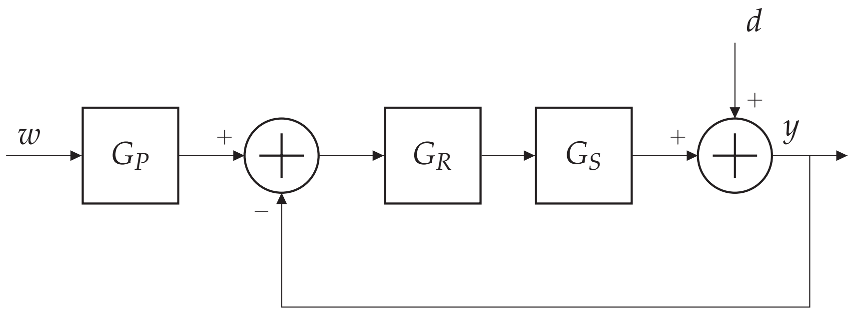


Figure 2.10.: Structure of a prefilter

Remark 2.30
Since the feed forward does not interact with the feedback, the stability properties of the closed loop remain unchanged.

The structures of the precontrol in Figure 2.9 and of the prefilter in Figure 2.10 allow to simultaneously treat reference tracking via the feedback and disturbance suppression via the feed forward. Hence, the choice of T does not depend on the necessity to find a compromise between sensitivity wrt. noise and speed of the control anymore.
More formally, we define the following:

Definition 2.31 (Transfer function precontrol).
Consider a system (1.20), which is SISO. Let $G_R(s)$, $G_F(s)$ and $G_S(s)$ denote the feedback, feed forward and system transfer functions. Moreover, let $\mathbf{w}(t)$ denote the tracking reference and $\mathbf{d}(t)$ denote the known disturbance. Last, suppose the structure of Figure 2.9 for these controllers and systems to hold. Then we call

$$\hat{\mathbf{y}}(s) = \frac{G_R(s) \cdot G_S(s) + G_F(s) \cdot G_S(s)}{1 + G_R(s) \cdot G_S(s)} \cdot \hat{\mathbf{w}}(s) \tag{2.21}$$

transfer function of a precontrolled system.

Definition 2.32 (Transfer function prefilter).
Consider a system (1.20), which is SISO. Let $G_R(s)$, $G_F(s)$ and $G_S(s)$ denote the feedback, feed forward and system transfer functions. Moreover, let $\mathbf{w}(t)$ denote the tracking reference and $\mathbf{d}(t)$ denote the known disturbance. Last, suppose the structure of Figure 2.10 for these controllers and

systems to hold. Then we call

$$\hat{\mathbf{y}}(s) = \frac{G_P(s) \cdot G_R(s) \cdot G_S(s)}{1 + G_R(s) \cdot G_S(s)} \cdot \hat{\mathbf{w}}(s) \tag{2.22}$$

transfer function of a prefiltered system.

If we compare both approaches, then we have identical behavior if the transfer functions are identical, i.e.

$$\begin{aligned} \frac{G_R(s) \cdot G_S(s) + G_F(s) \cdot G_S(s)}{1 + G_R(s) \cdot G_S(s)} &= \frac{G_P(s) \cdot G_R(s) \cdot G_S(s)}{1 + G_R(s) \cdot G_S(s)} \\ \iff G_R(s) \cdot G_S(s) + G_F(s) \cdot G_S(s) &= G_P(s) \cdot G_R(s) \cdot G_S(s) \\ \iff G_R(s) + G_F(s) &= G_P(s) \cdot G_R(s). \end{aligned}$$

The latter reveals the following equivalency result:

Theorem 2.33 (Equivalency precontrol and prefilter).
Consider the precontrol and prefilter as given in Definition 2.31 and 2.32 respectively. If

$$G_P \equiv 1 + \frac{G_F(s)}{G_R(s)} \tag{2.23}$$

holds, then the transfer functions of precontrol and prefilter are identical.

The design of a precontrol is done with the following two steps:

Algorithm 2.34 (Design precontrol)
Consider a control system as illustrated in Figure 2.9.

(1) Design the feed forward $G_F(s)$ to be (approximately) the inverse of the system $G_S(s)$.

(2) Design the feedback $G_R(s)$ such that the disturbance $\mathbf{d}(t)$ is suppressed as good as possible.

The idea to design a prefilter is exactly the other way around:

Algorithm 2.35 (Design prefilter)
Consider a control system as illustrated in Figure 2.10.

(1) Design the feedback $G_R(s)$ such that the disturbance $\mathbf{d}(t)$ is suppressed as good as possible.

(2) Design the feed forward $G_F(s)$ to be (approximately) the inverse of the closed loop system.

Remark 2.36
Note that the precontrol — in contrast to the prefilter — does not depend on the closed loop. Consequently, there is no need to adapt it in case the closed loop is optimized or changed after the design process is finished.

Remark 2.37
The prefilter can be used to compensate for unwanted behavior of the closed loop caused by large values K_D of the D part of a closed loop control. Large K_D may be wanted to stabilize or improve performance of the closed loop, yet its impact on zeros of the closed loop may lead to large overshoots. The prefilter can be used to cancel out these zeros, hence the D part can be designed without having to worry about possible negative impacts.

Task 2.38
Consider the (critically stable) system given by the transfer function

$$G_S(s) = \frac{5}{s \cdot (s+5)^2}.$$

Since the system exhibits an explicit I part, the feedback is designed as a PD controller following

$$G_R(s) = K_P \cdot (s+1)\,.$$

Design a prefilter, which leaves the amplitude response unchanged.

Solution to Task 2.38: For the closed loop we obtain the transfer function

$$G_w(s) = \frac{\frac{5 \cdot K_P \cdot (s+1)}{s \cdot (s+5)^2}}{1 + \frac{5 \cdot K_P \cdot (s+1)}{s \cdot (s+5)^2}} = \frac{5 \cdot K_P \cdot (s+1)}{s^3 + 10s^2 + (5K_P + 25)\,s + 5K_P}.$$

Hence, we observe an unwanted zero in the nominator given by $(s+1)$, which need to be included in the prefilter. To ensure unchanged amplitude response, we require $G_P(0) = 1$. Consequently, we design

$$G_P(s) = \frac{1}{(s+1)}.$$

A different approach can be used if the disturbance can be measured. In this case, the system no longer exhibits only one but two outputs and may additionally show a disturbance transfer function. Since the system is disturbed, a feedback is required to stabilitze it. The feedback is based on the output of the system. As the disturbance itself cannot be influenced, no closed loop can be used to compensate for the disturbance. Yet, a feed forward can be applied based on the output of the disturbance to update the feedback. A sketch of the system is given in Figure 2.11.

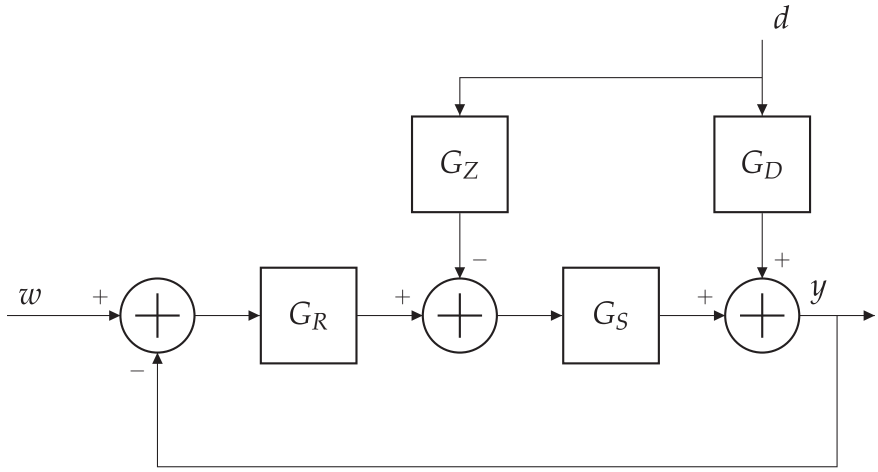


Figure 2.11.: Structure of a disturbance control

More formally, we define the following:

Definition 2.39 (Transfer function disturbance control).
Consider a system (1.20) with one input and two outputs. Moreover, let $\mathbf{w}(t)$ denote the tracking reference and $\mathbf{d}(t)$ denote the disturbance, which can be measured by one of the outputs. Let $G_D(s)$ and $G_S(s)$ denote the system and disturbance transfer functions and let $G_R(s)$ and $G_Z(s)$ denote the feedback and feed forward control. Last, suppose the structure of Figure 2.11 for these controllers and systems to hold. Then we call

$$\hat{\mathbf{y}}(s) = \frac{G_R(s) \cdot G_S(s)}{1 + G_R(s) \cdot G_S(s)} \cdot \hat{\mathbf{w}}(s) + \frac{G_D - G_S \cdot G_Z}{1 + G_R(s) \cdot G_S(s)} \cdot \hat{\mathbf{d}}(s) \tag{2.24}$$

transfer function of a disturbance control system.

We directly observe that

1. the ideal compensation of the disturbance $G_D(s)$ is given by

$$G_Z(s) = \frac{G_D(s)}{G_S(s)},$$

which cancels out the disturbance dependent component of (2.24), and

2. the stability of the system is not influenced by the additional component if the disturbance influence is canceled out.

Remark 2.40
In contrast to precontrol, a disturbance control is (typically) more easily realizable. The reason is that (at least typically) the impact of the disturbance is lagged. The more lagged $G_D(s)$ is compared to $G_S(s)$, the higher will be its low pass behavior and the easier will be the realization of $G_Z(s)$.

Task 2.41
Consider the system transfer function

$$G_S(s) = \frac{K}{(1 + T_1 s) \cdot (1 + T_2 s)} \cdot \exp(-2s).$$

Compute the ideal and realizable disturbance controls for the cases $G_D(s) = 1$ and $G_D(s) = \exp(-3s) / (1 + T_3 s)$.

Solution to Task 2.41: Considering $G_D(s) = 1$ we obtain

$$G_Z(s) = \frac{(1 + T_1 s) \cdot (1 + T_2 s)}{K} \cdot \exp(2s).$$

Ignoring the negative lag time and equalizing degree of nominator / denominator reveals the possible realization

$$G_Z(s) = \frac{(1 + T_1 s) \cdot (1 + T_2 s)}{K \cdot (1 + Ts)^2}.$$

Considering $G_D(s) = \exp(-3s) / (1 + T_3 s)$ we get

$$G_Z(s) = \frac{(1 + T_1 s) \cdot (1 + T_2 s)}{K \cdot (1 + T_3 s)} \cdot \exp(-s).$$

Since the lag time is positive, we only need to equalize degree of nominator / denominator, which reveals the possible realization

$$G_Z(s) = \frac{(1 + T_1 s) \cdot (1 + T_2 s)}{K \cdot (1 + T_3 s) \cdot (1 + Ts)}.$$

Recall that for $T \to 0$ the ideal inverse, and respectively the perfect disturbance suppression, is obtained. Yet, for smaller T the derivative character of the control is increasing and hence the input magnitude is inverse proportionally rising with the decrease of T. The latter may quickly result in violating input constraints.
To summarize, if compared to prefilter/precontrol a disturbance control shows the following advantages/disadvantages given in Table 2.2.

Table 2.2.: Advantages and disadvantages of disturbance control as compared to prefilter/precontrol

Advantage	Disadvantage
✓ Identical design of feed forward and feedback as for SISO	✗ No compensation of D character in closed loop
✓ Possible integration of multiple outputs	✗ D character for suppression
✓ Improved disturbance suppression	✗ More sensors required
✓ Improved realizability	✗ More controllers required
✓ Consideration of disturbance dynamic	✗ Possible violation of input bounds

Remark 2.42
In the literature, two special cases of the cascade and the precontrol are known. If the cascade exhibits only two loops, it is also referred to as auxiliary feedback. If a second input u is available, the concept of precontrol can be applied to the controllable subsystem of the second input which is referred to as auxiliary feed forward.

CHAPTER 3

COMPLEX CONTROL STRUCTURES

In the upcoming chapter, we consider control structures, which extend the standard setting of P, PI, PID, PIDT and PIDT-T_t considered so far in different directions. First, for practical applications, the electric/electronic realization of these controllers is (despite their simplicity) too complex and expensive. To address this issue, we consider *bang-bang* and *double-setpoint* control in the following Section 3.1. The simplification of these controllers lies in their operating range, which consists of 2 (or 3) operating points only, e.g. on/off switches or gear shifts.
To address more complex control architectures with several inputs and outputs, we already saw in the previous Chapter 2 how prefilter, precontrol, cascade control and disturbance control can be applied. The more general setting of multi-input multi-output systems will be addressed in Section 3.3.

3.1. Bang-bang and double-setpoint control

The first class of complex control structures we consider are switches. These elements represent discontinuous controls and are therefore by definition nonlinear and cannot be linearized. For a *bang-bang control*, a realization could be a switch with „on“ and „of“ settings. For an engine, a *double-setpoint control* may be the settings „drive“, „reverse“ and „neutral“. The typical applications for bang-bang control are in the range of simple temperature and pressure regulation whereas double-setpoint control is applied to motors.
Switching devices exists in a variety of realizations, cf. Table 3.1.

Table 3.1.: Technical possibilities of continuous and switching actuators

	Continuous actuator	**Switching actuator**
Valve	Proportional value, servo valve, nozzle, slit	Shift valve
Resistor	Potentiometer	Relay, switch, contactor, transistor, thyristor
Clutch	Friction clutch, converter	Clutch coupling

Practitioners use such devices as they are very cheap, more robust, require less maintenance, are smaller, simpler, consist of less parts and exhibit a higher degree of efficiency. On the downside, however, bang-bang and double-setpoint controllers induce oscillations, which may lead to resonances, noise and degrade comfort. Moreover, due to switching, these elements show higher wearing and a limited lifespan. Additionally, the control shows a slower response on the system due to integrating the pulse waves of the input. The most dreadful disadvantage is yet the complexity of modeling and evaluation. Here, we will particularly focus on the last issue.
The block diagrams of bang-bang and double-setpoint control are given in Figures 3.1 and 3.2 respectively.

Remark 3.1
Note that the right part of each figure represents the control with hysteresis, i.e. the case when shifting up/down is not done at the same value. The idea of the latter is to avoid repeated and fast switching (at the cost of potentially larger oscillations).

(a) Bang bang control   (b) Bang bang control with hysteresis

Figure 3.1.: Block diagram of a bang bang control

Formally, a bang-bang controller is a system with the following properties:

Definition 3.2 (Bang-bang control).
Consider a system $f : \mathcal{U} \to \mathcal{Y}$ with $\mathcal{Y} := [y_{\min}, y_{\max}] \subset \mathbb{R}$ such that $\mathcal{Y} \neq \emptyset$. Furthermore,

(a) Double-setpoint control

(b) Double-setpoint control with hysteresis

Figure 3.2.: Block diagram of a double-setpoint control

consider a threshold $\theta \in \mathcal{U}$ to be given. Then we call f to be bang-bang iff

$$f(u) = \begin{cases} y_{\min}, & \text{if } u \leq \theta \\ y_{\max}, & \text{if } u > \theta \end{cases} \tag{3.1}$$

is the setting command of the system.

Remark 3.3

For simplicity reasons, the threshold value is typically set to zero.

Remark 3.4

We like to stress that — in contrast to all control types considered so far — a bang-bang control is by definition nonlinear. Hence, the principles of superposition, amplification and commutativity do not hold in general.

The output of a bang-bang control as given in Figure 3.1(left) takes the form of a so called *pulse modulation function*. To generate such a signal, several possibilities exist, which include

- pulse width modulation (PWM),
- pulse frequency modulation (PFM), and
- pulse amplitude modulation (PAM).

The ideas of these modulations can be captured as follows:

- PWM: The amplitude of output is fixed. The length of the impulse depends on the amplitude of the input signal.
- PFM: The amplitude of output is fixed. The frequency of the pulse depends on the amplitude of the input signal.

- PAM: The amplitude of output depends on amplitude of input. the frequency of the pulse is fixed.

In practice, all three options are applied, yet due to its similarity to electronics, PWM is the most common one. Here, we focus on PWM only.
The idea of pulse modulation is to generate a switching function to mimic a continuous control like a PID. To this end, the signal to be mimiced is required as an input and is compared to a so called generator function, cf. Figure 3.3 for the general setting.

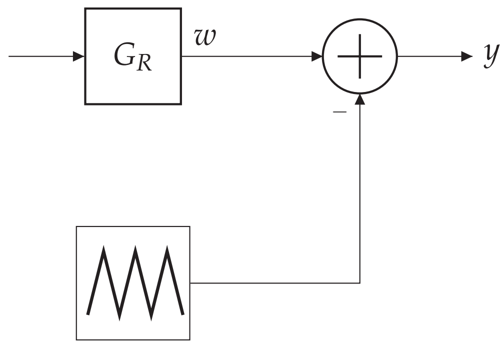


Figure 3.3.: Mimicry of a continuous input function

The easiest way is to generate a PWM signal is to apply a triangle function and compare it to a reference signal, cf. Figure 3.4. Note that the reference $w(t)$ in Figures 3.3, 3.4 is the input for our controller. Whenever the triangle function is less than the reference, the lower bound is applied. If it is higher, then the upper bound is used.

Remark 3.5
Note that other funtions like a sawtooth or delta function with respect to limits can be applied.

If a bang-bang control is applied, one has to face the problem that the reference w is typically not asymptotically stabilized. The only exception is if either $(w, u_{\min})$ or $(w, u_{\max})$ is an operating point of the system. In any other case, applying either $u_{\min}$ or $u_{\max}$ to the system results in a deviation. Once the error passes the threshold, the deviation in turn leads to a switch of the control. The switch of the control induces a change of direction of the development of the error, which at some point results in another switch of the control.
As a result, the control *chatters*. To reduce such a behavior, hysteresis is introduced:

Definition 3.6 (Bang-bang control with hysteresis).
Consider a system $f : \mathcal{U} \to \mathcal{Y}$ with $\mathcal{Y} := [y_{\min}, y_{\max}] \subset \mathbb{R}$ such that $\mathcal{Y} \neq \emptyset$. Furthermore,

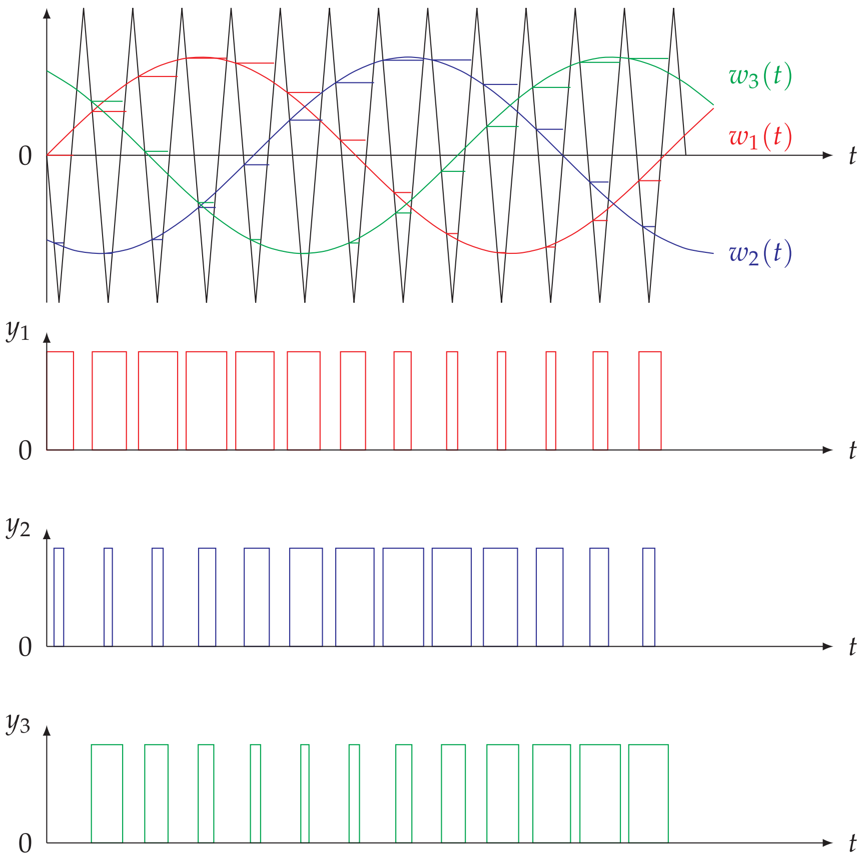


Figure 3.4.: Sketch of a pulse width modulation using triangle functions

consider a threshold $\theta \in \mathcal{U}$ and a hysteresis width $\Delta > 0$ to be given. Then we call f to be bang-bang with hysteresis iff

$$f(u) = \begin{cases} y_{\min}, & \text{if } u \leq \theta - \Delta \\ y_{\max}, & \text{if } u \geq \theta + \Delta \end{cases} \tag{3.2}$$

is the setting command of the system.

Remark 3.7

Note that bang-bang with hysteresis is not a function. It exhibits a region where the output may either be $y_{\min}$ or $y_{\max}$.

Note that even with hysteresis, the closed loop for any system (cf. Figure 3.5) will still oscillate around the reference.

Here, the following holds:

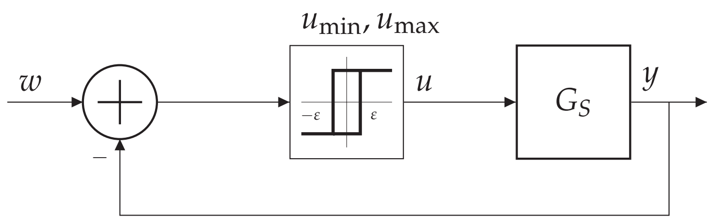


Figure 3.5.: Closed loop with bang-bang control

Theorem 3.8 (Oscillation for bang-bang control w/o hysteresis).
Consider a closed loop system as given in Figure 3.5 and suppose a controllable operating point $x^\star$ to exist for system G_S and the corresponding control $u(\cdot)$ to satisfy $u(\cdot) \in \{u_{\min}, u_{\max}\}$. Then the closed loop from Figure 3.5 is stable and the amplitude of the oscillation of the closed loop is directly proportional to the hysteresis width Δ whereas the frequency of the oscillation is directly proportional to the control bounds $(u_{\max} - u_{\min})/\theta$.

In order to still be fairly close to the reference, a fast reaction to even little changes is required. To this end, the deviation from the reference needs to be amplified to cross the threshold of the bang-bang controller even for small deviations. Still, we don't want the control to react too often. To this end, high frequency changes are filtered out.

Remark 3.9
High frequency changes are often subject to measurement errors or unmodeled elements of the system. In practice, a low pass can be applied to filter these occurrences and smooth the system output.

Hence, we end up with a combination of a bang-bang control with a low pass and a control amplifier as illustrated in Figure 3.6

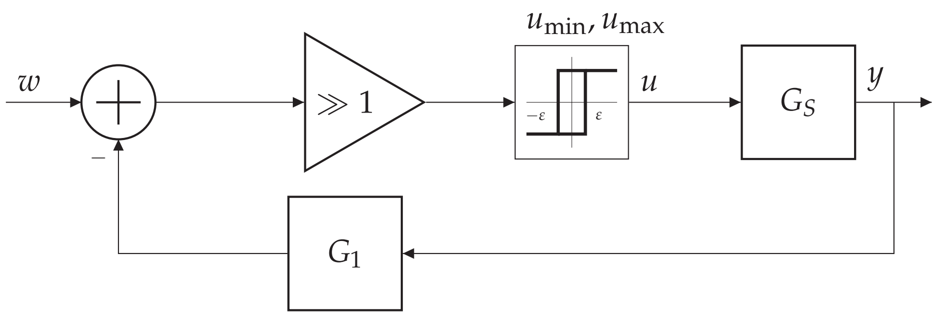


Figure 3.6.: Closed loop with bang-bang control with low pass and amplifier

Considering the impact of the gain and low pass elements, we want to assess the impact on Theorem 3.8. To this end, we denote the transfer function of the gain by G_{gain} and of the combination by G_{adapt}. Since G_{gain} is chosen large it will dominate the control and system transfer function, which allows us to neglect them and obtain

$$G_{\text{adapt}} = \frac{G_{\text{gain}}}{1 + G_1 \cdot G_{\text{gain}}} = \frac{G_{\text{gain}}}{G_{\text{gain}} \cdot \left(\underbrace{\frac{1}{G_{\text{gain}}}}_{\approx 0} + G_1 \right)} \approx \frac{1}{G_1}$$

Hence, we directly get the following:

Theorem 3.10 (Oscillation for bang-bang control with low pass and amplifier).
Consider a closed loop system as given in Figure 3.6 and suppose a controllable operating point $x^\star$ to exist for system G_S and the corresponding control $u(\cdot)$ to satisfy $u(\cdot) \in \{u_{\min}, u_{\max}\}$. Then the closed loop from Figure 3.6 is stable and the amplitude of the oscillation of the closed loop is directly proportional to Δ / G_{gain} whereas the frequency of the oscillation is directly proportional to $(u_{\max} - u_{\min}) / (\theta \cdot G_1)$.

Remark 3.11
A direct conclusion from Theorem 3.10 is that systems with only slow frequencies are ideally suited for control via bang-bang controllers. For such systems, the output is almost linear and performance analysis can be done via the mean of the output.

Still, the chattering behavior only be reduced by these extensions, yet not avoided. A complete and general avoidance is also not possible, yet in particular cases a resolution can be found. These cases refer to systems which exhibit an I like behavior close to the reference. The reason why a resolution is possible is that for I like behavior close to the reference the control input satisfies $u \equiv u^\star$. Hence, we can introduce a third state into the bang-bang controller, which reveals exactly that value within a so called *dead zone* of the system, i.e. the neighborhood of the reference. We define the following:

Definition 3.12 (Double-setpoint control with hysteresis).
Consider a system $f : \mathcal{U} \to \mathcal{Y}$ with $\mathcal{Y} := [y_{\min}, y_{\max}] \subset \mathbb{R}$ such that $\mathcal{Y} \neq \emptyset$. Furthermore, consider a control value $y^\star \in \mathcal{Y}$, a threshold $\theta \in \mathcal{U}$ and two hysteresis widths $\Delta_2 > \Delta_1 > 0$ to

be given. Then we call f to be double-setpoint with hysteresis iff

$$f(u) = \begin{cases} y^\star, & \text{if } u \leq \theta - \Delta_1 \\ y^\star, & \text{if } u \geq \theta + \Delta_1 \\ y_{\min}, & \text{if } u \leq \theta - \Delta_2 \\ y_{\max}, & \text{if } u \geq \theta + \Delta_2 \end{cases} \tag{3.3}$$

is the setting command of the system.

Remark 3.13
Note that the I like behavior of a system can be forced to exist by adding an integrator between the double-setpoint controller and the system.

Extending our system setup from Figure 3.6 by a double-setpoint controller and forcing applicability by incorporating an integrator in Figure 3.7, we can actually show the following remarkable result:

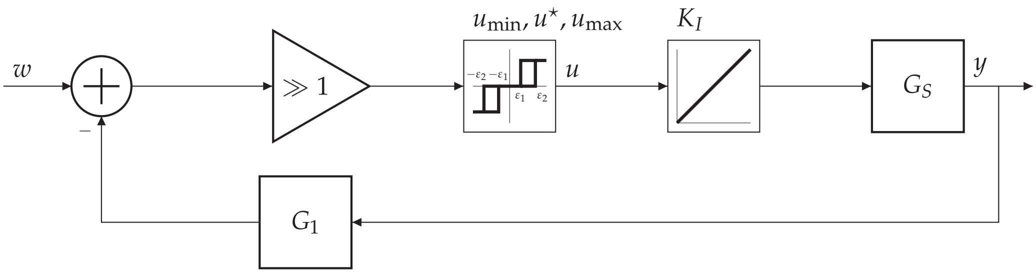


Figure 3.7.: Closed loop with double-setpoint control with low pass, amplifier and integrator

Theorem 3.14 (Asymptotic stability for double-setpoint control with hysteresis).
Consider a closed loop system as given in Figure 3.7 and suppose a controllable operating point $(x^\star, u^\star)$ to exist for system G_S and the corresponding control $u(\cdot)$ to satisfy $u(\cdot) \in \{u_{\min}, u_{\max}\}$. Then the closed loop from Figure 3.7 is asymptotically stable.

One can even go one step further and minimize the number of switches that are necessary to reach the reference value. To this end, a latency can be introduced to decelerate the speed of the control, cf. Figure 3.8.
For such a structure, we can show an extension of Theorem 3.14 revealing:

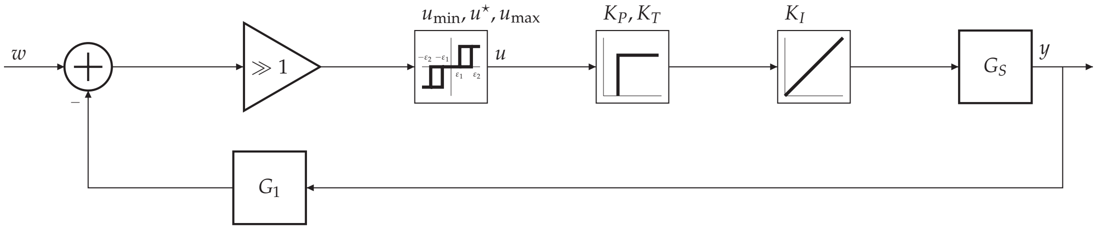


Figure 3.8.: Closed loop with double-setpoint control with low pass, amplifier, latency and integrator

Theorem 3.15 (Minimal asymptotic stability for double-setpoint control with hysteresis).
Consider a closed loop system as given in Figure 3.8 and suppose a controllable operating point $(x^\star, u^\star)$ to exist for system G_S and the corresponding control $u(\cdot)$ to satisfy $u(\cdot) \in \{u_{\min}, u_{\max}\}$. Then the closed loop from Figure 3.8 is asymptotically stable and only a finite number of switches of the control occur.

Including the latency, however, comes at the price of reduced convergence speed of the closed loop. Additionally, the dead zone depends on the latency leading to the problem of balancing the low pass and the latency parameters.
Apart from influencing the switching behavior, one can also adapt the structure to mimic the behavior of a continuous controller. The following three structures are common:

Theorem 3.16 (Mimic continuous PD, PID, PI control).
Consider the system given in Figures 3.9, 3.10 and 3.11. Let the reference $w(\cdot)$ be generated by a PD, PID and PI controller according to Figure 3.3. Then the transfer function is approximately equal to a PD, PID and PI controller respectively.

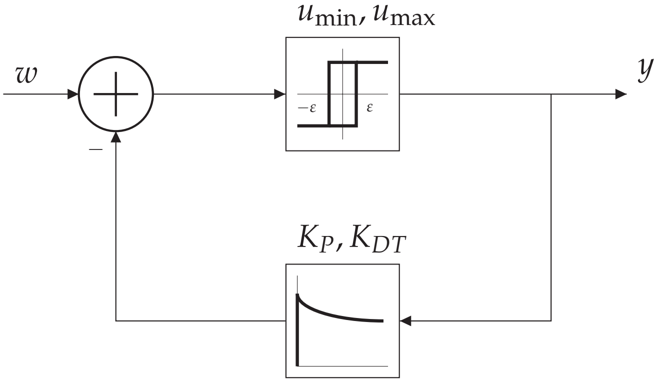


Figure 3.9.: Mimic PD control

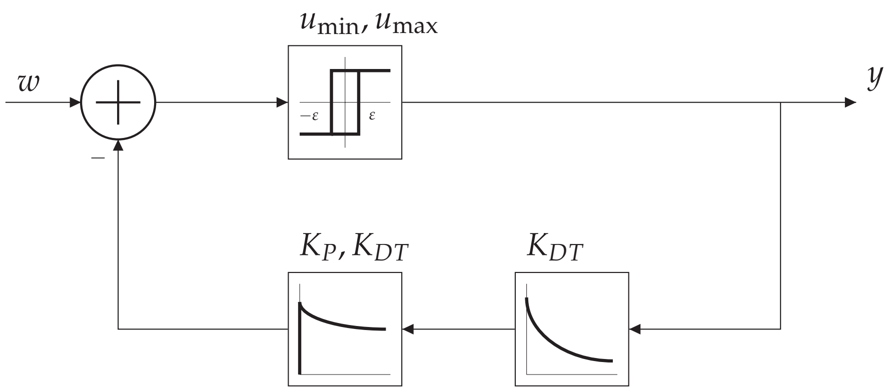


Figure 3.10.: Mimic PID control

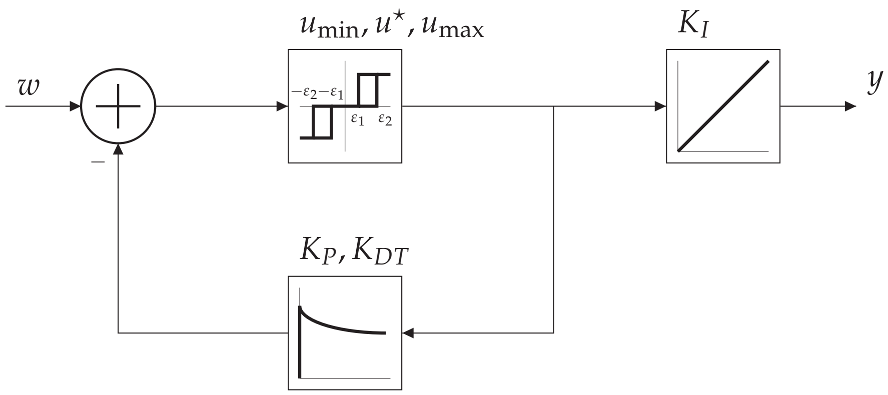


Figure 3.11.: Mimic PI control

Task 3.17

Evaluate the transfer functions from Figures 3.9, 3.10 and 3.11 to show the approximate equivalence with respect to the continuous controllers.

Solution to Task 3.17: Left to the user.

Summarizing, bang-bang and double-setpoint controllers show the advantages and disadvantages given in Tables 3.2 and 3.3.

Table 3.2.: Advantages and disadvantages of bang-bang

Advantage	Disadvantage
✓ Proven stability	✗ Remaining oscillations
✓ Full dynamics using control bounds	✗ High wear of components
✓ Reduction of oscillations	✗ Reduction increases switching frequency
✓ Low pass compensation possible	✗ Reduction may require low pass feedback
✓ Analysis via mean	✗ Mean requires low pass feedback

Table 3.3.: Advantages and disadvantages of double-setpoint control

Advantage	Disadvantage
✓ Proven asymptotic stability	✗ More cost intensive
✓ Avoidance of oscillations	✗ Limited to I type systems
✓ Full dynamics using control bounds	✗ May require I component
✓ Minimal number of switches	✗ Reduced control dynamic
✓ Reduced wear of components	✗ Balance of low pass and latency
✓ Analysis via mean	✗ Mean requires low pass feedback

At this point, we like to come back to our Remark 3.4 stating that bang-bang and therefore also double-setpoint are nonlinear components. The ideas used in the results shown before follow one idea only: Additional components are included in the closed loop to simplify, transform and compensate nonlinearities and map the system to a linear one. To this end, adding low pass and amplifier in Figure 3.6 is equivalent to reducing the neighborhood where a linearization is applied and at the same time making the linear reaction to dominate the nonlinear parts. Adding the integrator and latency in Figure 3.8 basically increases the order of the system by integration, i.e. the control is applied as to a derivative of the system, and compensating for tardiness of the system wrt. the control.

In general, even more complex connections can be drawn as we will see in the following section.

3.2. Characteristic map

From Section 3.1 we have seen a nonlinear yet very special control component. In practice, more complex systems connecting input and output are possible as illustrated in Figure 3.12.

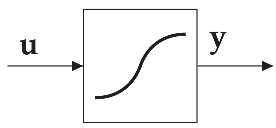


Figure 3.12.: Nonlinear static system

The map of the system is not necessarily given by a function as we already indicated in Definition 1.1 of a system. Instead, also (possibly multidimensional) data sets can be used. In that case, the map is defined as follows:

Definition 3.18 (Characteristic map).
Consider two sets $\mathcal{U}$ and $\mathcal{Y}$ and a set $\mathfrak{U} \times \mathfrak{Y} \subsetneq \mathcal{U} \times \mathcal{Y}$. Then we call $f : \mathcal{U} \to \mathcal{Y}$ satisfying

$$f(\mathbf{u}) = \begin{cases} \mathbf{y}, & \forall\ (\mathbf{u}, \mathbf{y}) \in \mathfrak{U} \times \mathfrak{Y} \\ g(\mathbf{u}, \mathfrak{U} \times \mathfrak{Y}), & \forall\ (\mathbf{u}, \mathbf{y}) \notin \mathfrak{U} \times \mathfrak{Y} \end{cases} \tag{3.4}$$

characteristic map.

For the realization of a map, the function $g(\cdot, \cdot)$ is typically implemented as an interpolation. A typical example of a characteristic map can be found in motor control for combustion engines. For such devices, the engine torque depends on both the engine speed and setting of the throttle valve (for Otto) or quantity of injected fuel (for Diesel).
Since the required storage rises exponentially with the dimension of the data sets, other representations of characteristic maps via

- polynoms,
- splines,
- fuzzy logics,
- neural networks, and
- associative storage

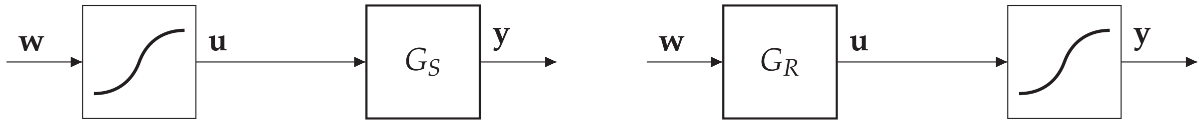
are used. In particular, polynoms and splines additionally exhibit the advantage of being differentiable instead of only continuous.

Remark 3.19
These maps (also called models) are typically identified offline using optimization methods such as regression or MINLP, or online via filter techniques such as Kalman. Modelling and identification is not within the scope of this lecture but instead treated in „Systemics“.

Such systems can often be separated into subsystems, which are either

- linear dynamic, or
- nonlinear static.

If such a separation into one of each subsystems is possible, then the structures given in Figure 3.13 are possible.



(a) Hammerstein structure (b) Wiener structure

Figure 3.13.: Separation of maps

Remark 3.20
The bang-bang and double-setpoint controller are special cases of the Hammerstein structure.

Both structures may be identical in the static and dynamic behavior in a neighborhood of an operating point, yet they exhibit very different large-signal behavior. The reason for the latter is that for nonlinear systems the principle of commutativity does not apply.
As outlined before, also superposition and amplification cannot be applied, which results in

- Blocks cannot be switched arbitrarily
- Analysis is more difficult and limited to special cases
- Complexity of results is increased
- Laplace transform is no longer applicable

As a consequence of the above, derivation of results in frequency domain is limited. Instead, the time domain is considered which we will focus on in Part II of the lecture.

3.3. Multi-input multi-output systems

The systems we considered so far exhibit two properties. For one, they are restricted to a single input and single outputs or can be treated independently. If both properties do not hold, then we call a system multi-input multi-output or short MIMO.
In particular, we define the following:

Definition 3.21 (MIMO system).
Consider a system $f : \mathcal{U} \to \mathcal{Y}$ with input and output sets $\mathcal{U}$ and $\mathcal{Y}$. If

$$\left\| \frac{\partial^2 f}{\partial u_j \partial u_k}(\mathbf{u}) \right\| \geq \theta \qquad \text{or} \qquad \left\| \frac{\partial^2 f}{\partial y_j \partial y_k}(\mathbf{u}) \right\| \geq \theta \qquad \text{or} \qquad \left\| \frac{\partial^2 f}{\partial u_j \partial y_k}(\mathbf{u}) \right\| \geq \theta \tag{3.5}$$

for indexes j, k with $\theta \in \mathbb{R}^+$ holds, then we call the system to be MIMO.

The threshold parameter θ indicates the degree of coupling of the inputs and outputs. In case the coupling is larger than the threshold, we call the coupling to be strong, otherwise weak. In case of weak coupling, the system can be decoupled into multiple single-input single-output systems (SISO), for which the coupling can be neglected. In order to be neglectable, the control needs to be designed to suppress disturbances emanating from other systems using one of the methods discussed so far. For MIMO systems, we therefore need to focus on strongly coupled systems only.
Examples for strongly coupled outputs are

- pressure and temperature for steamers,
- temperature and humidity for air conditioners,
- temperature, height of flame and mixture of stack gas for burners, or
- position, velocity and force for robot arms.

Coupled inputs may be

- position control of multiple drives for robots,
- roll and yaw angle control for flying curves with an aircraft, or
- temperature, pH measurement and biomass distribution for bio reactors.

For such systems, we utilize the concept of a transfer matrix introduced in Definition 2.8. As an extension is easily possible, we consider the setting of two inputs and two outputs only. Therefore, the setting is given as shown in Figure 3.14.

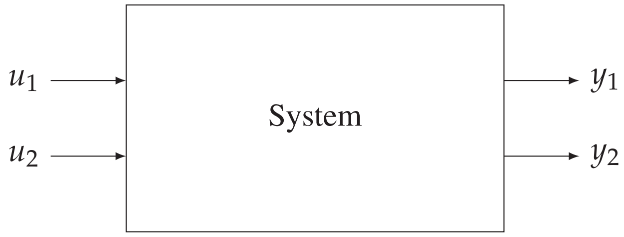
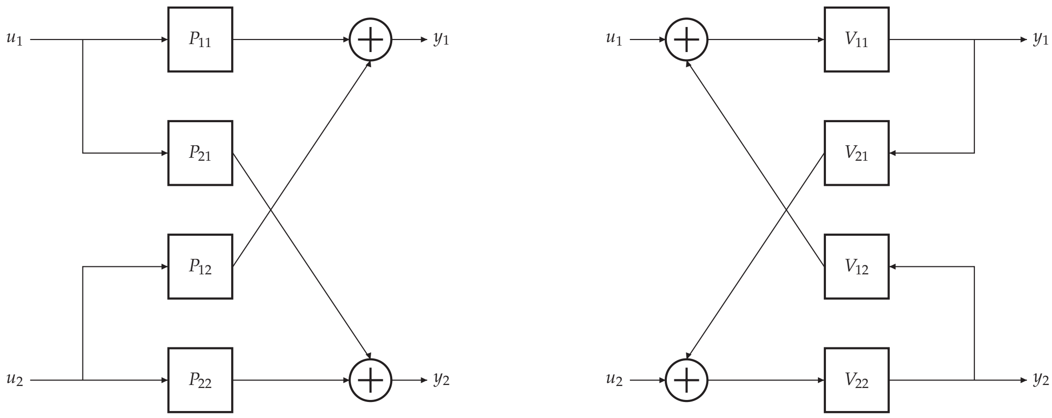


Figure 3.14.: MIMO system with two inputs and two outputs

Zooming into this setting and recalling that the inputs or outputs are strongly coupled, there are two different structures resembling feed forward and feed back connectivity.

Definition 3.22 (P canonical structure).
Consider a system with two inputs and two outputs. If the coupling of inputs to outputs exhibits a feed forward structure as shown in Figure 3.15a, then we call it P canonically structured.

Definition 3.23 (V canonical structure).
Consider a system with two inputs and two outputs. If the coupling of inputs to outputs exhibits a feedback structure as shown in Figure 3.15b, then we call it V canonically structured.



(a) P canonical structure

(b) V canonical structure

Figure 3.15.: Canonical structures of MIMO systems with two inputs and two outputs

Both structures are often found in practice showing the properties given in Table 3.4.

Table 3.4.: Properties of P and V canonical structure

P canonical structure		**V canonical structure**	
✓	Direct correspondence to transfer matrix	✗	Required transformation of transfer matrix
✗	Typically no connection to modeling	✓	Direct derivation via modeling
✓	Easy to treat	✗	Difficult to treat
✗	Typically no equivalent of P_{jk} in real system	✓	Equivalent of V_{jk} in real system
✗	Physical interpretation questionable	✓	Physical interpretation given

As the P canonical structure is more easily treatable via control methods, one typically transforms V canonical systems to P canonical structure. To this end, we have

$$\begin{aligned}
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} \cdot \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & V_{12} \\ V_{21} & 0 \end{bmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \\
&= \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & V_{11} \cdot V_{12} \\ V_{22} \cdot V_{21} & 0 \end{bmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\end{aligned}$$

revealing the linear equation system

$$\begin{aligned}
\begin{bmatrix} 1 & -V_{11} \cdot V_{12} \\ -V_{22} \cdot V_{21} & 1 \end{bmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
\iff \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{bmatrix} 1 & -V_{11} \cdot V_{12} \\ -V_{22} \cdot V_{21} & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\end{aligned}$$

Theorem 3.24 (Equivalence P and V canonical structure).
Consider two systems with two inputs and two outputs to be given. Suppose one system is in P canonical structure and one in V canonical structure. If

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1 & -V_{11} \cdot V_{12} \\ -V_{22} \cdot V_{21} & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} \tag{3.6}$$

holds, then the transfer matrices of both systems are equivalent.

To treat such MIMO systems, one can distinguish between three concepts.

- Decentralized control: For each input/output pair we design exactly one control. For each pair, the input of the other pair is considered to be a disturbance.
- Decoupling control: For each input/output pair we design one main control and for all couplings one decoupling control. The task of the latter is to reduce or eliminate the input of other pairs such that the pairs can be treated separately.
- Multivariable control: The control exhibits as many inputs and outputs as the system does.

Note that decentralized control can only be applied for weakly coupled systems. The reason for that is due to the wrong assumption of the coupling to be a disturbance, i.e. an independent input. Since the coupling is driven by the variables of the control loop, they are, however, not independent. As a consequence, stability issues (e.g. by shifting poles) may arise.
In the following, we focus on decoupling control and consider multivariable control in the time domain setting.

Definition 3.25 (Decoupling control).
Consider a system with two inputs and two outputs in P canonical structure. If the control exhibits the structure given in Figure 3.16, then we call it decoupling control.

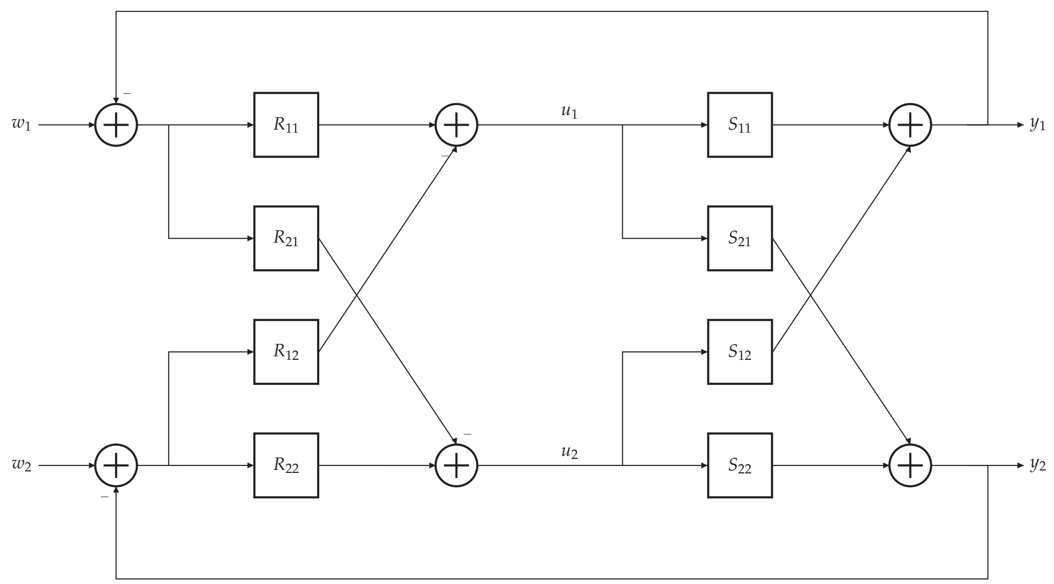


Figure 3.16.: Decoupling structure of MIMO system with P canonical structure

The idea of decoupling control is a special case of disturbance rejection, i.e. we eliminate or at least reduce the impact of the systems on one another, which allows us to apply standard methods for the decoupled circuits.

Within Figure 3.16, there are four controllers which need to be designed. While designing, the intention is that

- R_{11} shall control y_1 using u_1 (main system S_{11}),
- R_{12} shall eliminate the impact of u_2 on y_1 (coupling system S_{12}),
- R_{21} shall eliminate the impact of u_1 on y_2 (coupling system S_{21}), and
- R_{22} shall control y_2 using u_2 (main system S_{22}).

We now focus on eliminating the impact of the second system on the first, cf. Figure 3.17.

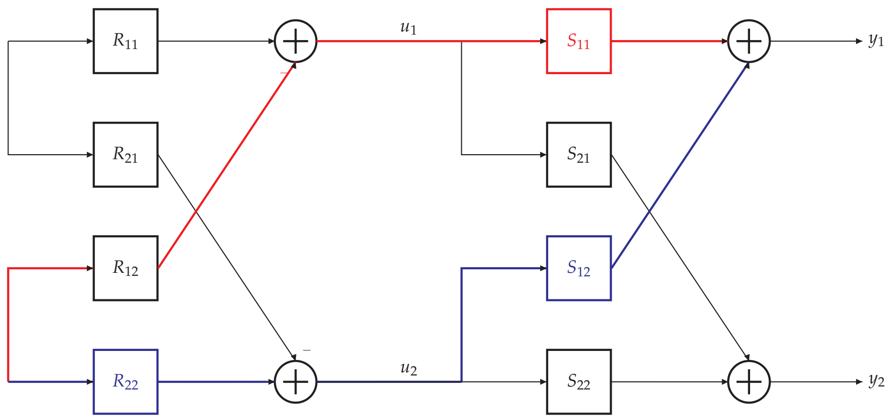


Figure 3.17.: Elimination of coupling

In order to eliminate one another, the blue and red paths in Figure 3.17 need to be identical. Hence, we directly obtain

Theorem 3.26 (Decoupling condition).
Consider a MIMO system with two inputs and two outputs in P canonical structure subject to a decoupling control. If the conditions

$$R_{12} = R_{22} \cdot \frac{S_{12}}{S_{11}} \tag{3.7}$$

$$R_{21} = R_{11} \cdot \frac{S_{21}}{S_{22}} \tag{3.8}$$

hold, then the system is decoupled.

Remark 3.27
Regarding realization, the same considerations as for disturbance control apply. If a perfect decoupling is not possible, then its impact should be reduced. In general, a decoupling is more easily achieved if the coupling is slow. Since we have

$$\textit{decoupling control} = \textit{main control} \cdot \frac{\textit{coupling system}}{\textit{main system}}$$

and the main control must satisfy degree of nominator is equal to degree of denominator, then the decoupling control is realizable if and only if the number of poles of the coupling system is larger than number of poles of the main system. Additionally, the known limitations for realizability apply, cf. Section 2.4.

As an alternative to a pathwise comparison as in Theorem 3.26, we can use the transfer matrix for decoupling. From Figure 3.16 we obtain

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} R_{11} & -R_{12} \\ -R_{21} & R_{22} \end{bmatrix} \cdot \begin{pmatrix} w_1 - y_1 \\ w_2 - y_2 \end{pmatrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

which gives us

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} R_{11}S_{11} - R_{21}S_{12} & -R_{12}S_{11} + R_{22}S_{12} \\ R_{11}S_{21} - R_{21}S_{22} & -R_{12}S_{21} + R_{22}S_{22} \end{bmatrix} \cdot \begin{pmatrix} w_1 - y_1 \\ w_2 - y_2 \end{pmatrix}. \tag{3.9}$$

Hence, the system is decoupled if the following holds:

Theorem 3.28 (Decoupling transfer matrix).
Consider a MIMO system with two inputs and two outputs in P canonical structure subject to a decoupling control. If the matrix in Equation 3.9 is diagonal, then the system is decoupled.

Remark 3.29
Note that both approaches reveal identical conditions.

It is of particular importance that even in the case of ideal decoupling, each system depends on both the main and the decoupling control. As a consequence, if we want to adapt the main controller in a later stage of the development, the decoupling controller needs to be adapted as well. One way to circumvent this problem is to consider a slight modification of the decoupling circuitry, cf. Figure 3.18.

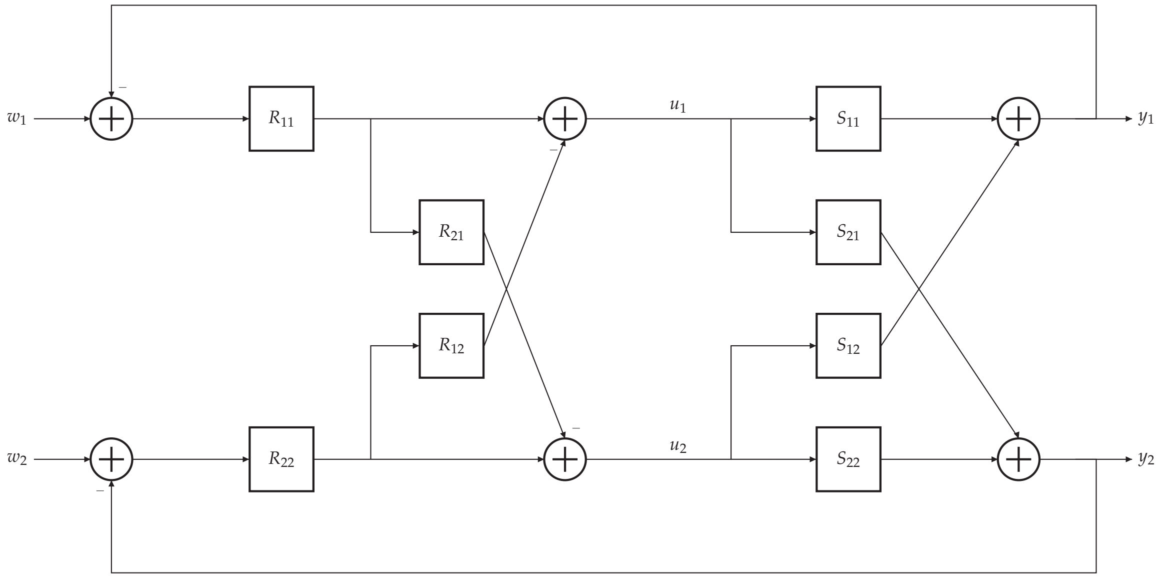


Figure 3.18.: Adaptable decoupling structure of MIMO system with P canonical structure

In this case, compensation is achieved if

$$R_{12} = \frac{S_{12}}{S_{11}}$$
$$R_{21} = \frac{S_{21}}{S_{22}},$$

i.e. the decoupling control is only subject to the input of the main system and therefore independent from the main controller.

Remark 3.30
Alternatively, the decoupling control can be designed as a V canonical structure. The advantage of that approach is that for the design of the main control no aspect of the respective other system needs to be considered.

Table 3.5.: Advantages and disadvantages of MIMO control

Advantage	Disadvantage
✓ Allows for MIMO structure	✗ Computationally involved
	Continued on next page

Table 3.5 – continued from previous page

Advantage	**Disadvantage**
✓ Standardized P and V structures	✗ Any structure limited in either usage or derivation
✓ Decoupling possible	✗ Require additional controllers
✓ Allows for independent design	✗ Requires specific decoupling structure
✓ Allows for SISO methods	✗ Not suitable for multivariate control

Yet, as we can see from the involved design process of main and decoupling controller, this approach does not allow for scalability to high dimensional MIMO systems. To consider the latter, we will shift our solution approach to the time domain.

Part II.

Time Domain

CHAPTER 4

NONLINEAR CONTROL SYSTEMS

In Chapter 1, we introduced the concept of a generic system and thereafter discussed the property of stability and how to design feed forward and feedback control laws to enforce this property of a system. In frequency domain, we observed that for more complex structures of the system and/or of the control, the design of the latter becomes more and more involved. Additionally, we saw that asymptotic stability was in most cases out of scope for the design methods. Now, we shift our view to the time domain and in particular on nonlinear systems. Again, our interest lies in showing (asymptotic) stability of a system, and respective ideas of designing controls to guarantee this property. Starting point in Section 4.1 will be the notion of asymptotic stability, where we will directly jump to nonlinear systems. As we will see, if we move from linear to nonlinear systems, it is not entirely clear whether of not a continuous stabilizing control exists. To this end, we discuss Brockett's condition [2] and Artstein's counterexample [1]. In Section 4.2, we then discuss alternative concepts for equivalent definitions of stability and controllability. Here, we follow the approach of Khalil [9] and Isidori [8]. These concepts allow us to foster properties of the system dynamics to derive stabilizing controls. In Section 4.3, we will utilize one of these properties to derive the so called *Sontag's formula* for computing an asymptotically stabilizing feedback based on a Control-Lyapunov function [18]. Moreover, we will introduce the concept of *backstepping* to construct such a Control-Lyapunov function [8].

4.1. Necessary conditions for controllability

Let us first recall the term (asymptotic) controllability from Definition 1.29. In particular, for a system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \tag{4.1}$$

we require that there exists a control $\mathbf{u}$ such that bounded deviations from an operating point result in bounded behavior, that is

$$\|\mathbf{x}_0\| \leq \delta \implies \|\mathbf{x}(t)\| \leq \varepsilon \qquad \forall t \geq 0, \tag{4.2}$$

and that any bounded deviation will be controlled to the operating point, i.e.

$$\lim_{t\to\infty} \mathbf{x}(t) = 0 \qquad \forall \|\mathbf{x}_0\| \leq r. \tag{4.3}$$

In the linear case, we saw that a linear control $\mathbf{u}$ in both feed forward and feedback form can always be constructed in „Control Engineering 1“, i.e. for each $\mathbf{u} : \mathcal{T} \to \mathcal{U}$ we can construct $\mathbf{u} : \mathcal{X} \to \mathcal{U}$ and vice versa. For the nonlinear case, the latter does not hold true. In particular, we only have the following quite intuitive connection between existence of feedback and feed forward control to still hold true:

Lemma 4.1
Consider a system (4.1) *and let* $(\mathbf{x}^\star, \mathbf{u}^\star)$ *be an operating point. If a feedback* $\mathbf{u} : \mathcal{X} \to \mathcal{U}$ *exists such that the closed loop is asymptotically stable and additionally both the feedback and the closed loop are Lipschitz, then there exists a feed forward* $\mathbf{u} : \mathcal{T} \to \mathcal{U}$ *such that the system is asymptotically controllable.*

Remark 4.2
Lipschitz continuity is a crucial element within Lemma 4.1 as it allows us to invert the system and derive the controllability property.

So the first question to be answered is how to construct a Lipschitz continuous feedback in time domain. Here, the following core result holds true:

Theorem 4.3 (Brockett's condition).
Consider a system (4.1) *and let* $(\mathbf{x}^\star, \mathbf{u}^\star)$ *be an operating point. Moreover, suppose a Lipschitz*

continuous feedback $\mathbf{u} : \mathcal{X} \to \mathcal{U}$ *to be given. Then the set*

$$f(\mathcal{X}) := \{\mathbf{y} \in \mathcal{X} \mid \mathbf{y} = f(\mathbf{x}, \mathbf{u}(\mathbf{x}))\} \tag{4.4}$$

contains a neighborhood of $\mathbf{x}^\star$.

The latter result allows us to utilize Taylor's theorem and obtain:

Theorem 4.4 (Exponential Stability).
Consider an operating point $(\mathbf{x}^\star, \mathbf{u}^\star)$ *of a system* (4.1) *with feedback* $\mathbf{u} : \mathcal{X} \to \mathcal{U}$ *resulting in a autonomous vector field* $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$. *Suppose* f *is continuously differentiable in a neighborhood of* $\mathbf{x}^\star$ *and that* $Df(\mathbf{x}^\star) \in \mathbb{R}^{n_x \times n_x}$ *represents the Jacobian of* f *at* $\mathbf{x}^\star$. *Then the following holds:*

1. *The operating point* $\mathbf{x}^\star$ *is (locally) exponentially stable if and only if the real parts of all Eigenvalues* $\lambda_i \in \mathbb{C}$ *of* $Df(\mathbf{x}^\star)$ *are negative.*
2. *The operating point* $\mathbf{x}^\star$ *is exponentially unstable if and only if there exists one Eigenvalue* $\lambda_i \in \mathbb{C}$ *of* $Df(\mathbf{x}^\star)$ *with positive real part.*
3. *The operating point* $\mathbf{x}^\star$ *is exponentially antistable if and only if the real parts of all Eigenvalues* $\lambda_i \in \mathbb{C}$ *of* $Df(\mathbf{x}^\star)$ *are positive.*

Theorem 4.4 gives us a remarkable insight: in a neighborhood of an operating point the first moment of the dynamics dominates all higher moments. Based on Theorem 4.4 and the definition of equilibria / working points together with Taylor's approximation theorem, we obtain the following:

Theorem 4.5 (Linearization).
Consider system (4.1) *and let* $(\mathbf{x}^\star, \mathbf{u}^\star)$ *be an operating point. If the deviation* $(\triangle\mathbf{x}, \triangle\mathbf{u}) := (\mathbf{x}_0 - \mathbf{x}^\star, \mathbf{u} - \mathbf{u}^\star)$ *is sufficiently small, then the solution of system*

$$\triangle\dot{\mathbf{x}}(t) = A \cdot \triangle\mathbf{x}(t) + B \cdot \triangle\mathbf{u}(t), \qquad \triangle\mathbf{x}(0) = \triangle\mathbf{x}_0 - \mathbf{x}^\star \tag{4.5a}$$

$$\triangle\mathbf{y}(t) = C \cdot \triangle\mathbf{x}(t) + D \cdot \triangle\mathbf{u}(t). \tag{4.5b}$$

with

$$A = \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}^\star, \mathbf{u}^\star), \qquad B = \frac{\partial}{\partial \mathbf{u}} f(\mathbf{x}^\star, \mathbf{u}^\star)$$
$$C = \frac{\partial}{\partial \mathbf{x}} h(\mathbf{x}^\star, \mathbf{u}^\star), \qquad D = \frac{\partial}{\partial \mathbf{u}} h(\mathbf{x}^\star, \mathbf{u}^\star)$$

is also the solution of the nonlinear system (4.1)*. The system* (4.5) *is called linearization around the operating point* $(\mathbf{x}^\star, \mathbf{u}^\star)$.

Remark 4.6
As a consequence of Theorem 4.5 we can transfer results from linear systems to nonlinear systems at least in a neighborhood of an operating point.

Task 4.7 (Inverted pendulum)
Consider the inverted pendulum on a cart given by the system

$$\dot{x}_1(t) = x_2(t) \tag{4.6a}$$
$$\dot{x}_2(t) = -kx_2 + g\sin(x_1(t)) + u(t)\cdot\cos(x_1(t)) \tag{4.6b}$$
$$\dot{x}_3(t) = x_4(t) \tag{4.6c}$$
$$\dot{x}_4(t) = u(t) \tag{4.6d}$$

and compute its linearization. Design a feedback such that the Eigenvalues of the closed loop are -1.

Solution to Task 4.7: The linearization reads

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let F be the feedback matrix we are looking for. Then we obtain

$$A + B \cdot F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ g + F_1 & -k + F_2 & F_3 & F_4 \\ 0 & 0 & 0 & 1 \\ F_1 & F_2 & F_3 & F_4 \end{pmatrix}$$

Now, we can compute the entries of F such that all Eigenvalues are equal to -1 revealing

$$\begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix} = \begin{pmatrix} -\frac{g+k^2}{g^2} - \frac{4k}{g} - 6 - g & -\frac{k}{g^2} - \frac{4}{g} - 4 + k & \frac{1}{g} & \frac{k}{g^2} + \frac{4}{g} \end{pmatrix}.$$

Unfortunately, the inverse of Lemma 4.1 does not hold true. To illustrate the latter, consider the following:

Task 4.8 (Nonholonomic integrator)
Consider a steerable vehicle from Figure 4.1 given by the dynamics

$$\begin{aligned} \dot{x}_1(t) &= u_1(t) \\ \dot{x}_2(t) &= u_2(t) \\ \dot{x}_3(t) &= x_2(t) \cdot u_1(t) \end{aligned}$$

where the angle of heading is given by x_1 *and the position by* $(x_2 \cdot \cos(x_1) + x_3 \cdot \sin(x_1), x_2 \cdot \sin(x_1) - x_3 \cdot \cos(x_1))$. *Show that Theorem 4.3 does not apply.*

Solution to Task 4.8: From the dynamic, we directly obtain that no point $(0, r, \varepsilon)$ with $\varepsilon \neq 0$ and $r \in \mathbb{R}$ is in the image of f.

The displayed example is also called Brockett's nonholonomic integrator and systems showing this property are called nonholonomic. As a consequence of not being feedback stabilizable, we can also not apply linearization to the nonholonomic integrator.

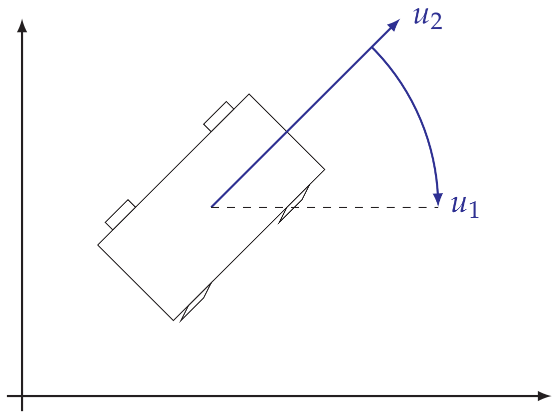


Figure 4.1.: Sketch of a nonholonomic car

Task 4.9

Consider the dynamics from Task 4.8. Show that the linearization is not controllable.

Solution to Task 4.9: Computing the linearization using Theorem 4.5 we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u_1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_2 & 0 \end{bmatrix}.$$

For the operating point $(\mathbf{x}^\star, \mathbf{u}^\star) = (0, 0)$ we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Consequently, the third line results in

$$\dot{x}_3(t) = 0.$$

Hence, the solution cannot converge to $\mathbf{x}^\star = 0$ and similarly no stabilizing feedback exists for the linearization.

Technically speaking, Task 4.8 refers to parallel parking using a steered vehicle. Since a parallel movement is not possible, such a feedback cannot exists. Yet we know that there exists a solution to transport a vehicle into a parallel parking spot using a non-parallel movement. Such a trajectory, however, cannot be described using a Lipschitz continuous feedback.

Solution to Task 4.8: Consider the operating point $(0,0)$ or shift the operating point respectively. If we apply

$$u_1 = \begin{cases} 0, & t \in [0,1] \\ -\operatorname{sgn}(x_3) \cdot \sqrt{|x_3|}, & t \in [1,2] \\ 0, & t \in [2,3] \\ -\left(x_1 - \operatorname{sgn}(x_3) \cdot \sqrt{|x_3|}\right), & t \in [3,4] \\ 0, & t > 4 \end{cases}$$

and

$$u_1 = \begin{cases} \operatorname{sgn}(x_3) \cdot \sqrt{|x_3|}, & t \in [0,1] \\ 0, & t \in [1,2] \\ -\operatorname{sgn}(x_3) \cdot \sqrt{|x_3|}, & t \in [2,3] \\ 0, & t > 3 \end{cases}$$

then the system will reach $(0,0)$ at $t = 4$ and remain there. Hence, the system is asymptotically controllable.

Based on this counterexample, we have the following:

Corollary 4.10
Consider a system (4.1) *and let* $(\mathbf{x}^\star, \mathbf{u}^\star)$ *be an operating point. If for the set*

$$f(\mathcal{X}) := \{\mathbf{y} \in \mathcal{X} \mid \mathbf{y} = f(\mathbf{x}, \mathbf{u}) \textit{ for all } \mathbf{u} \in \mathcal{U}\}$$

there exists no open ball $\mathbf{B}_r(\mathbf{x}^\star)$ *with radius* $r > 0$ *around* $\mathbf{x}^\star$ *such that* $\mathbb{B}_r(\mathbf{x}^\star) \subset f(\mathcal{X})$*, then no Lipschitz continuous feedback exists.*

In the nonlinear setting, we therefore know that Brockett's condition is at least necessary. Unfortunately, the following example will show that it is not sufficient. The idea of the example is to design a system such that Brockett's conditions holds, yet no Lipschitz continuous feedback does exist.

Task 4.11 (Artstein's circles)
Consider the system

$$\dot{x}_1 = \left(-x_1(t)^2 + x_2(t)^2\right) \cdot u(t)$$
$$\dot{x}_2 = (-2x_1(t) \cdot x_2(t)) \cdot u(t).$$

Show that Brockett's condition applies.

Solution to Task 4.11: For $v = (v_1, v_2)$ we consider three cases:

- for $v_2 \neq 0$, we choose $x_1 = 1$, $x_2 = \frac{v_1}{v_2} + \sqrt{\frac{v_1^2}{v_2^2} + 1}$ and $u = -\frac{v_2}{2x_2}$. Then we obtain

$$f(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} -1 + x_2^2 \\ -2x_2 \end{pmatrix} \cdot u = \begin{pmatrix} -\frac{v_2}{2x_2} \cdot \left(-2\frac{v_1}{v_2} x_2\right) \\ -\frac{v_2}{2x_2} \cdot (-2x_2) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

  and therefore $f(\mathbf{x}, \mathbf{u}) \in f(\mathcal{X}, \mathcal{U})$.

- For $v_1 \neq 0$, we set $x_1 = 0$, $x_2 = \sqrt{|v_1|}$ and $u = \operatorname{sgn}(v_1)$. Again, we obtain $f(\mathbf{x}, \mathbf{u}) \in f(\mathcal{X}, \mathcal{U})$.

- For $v_1 = v_2 = 0$ we set $u = 0$ and directly obtain $f(\mathbf{x}, \mathbf{u}) \in f(\mathcal{X}, \mathcal{U})$.

Hence, Brockett's condition holds.

Remark 4.12
Since the proof of nonexistence of a Lipschitz continuous feedback is rather involved, we refer to [1]. The idea of Artstein is to utilize that the dynamic forms a circle and dependency on the state results in a contradiction to the uniqueness of an operating point. In particular, the convergence is getting so slow, that the operating point is never reached, not even for $t \to \infty$.

Utilizing the examples from Brockett and Artstein, we can only state the following:

Corollary 4.13
Consider a system (4.1) *and let* $(\mathbf{x}^\star, \mathbf{u}^\star)$ *be an operating point. If the system is asymptotically controllable, existence of a Lipschitz continuous feedback is not guaranteed.*

From the discussion so far we obtain that the ε-δ definition of controllability does not provide us with enough insight to design a control for a given nonlinear system. In the following section, we will introduce equivalent notions of asymptotic stability and controllability, which allow for further interpretation of the system behavior.

4.2. Equivalent concepts of controllability

Apart from Definition 1.29, alternative definitions can be found in the literature, and these definitions are all equivalent. The intention of these definitions is to foster information of the system dynamics into abstract concepts for stability and controllability.

Remark 4.14
We already like to note that identical considerations hold true for observability and detectability. The latter are, however, related to gathering information about the state of the system, not about control of the system, and are therefore outside the scope of this lecture.

The first alternative concept for stability/controllability utilizes so called comparison functions, cf. Figure 4.2 for an illustration.

Definition 4.15 (Comparison Functions).
The following classes of functions are called comparison functions:

- A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing.
- A function is of *class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also unbounded.
- A function is of *class* $\mathcal{L}$ if it is strictly positive and it is strictly decreasing to zero as its argument tends to infinity.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if for every fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s > 0$ the function $\beta(s, \cdot)$ is of class $\mathcal{L}$.

The functions allow us to geometrically include solutions emanating from a given initial value by inducing a bound for the worst case. This directly leads to the following result, cf. [9]:

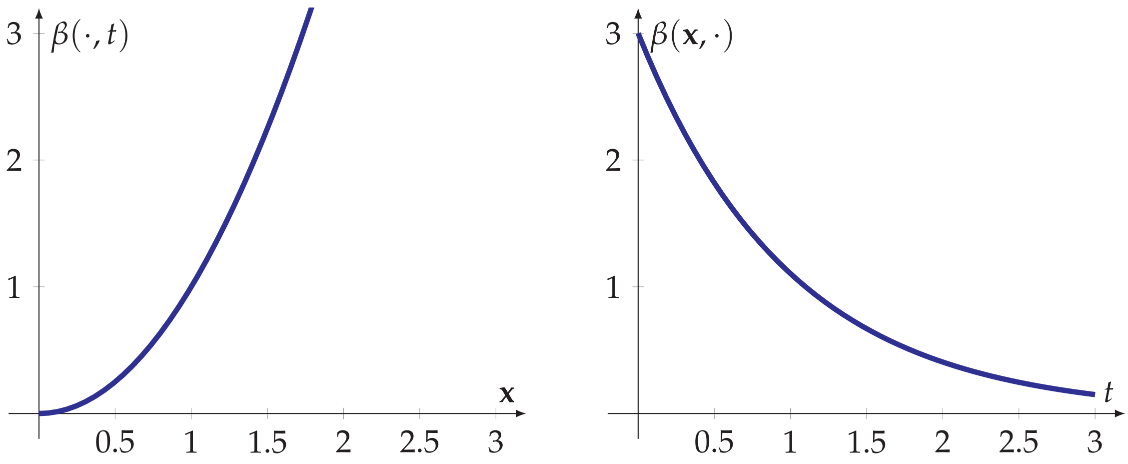


Figure 4.2.: Sketch of classes of comparison functions

Theorem 4.16 (Stability Concepts).
Consider a system (4.1).

(i) An operating point $\mathbf{x}^\star = 0$ *is strongly asymptotically stable or robustly asymptotically stable if there exists an open neighborhood* $\mathcal{N}$ *of* $\mathbf{x}^\star$ *and a function* $\beta \in \mathcal{KL}$ *such that*

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}_0\|, t) \tag{4.7}$$

holds for all $\mathbf{x}_0$, $\mathbf{u}$ *and all* $t \geq 0$.

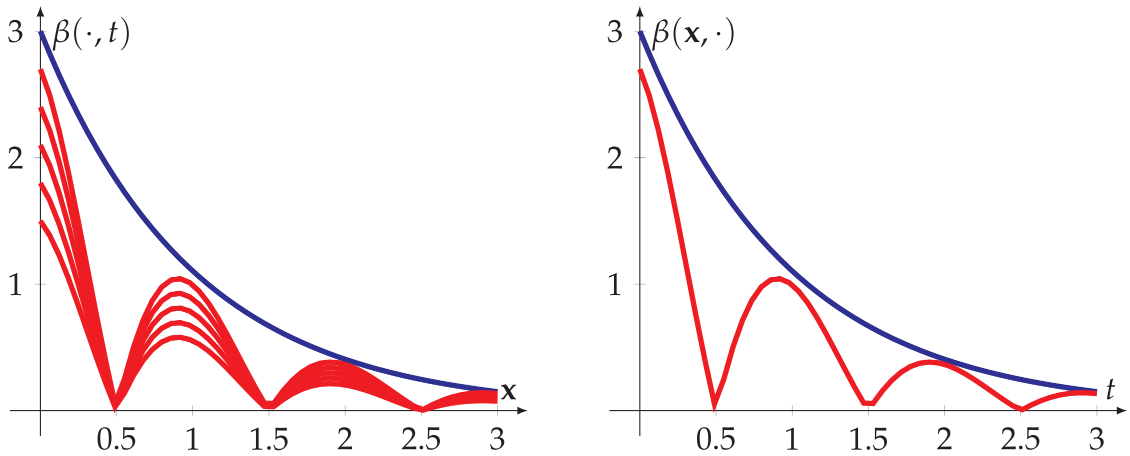
(ii) An operating point $\mathbf{x}^\star = 0$ *is weakly asymptotically stable or asymptotically controllable if there exists an open neighborhood* $\mathcal{N}$ *of* $\mathbf{x}^\star$ *and a function* $\beta \in \mathcal{KL}$ *such that for every* $\mathbf{x}_0$ *there exists a control law* $\mathbf{u}$ *such that*

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}_0\|, t) \tag{4.8}$$

holds for all $t \geq 0$.

Remark 4.17
The result from Theorem 4.16 can be generalized to arbitrary operating points by subtracting $\mathbf{x}^\star$ *within the norm operator on both sides of equation* (4.7) *or* (4.8).

Task 4.18
Draw solutions of systems to visualize Theorem 4.16.



(a) Sketch of a robustly asymptotically stable system (b) Sketch of a weakly asymptotically stable system

Figure 4.3.: Sketch of asymptotically stable systems

The idea of the comparison function concept is to establish a bound on the system behavior. Once we can verify/compute how such a bound looks like, we know that even in the worst case, the system behavior will be better, i.e. the trajectories will be closer to the operating point than the comparison bound. Secondly, we also know that for any control function to be used satisfying this bound, then the solution will converge and the system will be asymptotically stabilized by the control.

Remark 4.19
Note that Theorem 4.16 makes no assumption as to whether the control is feed forward or feedback, nor whether the control is continuous or discontinuous.

The second concept utilizes the Lyapunov functions, which can be interpreted as energy function of the system state. The main difference lies in considering a minimizing control in the neighborhood of the considered state.

Definition 4.20 (Control-Lyapunov function).
Consider a system (4.1) with operating point $(\mathbf{x}^\star, \mathbf{u}^\star) = (0, 0)$ and a neighborhood $\mathcal{N}(\mathbf{x}^\star)$. Then a continuous function $V : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ is called a *Control-Lyapunov function* if there exist

functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that there exists a control function $\mathbf{u}$ satisfying the inequalities

$$\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|) \tag{4.9}$$

$$\inf_{\mathbf{u}\in\mathcal{U}} \frac{d}{dt} V(\mathbf{x}) = \inf_{\mathbf{u}\in\mathcal{U}} \frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}, \mathbf{u}) \leq -\alpha_3(\mathbf{x}) \tag{4.10}$$

for all $\mathbf{x} \in \mathcal{N} \setminus \{\mathbf{x}^\star\}$.

The idea of the Lyapunov function is comparable to a salad bowl: If we put a ball into the bowl, it will run downhill and remain at the lowest point. Using this metaphor, the lowest point regarding the Lyapunov function marks the desired equilibrium. The Control-Lyaponov function itself can be seen as energy of a system. Hence, the first inequalities (4.9) are bounds on the behavior of the system. In contrast to comparison functions, however, both lower and upper bounds are required. The reason for this necessity emanates from the second inequality (4.10). This inequality basically says that energy is drawn from the system continuously. Yet, even if energy is continuously drawn from the system, it may come to a rest far away from the operating point. To avoid such cases, the bounds on V are required.
In our last step, we apply this energy concept to obtain stability by energy draining arguments:

Theorem 4.21 (Asymptotic Stability).
Consider a system (4.1) *where* $f(0,0) = 0$, *a neighborhood* $\mathcal{N}$ *of* $\mathbf{x}^\star$ *and a continuous function* $V : \mathcal{N} \to \mathbb{R}_0^+$.

(i) An equilibrium $\mathbf{x}^\star = 0$ *is strongly asymptotically stable or robustly asymptotically stable if* (4.9) *and*

$$\sup_{\mathbf{u}\in\mathcal{U}} \frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}, \mathbf{u}) \leq -\alpha_3(\mathbf{x})$$

hold for $\alpha_3 \in \mathcal{K}_\infty$ *and all* $\mathbf{x} \in \mathcal{N}$.

(ii) An equilibrium $\mathbf{x}^\star = 0$ *is weakly asymptotically stable or asymptotically controllable if for all* $\mathbf{x} \in \mathcal{N}$ *there exists a control* $\mathbf{u}$ *such that* (4.9), (4.10) *hold.*

Task 4.22
Draw solutions of systems to visualize Theorem 4.21.

Remark 4.23
Again, the strong or robust concept means that no matter which control we consider, energy is drawn from the system. The weak concept requires additional work to design a control such that the stability property is induced.

In contrast to Lipschitz continuous feedback stabilizability, cf. Lemma 4.1 and Corollaries 4.10, 4.13, using the notion of Control-Lyapunov functions an inversion of the statement is possible.

Theorem 4.24 (Existence of Control-Lyapunov function).
Consider a system (4.1) *where* $f(0,0) = 0$. *Suppose that* $\mathbf{x}^\star = 0$ *is weakly asymptotically stable or asymptotically controllable. Then there exists a continuous function* $V : \mathcal{N} \to \mathbb{R}_0^+$ *satisfying the properties of a Control-Lyapunov function given in Definition 4.20.*

Theorem 4.25 (Existence of Control-Lyapunov function).
Consider a system (4.1) *where* $f(0,0) = 0$. *If* $\mathbf{x}^\star = 0$ *is asymptotically stabilizable/controllable on a neighborhood* $\mathcal{N}(\mathbf{x}^\star)$ *by a Lipschitz continuous feedback* $\mathbf{u} : \mathcal{N}(\mathbf{x}^\star) \to \mathcal{U}$*, then the Control-Lyapunov function is arbitrarily often continuously partially differentiable on* $\mathcal{N}(\mathbf{x}^\star)$.

Remark 4.26
Consider Artstein's circles, cf. Task 4.11, one can show that

$$V(\mathbf{x}) = \sqrt{4x_1^2 + 3x_2^2} - |x_1|$$

is a Control-Lyapunov function, for which the conditions (4.9), (4.10) *hold true for either* $\mathbf{u} \equiv -1$ *or* $\mathbf{u} \equiv 1$.

Based on the latter remark, we already see that the concept of Control-Lyapunov function allows us to conclude asymptotic stability directly once such a function is known. Hence, the concept of a Control-Lyapunov function allows for a broader class of feedbacks to be computed as compared to linearization (Theorem 4.5) or characteristic maps (Section 3.2). The map in Figure 4.4 characterizes the connection between the last results:
From Figure 4.4, we directly obtain the weak links in the nonlinear setting: From Brockett and Artstein, it is clear that the arrow from controllability to existence of a Lipschitz feedback is not present. Additionally, from a differentiable Control-Lyapunov function, the existence of a Liptschitz feedback cannot be guaranteed.

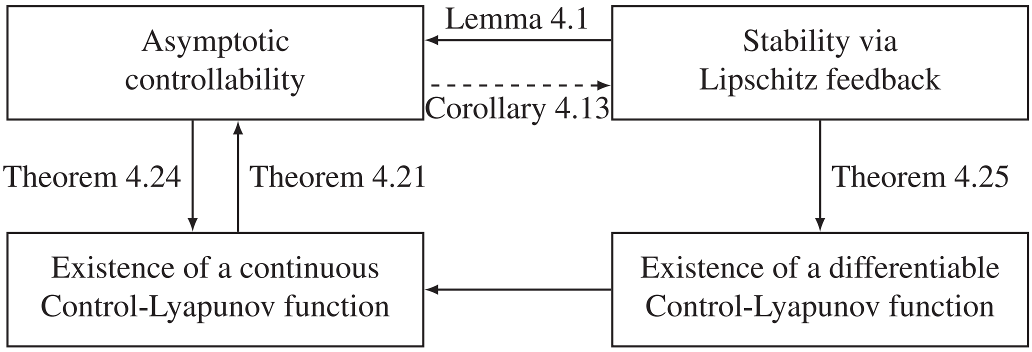


Figure 4.4.: Schematic connection of stability results

Combining the results of this section, we obtain the following:

Table 4.1.: Advantages and disadvantages of Control-Lyapunov functions

Advantage	**Disadvantage**
✓ Allows energy based approach	✗ Derivation complicated
✓ Requires no knowledge of system	✗ Limited to neighborhood
✓ Allows broader class of systems	✗ Existence of feedback not guaranteed
✓ Stability induces existence	

In the following section, we specialize our setting and aim to close the gaps illustrated in Figure 4.4 to derive a concept to compute a nonlinear feedback for a nonlinear system.

4.3. Backstepping and Sontag's formula

The research are for constructive nonlinear control is still an open field of research and focuses on the derivation of explicit formulas for the computation of feedback controllers. A characteristic of this field is that approach are not valid for the entire class of nonlinear systems (4.1), but only for subclasses satisfying certain structural assumptions.
The aim of this section is to invert the result from Theorem 4.24, that is for a given Control-Lyapunov function we want to derive a feedback such that the closed loop is asymptotically stable.

Within this section, we focus on so called control-affine systems, which take the following general form:

Definition 4.27 (Control-affine system).
Consider a system (4.1). If the dynamic is given by

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) = f_0(\mathbf{x}(t)) + \sum_{j=1}^{n_u} f_j(\mathbf{x}(t)) \cdot u_j(t) \tag{4.11}$$

with locally Lipschitz continuous functions $f_j : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ for all $j = 1, \ldots, n_u$, then we call the system *control-affine*.

Note that the controls $u_j : \mathcal{T} \to \mathbb{R}$ are scalar and the sum ranges across the dimension of the control $\mathbf{u} \in \mathbb{R}^{n_u}$.
Moreover, we need to restrict ourselves to feedbacks, which are bounded for every bounded input. The easiest way to obtain such a restriction is to impose the following assumption:

Assumption 4.28
Consider a feedback $\mathbf{u} : \mathcal{X} \to \mathcal{U}$. Then we assume $\mathbf{u}$ to be Lipschitz continuous and satisfy $\mathbf{u}(0) = 0$.

Remark 4.29
Note that the condition $\mathbf{u}(0) = 0$ *can be easily satisfied by transforming the dynamic* $\tilde{f}(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}, \mathbf{u} + \mathbf{u}(0))$ *and obtain* $\tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \mathbf{u}(0)$. *Hence, we will always have* $\tilde{\mathbf{u}}(0) = 0$.

Utilizing the latter assumption, we can show boundedness of the feedback for bounded input:

Lemma 4.30
Consider a feedback $\mathbf{u} : \mathcal{X} \to \mathcal{U}$ *satisfying Assumption 4.28. Then we have*

$$\mathbf{u}(\mathbf{x}) \leq \max_{\|\hat{\mathbf{x}}\| \leq \|\mathbf{x}\|} \|\mathbf{u}(\hat{\mathbf{x}})\| + \|\mathbf{x}\| =: \gamma(\mathbf{x}) \tag{4.12}$$

where $\gamma \in \mathcal{K}$ *is a bounding function of the feedback.*

Now, we can use the latter to invert Theorem 4.24 and define an asymptotically stabilizing feedback. The result itself dates back to Artstein [1], yet here we use the explicit formula for the feedback derived by Sontag [18].

Theorem 4.31 (Sontag's formula).
Consider a control-affine system (4.11) *and suppose* $V : \mathcal{X} \to \mathbb{R}_0^+$ *is a Control-Lyapunov function for this system satisfying inequality* (4.10) *for a control* $\mathbf{u}$ *satisfying Assumption 4.28. Let* γ *to be defined as in Lemma 4.30. Then, using the abbreviation* $F_j(\mathbf{x}, \mathbf{u}) = \frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f_j(\mathbf{x}, \mathbf{u})$, *for the feedback* $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_{n_u})^\top$ *defined via*

$$u_j(\mathbf{x}) := \begin{cases} 0, & \text{if } \mathbf{x} = 0 \text{ or } F_j(\mathbf{x}, \mathbf{u}) = 0 \\ -F_j(\mathbf{x}, \mathbf{u}) \cdot \Phi\left(F_0(\mathbf{x}, \mathbf{u}), \sum_{j=1}^{n_u} \left(F_j(\mathbf{x}, \mathbf{u})\right)^2\right), & \text{if } F_j(\mathbf{x}, \mathbf{u}) \neq 0 \end{cases} \tag{4.13}$$

with

$$\Phi(a, b) = \begin{cases} \frac{a + \sqrt{a^2 + b^2}}{b}, & b \neq 0 \\ 0, & b = 0 \end{cases}$$

is Lipschitz continuous in the range $\mathbf{u}(\mathbf{x}) \leq \gamma(\mathbf{x})$ *and the closed loop is globally asymptotically stable.*

As a consequence, we can now compute an asymptotically stabilizing feedback explicitly using equation (4.13) if a Control-Lyapunov function is known. The latter part, however, remains a difficult task. To illustrate the idea, we consider the following example:

Task 4.32
Consider the controllable inverted pendulum

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + \sin(x_1(t)) + u(t) \end{aligned}$$

where the control represents the force of a motor acting on the angular velocity. Show that

$$V(\mathbf{x}) = \frac{1}{2}\left((x_1 + x_2)^2 + x_1^2\right)$$

is a Control-Lyapunov function and compute a feedback via Sontag's formula.

Solution to Task 4.32: To show that $V(\mathbf{x})$ is a Control-Lyapunov function, we use

$$V(\mathbf{x}) = \frac{1}{2}\left(x_1^2 + \underbrace{2x_1x_2}_{\geq -3x_1^2/2 - x_2^2/2} + x_2^2 + x_1^2\right) \geq \frac{1}{4}\left(x_1^2 + x_2^2\right) = \frac{1}{4}\|\mathbf{x}\|^2$$

to obtain $\alpha_1(\mathbf{x}) = \|\mathbf{x}\|^2/4$. Similarly, we have

$$V(\mathbf{x}) = \frac{1}{2}\left(\underbrace{(x_1 + x_2)^2}_{\leq 3x_1^2 + 4x_2^2} + x_1^2\right) \leq 2x_1^2 + 2x_2^2 \leq 2\|\mathbf{x}\|^2$$

and accordingly $\alpha_2(\mathbf{x}) = 2\|\mathbf{x}\|^2$. For the derivative, we obtain

$$\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}, \mathbf{u}) = (2x_1 + x_2)\, x_2 + (x_1 + x_2) \cdot (-x_2 + \sin(x_1) + u)\,.$$

As we require the descent only to hold in the infimum, we can choose a control wisely to cancel out inner parts of the latter expression. Here, we use $u = -x_1 - x_2 - \sin(x_1)$ and see

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}, \mathbf{u}) &= (2x_1 + x_2)\, x_2 + (x_1 + x_2) \cdot (-x_2 + \sin(x_1) + u) \\
&= (2x_1 + x_2)\, x_2 + (x_1 + x_2) \cdot (-x_1 - 2x_2) \\
&= 2x_1x_2 + x_2^2 - x_1^2 - 2x_2^2 - 3x_1x_2 \\
&= -x_1^2 - x_2^2 - \underbrace{x_1x_2}_{\geq -x_1^2/2 - x_2^2/2} \\
&\leq -\frac{1}{2}\left(x_1^2 + x_2^2\right) = -\frac{1}{2}\|\mathbf{x}\|^2 < 0
\end{aligned}$$

revealing $\alpha_3(\mathbf{x}) = \|\mathbf{x}\|^2/2$. Hence, $V(\mathbf{x})$ is a Control-Lyapunov function.
Considering the dynamic, we have

$$f_0(\mathbf{x}) = \begin{pmatrix} x_2 \\ -x_2 + \sin(x_1) \end{pmatrix} \qquad \text{and} \qquad f_1(\mathbf{x}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, we obtain

$$\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f_0(\mathbf{x}) = (2x_1 + x_2)\, x_2 + (x_1 + x_2) \cdot (-x_2 + \sin(x_1))$$

$$= x_1 x_2 + (x_1 + x_2) \cdot \sin(x_1)$$

and

$$\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}) \cdot f_1(\mathbf{x}) = x_1 + x_2.$$

Using Sontag's formula, we therefore obtain

$$u(\mathbf{x}) = -(x_1 + x_2) \cdot \frac{x_1 x_2 + (x_1 + x_2) \cdot \sin(x_1) + \sqrt{(x_1 x_2 + (x_1 + x_2) \cdot \sin(x_1))^2 + (x_1 + x_2)^4}}{(x_1 + x_2)^2}.$$

As we have seen, while evaluating Sontag's formula is straight forward, the derivation of a Control-Lyapunov function is a rather involved task. In this context, *backstepping* is an systematic approach to construct and compute such a function. The idea of backstepping is to iteratively construct the Control-Lyapunov function. To this end, the dynamic is split into parts forming a cascade. The split is made in such a way that the outer parts of the cascade are simple and exhibit a known Control-Lyapunov function for the outer part. The cascade is then formed using the following core result:

Theorem 4.33 (Backstepping).
Consider a nonlinear control system (4.1). *Suppose that f is continuously differentiable and that there exists a twice continuously differentiable feedback $\mathbf{u}_f(\mathbf{x})$ with $\mathbf{u}_f(0) = 0$ such that $\mathbf{x}^\star = 0$ is (locally) asymptotically stable. Let $V_f(\mathbf{x})$ be the corresponding continuously differentiable Control-Lyapunov function. Furthermore, suppose there exists a continuously differentiable function $h : \mathcal{X} \times \mathcal{U} \to \mathcal{U}$ with $h(0, \hat{\mathbf{u}}) = 0$ for $\hat{\mathbf{u}} \in \mathcal{U}$. Then for the coupled system*

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \hat{\mathbf{u}}(t)) \tag{4.14}$$

$$\dot{\hat{\mathbf{u}}}(t) = h(\mathbf{x}(t), \hat{\mathbf{u}}(t)) + \mathbf{u}(t) \tag{4.15}$$

with state $\hat{\mathbf{x}} = (\mathbf{x}, \hat{\mathbf{u}})$ and control $\mathbf{u}$ there exists a continuously differentiable feedback such that the closed loop is asymptotically stable and corresponding Control-Lyapunov function

$$V(\hat{\mathbf{x}}) = V(\mathbf{x}, \hat{\mathbf{u}}) := V_f(\mathbf{x}) + \frac{1}{2} \left\| \hat{\mathbf{u}} - \mathbf{u}_f(\mathbf{x}) \right\|^2. \tag{4.16}$$

In Figure 4.5, the block structure of the backstepping idea is depicted.

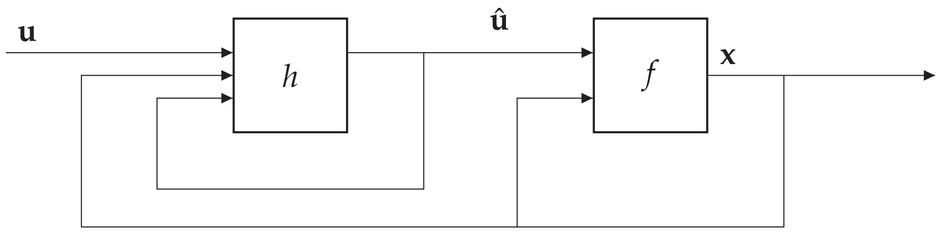


Figure 4.5.: System structure for backstepping

Note that different from the idea of a cascade control, the structure in the backstepping approach is already fixed by the model. As a consequence, the derivation of the controllers per loop is not done from inside to outside, but instead tracks down separability of simple systems from outside to inside. Hence, for an application, we first need to identify/fix an outer subsystem f such that the overall system exhibits the structure given in equations (4.14), (4.15).

Remark 4.34
We highlight that the aim of the added Lyapunov part $\left\|\hat{\mathbf{u}} - \mathbf{u}_f(\mathbf{x})\right\|^2 / 2$ *in* (4.15) *is to enforce a reference for the outer control* $\hat{\mathbf{u}}$ *such that it tracks the desired function* $\mathbf{u}_f(\mathbf{x})$*. Hence, the input* $\mathbf{u}(\mathbf{x})$ *shall be a reference tracking feedback. Such a feedback can be derived using Sontag's formula applied to the Control-Lyapunov function obtained via backstepping.*

To better understand the procedure of backstepping, we consider the example from Task 4.32 again.

Task 4.35
Again, consider the controllable inverted pendulum

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_2(t) + \sin(x_1(t)) + u(t)\end{aligned}$$

where the control represents the force of a motor acting on the angular velocity. Derive a Control-Lyapunov function via backstepping.

Solution to Task 4.35: Using the notation from Theorem 4.33, we set $\mathbf{x} = \mathbf{x}_1$ and $\hat{\mathbf{u}} = \mathbf{x}_2$. Then we obtain

$$f(\mathbf{x}, \hat{\mathbf{u}}) = \hat{\mathbf{u}}$$

$$h(\mathbf{x}, \hat{\mathbf{u}}) = -\hat{\mathbf{u}} + \sin(\mathbf{x}).$$

Now, we consider the first equation only. Since the system is linear, we can directly use the Control-Lyapunov function $V_f(\mathbf{x}) = \mathbf{x}^2/2$. Hence, we obtain the feedback $\mathbf{u}_f(\mathbf{x}) := -\mathbf{x}$ for the outer system. Note that due to the overall structure of the system, we have $\hat{\mathbf{u}} \neq \mathbf{u}_f(\mathbf{x})$. Applying the backstepping procedure, we use the latter to compute

$$\begin{aligned} V(\hat{\mathbf{x}}) = V(\mathbf{x}, \hat{\mathbf{u}}) &= V_f(\mathbf{x}) + \frac{1}{2} \left\| \hat{\mathbf{u}} - \mathbf{u}_f(\mathbf{x}) \right\|^2 \\ &= \frac{1}{2}\mathbf{x}^2 + \frac{1}{2} \left| \hat{\mathbf{u}} + \mathbf{x} \right|^2 \\ &= \frac{1}{2} \left(\mathbf{x}^2 + (\hat{\mathbf{u}} + \mathbf{x})^2 \right) \\ &= \frac{1}{2} \left(x_1^2 + (x_2 + x_1)^2 \right). \end{aligned}$$

This is exactly the Control-Lyapunov function given in Task 4.32.

Remark 4.36
Note that in the linear one dimensional case, we have that $V(\mathbf{x}) = \mathbf{x}^2/2$ is always a Control-Lyapunov function.

Using backstepping and Sontag's formula for the special case of control affine systems, we can complement our schematic from Figure 4.4. In Figure 4.6, the special case is highlighted in red.

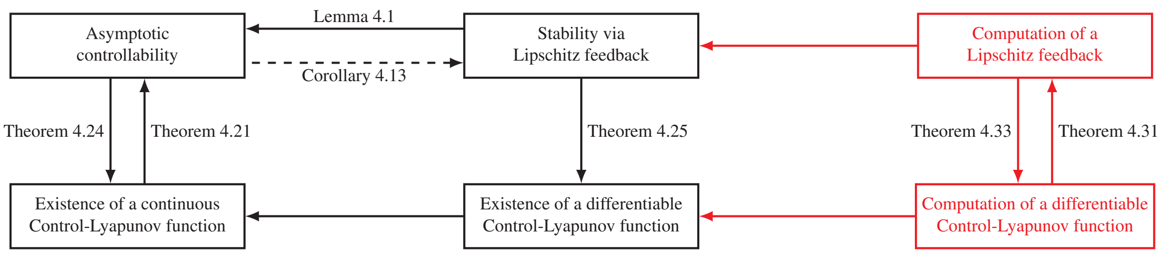


Figure 4.6.: Schematic connection of stability results to backstepping/Sontag's formula

At this point we like to note that stability is not necessarily restricted to equilibria, but can also apply to periodic orbits or areas. An illustration is given in Figure 4.7 indicating stability of the system yet not of a point.
To summarize, we considered one approach to treat nonlinear systems. We again like to stress that currently no generic concept to treat nonlinear systems is known. Modern approaches to

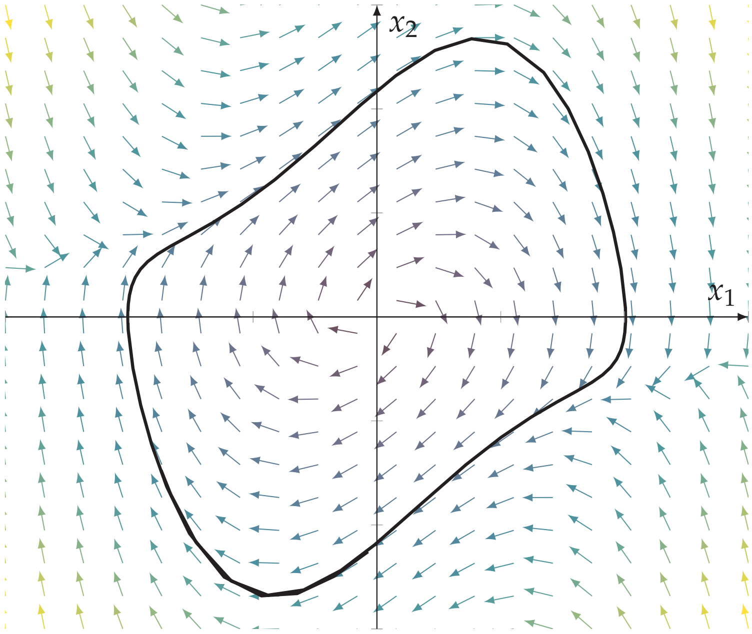


Figure 4.7.: Sketch of a stable system with periodic orbit

tackle this area utilize the concept of key performance indicators, exploit the dynamics and include constraints on the system. All of the latter will be part of the subsequent lecture *Control Engineering 3*.

Table 4.2.: Advantages and disadvantages of backstepping & Sontag's formula

Advantage	**Disadvantage**
✓ Direct derivation of feedback	✗ Requires separability
✓ Allows multivariate control	✗ Limited to control affine systems
✓ Recursive construction possible	✗ Results analytically involved
✓ Generic applicability	

CHAPTER 5

DIGITAL CONTROL SYSTEMS

In the previous Chapter 4 we observed that for an asymptotically controllable system Brockett's condition is a necessary yet not sufficient criterion for existence of a Lipschitz continuous feedback (see Corollary 4.10). As the system is asymptotically controllable, a possibly discontinuous feedback must exist.
Throughout this chapter, we again consider nonlinear control systems of the form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)). \tag{5.1}$$

5.1. Zero order hold control

The most simple case of a discontinuous feedback is given by the so called zero order hold. The idea is to sample the control, i.e. to fix a time grid $\{t_k\} \subset \mathbb{R}$ and define the control to be constant in between two sampling instances t_k and t_{k+1}. Here, we further simplify the setting by introducing a sampling period T and define the sampling instances to be equidistant, that is $t_k = k \cdot T$.

Remark 5.1
There are two more general cases: For one, the sampling times may be defined by a function of time, or secondly, the sampling times can be defined by a function of states. The first one is common in prediction and prescription of systems where action in the far future are significantly less important. Hence, one typically chooses between exactness of the prediction and computational complexity. The latter case is referred to a event driven control.

We still like to stress that in applications, the choice of T is not fixed right from the beginning, but depends on the obtainable solution and stability properties. Hence, we continue to formulate the following definitions of zero order hold control and solution as a parametrization of operators with respect to T.

Definition 5.2 (Zero order hold control).
Consider a nonlinear control system (5.1) and a feedback $\mathbf{u} : \mathcal{X} \to \mathcal{U}$ such that $\|\mathbf{u}(\mathbf{x})\| \leq \gamma(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$ and some continuous function $\gamma : \mathcal{X} \to \mathbb{R}_0^+$. Moreover suppose a *sampling period* $T > 0$ to be given, which defines the sampling times $t_k = k \cdot T$. Then we call the piecewise constant function

$$\mathbf{u}_T(t) \equiv \mathbf{u}(\mathbf{x}(t_k)), \qquad t \in [t_k, t_{k+1}) \tag{5.2}$$

zero order hold control.

As a consequence of the latter definition, the control $\mathbf{u}_T$ is not continuous but instead exhibits jumps at the sampling times t_k. Still, the function is integrable, which is a requirement for existence of a solution of (5.1) for such an input. This insertion directly leads to the following:

Definition 5.3 (Zero order hold solution).
Given a nonlinear control system (5.1) and a zero order hold control $\mathbf{u}_T : \mathcal{T} \to \mathcal{U}$. Then we call the function $\mathbf{x}_T : \mathcal{T} \to \mathcal{X}$ satisfying

$$\dot{\mathbf{x}}_T(t) = f(\mathbf{x}_T(t), \mathbf{u}_T(t)) \tag{5.3}$$

zero order hold solution.

In order to compute such a solution, we can simply concatenate solutions of subsequent sampling intervals $[t_k, t_{k+1})$. Here, we can use the endpoint of the solution on one sampling interval to be the initial point on the following one. Hence, the computation of $\mathbf{x}_T$ is well defined.

Remark 5.4
Since the system is Lipschitz continuous on each interval $[t_k, t_{k+1})$, the solution is also unique. Hence, identifying endpoint and initial point of subsequent sampling intervals is sufficient to show that teh zero order hold solution is unique. Yet, as a consequence of this concatenation, the solution is not differentiable at the sampling points t_k.

Remark 5.5
Note that despite $\mathbf{u}_T$ *to be piecewise constant, the zero order hold solution does not exhibit jumps and shows nonlinear behavior.*

Similar to the nonlinear case, we next introduce the concept of stability. Note that we did not fix the sampling period T, hence stability needs to be parametrized using this characterizing parameter of the control. Here, we use the same simplification to shift the operating point to the origin as in Chapter 4, cf. Remark 4.17.

Definition 5.6 (Practical stability/controllability).
Consider a nonlinear control system (5.1) with $f(0,0) = 0$. Then we call a family of feedbacks $\mathbf{u}_T$, $T \in (0, T^\star]$ to *semiglobally practically asymptotically stabilize* the operating point $(\mathbf{x}^\star, \mathbf{u}^\star) = (0,0)$ if there exists a function $\beta \in \mathcal{KL}$ and constants $R > \varepsilon > 0$ such that

$$\|\mathbf{x}_T(t)\| \leq \max\{\beta(\|\mathbf{x}_0\|, t), \varepsilon\} \tag{5.4}$$

holds for all $t > 0$, for all $T \in (0, T^\star]$ and all initial value satisfying $\|\mathbf{x}_0\| \leq R$.

Remark 5.7
The term „semiglobal" refers to the constant R*, which limits the range of the initial states for which stability can be concluded. The term „practical" refers to the constant* ε*, which is a measure on how close the solution can be driven towards the operating point before oscillations as in the case of the bang bang controller occur.*

As a direct conclusion of Definition 5.6, we can apply Lemma 4.1 and obtain:

Corollary 5.8
Consider a nonlinear control system (5.1) *with* $f(0,0) = 0$ *and suppose a family of feedbacks* $\mathbf{u}_T$*,* $T \in (0, T^\star]$ *to exist, which semiglobally practically asymptotically stabilize the operating point* $(\mathbf{x}^\star, \mathbf{u}^\star) = (0,0)$*. Then there exists a feed forward* $\mathbf{u} : \mathcal{T} \to \mathcal{U}$ *such that the system is practically asymptotically controllable.*

Definition 5.6 also shows the dilemma of digital control using fixed sampling periods: Both close to the desired operating point and for initial values far away from it, the discontinuous evaluation of the feedback $\mathbf{u}_T$ leads to a degradation of performance. Close to the operating point, a slow evaluation leads to overshoots despite the dynamics to be typically rather slow. Far away from the operating point, the dynamics is too fast to be captured in between two sampling points which leads to unstable behavior.

Still, it may even be possible to obtain asymptotic stability (not only practical asymptotic stability) using fixed sampling periods T as shown in the following task:

Task 5.9

Consider the system

$$\begin{aligned}\dot{x}_1(t) &= \left(-x_1(t)^2 + x_2(t)^2\right) \cdot u(t) \\ x_2(t) &= \left(-2 \cdot x_1(t) \cdot x_2(t)\right) \cdot u(t).\end{aligned}$$

Design a zero order hold control such that the system is practically asymptotically stable.

Solution to Task 5.9: We set

$$\mathbf{u}_T(t) = \begin{cases} 1, & x_1 \geq 0 \\ -1, & x_1 < 0 \end{cases}.$$

For this choice, the system is globally asymptotically stable for all $T > 0$ and even independent from T. The reason for the latter is that the solutions never cross the switching line $x_1 = 0$, i.e. the control to be applied is always constant, which leads to independence of the feedback from T.

As described before, the behavior observed in Task 5.9 is the exception. In practice, the limitations of semiglobality and practicality is typically the best we can expect in zero order hold control of nonlinear system.

In order to show that a stabilizing zero order hold control exists, we follow the path from Chapter 4 and adapt the concept of Control-Lyapunov functions from Definition 4.20 accordingly.

Definition 5.10 (Practical Control-Lyapunov functions).
Consider a nonlinear control system (5.1) with operating point $(\mathbf{x}^\star, \mathbf{u}^\star) = (0, 0)$ such that $f(\mathbf{x}^\star, \mathbf{u}^\star) = 0$ and a neighborhood $\mathcal{N}(\mathbf{x}^\star)$. Then the family of continuous functions $V_T : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ for $T \in (0, T^\star]$ is called a *semiglobal practical family of Control-Lyapunov functions* if there exist constants $\hat{R} > \hat{\varepsilon} > 0$ as well as functions $\alpha_1, \alpha_2, \in \mathcal{K}_\infty$ and a continuous function

$W : \mathcal{X} \to \mathbb{R}^+ \setminus \{0\}$ such that there exists a control function $\mathbf{u}$ satisfying the inequalities

$$\alpha_1(\|\mathbf{x}\|) \leq V_T(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|) \tag{5.5}$$

$$\inf_{\mathbf{u}\in\mathcal{U}} V_T(\mathbf{x}_T(t_{k+1})) \leq \max\left\{V_T(\mathbf{x}_T(t_k) - T \cdot W(\mathbf{x}_T(t_k)), \hat{\varepsilon})\right\} \tag{5.6}$$

for all $\mathbf{x} \in \mathcal{N} \setminus \{\mathbf{x}^\star\}$ with $V_T(\mathbf{x}) \leq \hat{R}$ and all $T \in (0, T^\star]$.

The latter definition extends the concepts of a Control-Lyapunov function is various ways. For one, as the zero order hold solution is not differentiable, we can no longer assume V_T to be differentiable. Hence, the formulation of decrease in energy in inequality (5.6) is given along a solution instead of its vector field. Secondly, the parametrization regarding T needs to be considered. This leads to a parametrization of the decrease in inequality (5.6) using the positive definite function $W(\cdot)$. Moreover, the ideas of semiglobality and practicality are integrated.

Remark 5.11
Comparing Definition 5.10 to Definition 5.6, we can identify the similarity of semiglobality between the constants R and $\hat{R}$ as well as ε and $\hat{\varepsilon}$. The difference between these two pairs lies in their interpretation: For $\mathcal{KL}$ function, we utilize the state space, whereas for Control-Lyapunov functions the energy space is used. Hence, both values are a transformation of one another using the Control-Lyapunov function V_T.

Now, supposingly that a practical Control-Lyapunov function exists, we can directly derive the existence of a zero order hold control.

Theorem 5.12 (Existence of feedback).
Consider a nonlinear control system (5.1) *with operating point* $(\mathbf{x}^\star, \mathbf{u}^\star) = (0, 0)$ *such that* $f(\mathbf{x}^\star, \mathbf{u}^\star) = 0$. *Let V_T to be a semiglobal practical family of Control-Lyapunov functions for* $T \in (0, T^\star]$. *Then the minimizer*

$$\mathbf{u}_T(t) := \underset{\mathbf{u}\in\mathcal{U}}{argmin}\, V_T(\mathbf{x}_T(t_{k+1})) \tag{5.7}$$

is a family of semiglobally practially asymptotically stabilizing feedbacks.

Note that in (5.7), the right hand side depends on $\mathbf{u}$ implicitly as $\mathbf{x}_T(t_{k+1})$ is defined using the initial value $\mathbf{x}_T(t_k)$ and the zero order hold control $\mathbf{u}$. Hence, the definition (5.7) is proper.

Remark 5.13
The transfer from infimum in (5.6) *to minimum in* (5.7) *is only possible as the control is constant in between two sampling instances t_k and t_{k+1} and therefore the solution $\mathbf{x}_T(\cdot)$ is continuous with respect to* $\mathbf{u}$.

Unfortunately, the pure existence of a feedback does not help us in computing it. Additionally, we still require the existence of a practical Control-Lyapunov function to conclude existence of such a feedback. Here, we first address existence of a Control-Lyapunov function, for which the following is known from the literature:

Theorem 5.14 (Existence of practical Control-Lyapunov function).
Consider a nonlinear control system (5.1) *with operating point $(\mathbf{x}^\star, \mathbf{u}^\star) = (0, 0)$ such that $f(\mathbf{x}^\star, \mathbf{u}^\star) = 0$. If the system is asymptotic controllable, then there exists a family of semiglobal practical Control-Lyapunov function.*

The most important aspect of Theorem 5.14 is the requirement regarding the control system. The result does only require the system to be asymptotically controllable, a task which we discussed in the previous Chapter 4, i.e. without digitalization. Hence, techniques such as backstepping or others depending on the structure of the control system may be applied.

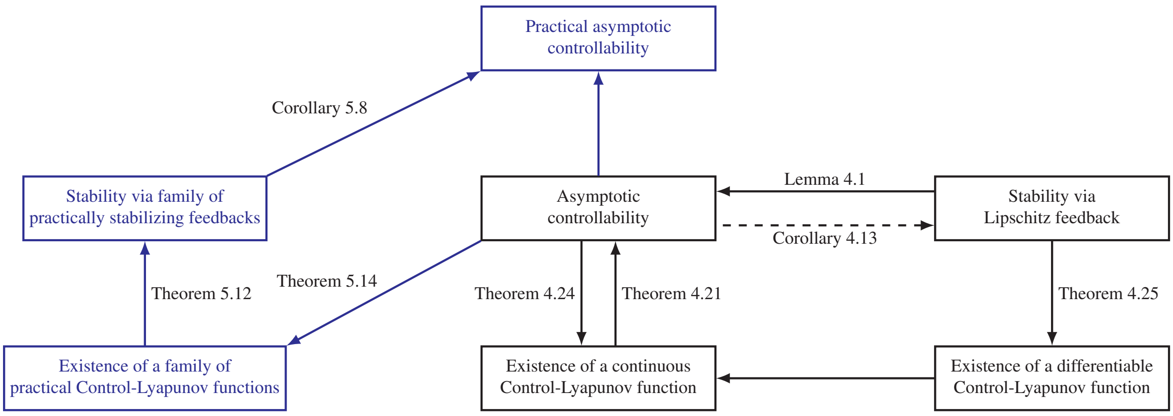


Figure 5.1.: Schematic connection of stability results to digitalization

In practice, however, the two tasks of deriving feedback $\mathbf{u}_T$ and Control-Lyapunov function V_T are often done in the inverse sequence. To this end, first a feedback $\mathbf{u}_T$ is derived, and then the inequality (5.6) is shown to hold for this feedback

$$V_T(\mathbf{x}_T(t_{k+1})) \leq \max\left\{V_T(\mathbf{x}_T(t_k) - T \cdot W(\mathbf{x}_T(t_k)), \hat{\varepsilon})\right\}.$$

The reason for using such a procedure is that Theorem 5.12 only requires a Control-Lyapunov function for fixed $\hat{R}$, $\hat{\varepsilon}$ to exists for some $T_0 > 0$ in order to conclude existence also for all smaller sampling periods. Hence, if we find a constructive way to derive a feedback, then a practical Control-Lyapunov function can be derived and stability properties of this feedback can be concluded for all $T \in (0, T_0]$.
Here, we follow this idea and show how a feedback can be derived, which exhibits the requires properties.

5.2. ISS – Input to state stability

The idea to obtain a feedback in a digital setting is to first derive a feedback in the continuous setting and „embed“ it in a digital setting. The simplest idea is to use the continuous feedback and apply the zero order hold idea to it. As the embedding comes from digitalization, the resulting feedback will exhibit an offset, which is similar to the difference between the continuous and the zero order hold feedback. This difference acts like a disturbance on the system. To formalize this, we consider the disturbed nonlinear control system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{d}(t) \tag{5.8}$$

where the disturbance $\mathbf{d}(t)$ is a measureable function and f is Lipschitz continuous in the disturbance.
Unfortunately, even small disturbances may lead to instability.

Task 5.15
Consider the system

$$\dot{\mathbf{x}}(t) = \begin{cases} -\exp^{-\mathbf{x}(t)+1}, & \mathbf{x}(t) \geq 1 \\ -x, & x \in [-1, 1] \\ \exp^{\mathbf{x}(t)+1}, & \mathbf{x} \leq -1 \end{cases}.$$

Show that the system is asymptotically stable using the Lyapunov function $V(\mathbf{x}) = \mathbf{x}^2/2$. *Show that the disturbed system*

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + \mathbf{d}(t)$$

is unstable.

Solution to Task 5.15: Using the Lyapunov function $V(\mathbf{x})$ we obtain $\alpha_1 = \alpha_2 = V(\mathbf{x})$ and the decrease via $\alpha_3(\mathbf{x}) = \mathbf{x} \cdot f(\mathbf{x})$. Hence, the system is asymptotically stable.
Considering $\mathbf{d}(t) \equiv \varepsilon > 0$, there always exists a $\delta > 0$ such hat $f(\mathbf{x}) + \varepsilon > \varepsilon/2$ for all $\mathbf{x} \geq \delta$. Hence, each solution with initial value $\mathbf{x} \geq \delta$ increases with at least constant rate, i.e. diverges to ∞.

Remark 5.16
Note that this possible instability is not present in the linear case. For systems of the form

$$\dot{\mathbf{x}}(t) = A \cdot \mathbf{x}(t) + D \cdot \mathbf{d}(t)$$

the solution is given by

$$\mathbf{x}(t) = \exp^{A \cdot t} \cdot \mathbf{x}(t_0) + \int_{t_0}^{t} \exp^{A \cdot (t-s)} \cdot D \cdot \mathbf{d}(s) ds.$$

Hence, using $\| \exp^{A \cdot t} \| \leq c \cdot \exp^{-\sigma \cdot t}$*, each solution satisfies*

$$\|\mathbf{x}(t)\| \leq c \cdot \exp^{-\sigma \cdot t} \|\mathbf{x}\| + \frac{c \cdot \|D\|}{\sigma} \cdot \|\mathbf{d}\|_\infty.$$

As a consequence, each solution converges towards a ball with radius $\frac{c \cdot \|D\|}{\sigma} \cdot \|\mathbf{d}\|_\infty$*, i.e. depends on the infinity norm of the disturbance.*

In the nonlinear setting, such a convergence cannot be expected, yet under certain conditions it can be assured. These conditions are known as ISS (input to state stability) and typically formulated for (uncontrolled) systems:

Definition 5.17 (ISS).
Consider the disturbed nonlinear system (5.8) with $\mathbf{u} \equiv 0$ and a neighborhood $\mathcal{N}(\mathbf{x}^\star)$ of $\mathbf{x}^\star = 0$. Then we call the system ISS if there exists function $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}_0\|, t) + \gamma(\|\mathbf{d}\|_\infty) \tag{5.9}$$

holds for all $\mathbf{x} \in \mathcal{N}(\mathbf{x}^\star)$ and all $t \geq 0$.

Having defined the ISS property, we can directly derive the following:

Corollary 5.18
Consider a nonlinear system (5.1) *and its connected disturbed system* (5.8) *with* $\mathbf{u} \equiv 0$*. If the disturbed system is ISS, then the undisturbed system is asymptotically stable*

In the other direction, however, we can use the result from Task 5.15 to derive the following:

Corollary 5.19
Consider a nonlinear system (5.1) *and its connected disturbed system* (5.8) *with* $\mathbf{u} \equiv 0$*. If the system is asymptotically stable, no conclusion regarding stability of the disturbed system can be drawn.*

However, by tightening the requirements on the system, the reverse direction can be concluded:

Theorem 5.20.
Consider a nonlinear system (5.1) *and its connected disturbed system* (5.8) *with* $\mathbf{u} \equiv 0$*. Suppose the undisturbed system to be asymptotically stable, the dynamic to be Lipschitz continuous with respect to state and disturbance. Then there exists a neighborhood* $\mathcal{N}(\mathbf{x}^\star)$ *such that the disturbed system is ISS.*

For our control setting, we can apply this definition if we consider the control to be given by a feedback $\mathbf{u} : \mathcal{X} \to \mathcal{U}$.

Definition 5.21 (η practical ISS).
Consider a disturbed nonlinear control system (5.8) with feedback $\mathbf{u}_T : \mathcal{X} \to \mathcal{U}$ and consider a neighborhood $\mathcal{N}(\mathbf{x}^\star)$ of $\mathbf{x}^\star = 0$.

- If for $\eta > 0$ there exists a function $\beta \in \mathcal{KL}$ such that

$$\|\mathbf{x}_T(t)\| \leq \beta(\|\mathbf{x}_0\|, t) + \eta \tag{5.10}$$

  holds for all $t > 0$ and all $\mathbf{x} \in \mathcal{N}(\mathbf{x}^\star)$, then the system is called η practically ISS.

- The family of systems $\mathbf{x}_T^j$ is called practically ISS if there exists a sequence $\eta^j \to 0$ and a function $\beta \in \mathcal{KL}$ such that

$$\|\mathbf{x}_T^j(t)\| \leq \beta(\|\mathbf{x}_0\|, t) + \eta^j \tag{5.11}$$

  holds for all $t > 0$ and all $\mathbf{x} \in \mathcal{N}(\mathbf{x}^\star)$.

We now want to use the ISS property to conclude under which conditions disturbances introduced by digitalization / zero order hold can be regarded as disturbances and asymptotic controllability of the undistrubed system can be carried over to its zero order hold solution.

5.3. Stability under digitalization

Within this section, we aim to derive conditions to show that the zero order hold solution from Definition 5.3 is practically asymptotically stable. To this end, we first introduce the concept of consistency and show that zero order hold reveals a consistent solution. Thereafter, we present that consistent solutions allow the preservation of asymptotic stability under digitalization, which is called embedding.
We start with the definition of consistency.

Definition 5.22 (Consistency).
Consider two systems $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ and a time $\Delta > 0$. We call both systems to be consistent on a set D if for $\varepsilon > 0$

$$\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq t \cdot \varepsilon \tag{5.12}$$

holds for all $t \in [0, \Delta]$ and all $\mathbf{x} \in D$.

For zero order hold systems, we directly obtain consistency for both the state and its derivative:

Lemma 5.23
Consider a nonlinear control system (5.8) *with feedback* $\mathbf{u}_T : \mathcal{X} \to \mathcal{U}$ *and its respective zero order hold.*

- *For each bounded set D and sampling intervals* $T^j \to 0$ *for* $j \to \infty$ *we have*

$$\|\mathbf{x}(t) - \mathbf{x}_{T^j}(t)\| \leq t \cdot \varepsilon^j \tag{5.13}$$

for all $t \in [0, T^j]$ *mit* $\varepsilon^j = O(T^j)$, *that is* $\varepsilon^j \leq C \cdot T^j$ *for some* $C > 0$.

- *For each bounded set D and sampling intervals* $T^j \to 0$ *for* $j \to \infty$ *we have*

$$\|\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_{T^j}(t)\| \leq t \cdot \varepsilon \tag{5.14}$$

for all $t \in [0, T^j]$ *mit* $\varepsilon^j = O(T^j)$.

Based on consistency, we can embed systems into one another. Our interest in the context of digitalization is to obtain parameters of the practical stability property based on the undisturbed/undigitalized version of the feedback. The idea of embedding is to express one system by another one and to express respective properties by one another. Here, we are particularly interested in the disturbance. In general, embedding is defined as follows:

Definition 5.24 (Embedding).
Consider two disturbed systems $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$. We call the systems to be embedded on a set $D \subset \mathcal{X}$ if for each disturbance $\mathbf{d}_2$ and each $\mathbf{x} \in D$ we have

$$\mathbf{x}_2(t) \in D \qquad \forall t \in [0, T] \tag{5.15}$$

and additionally if there exists a disturbance $\mathbf{d}_1$ such that $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all$t \in [0, T]$ and

$$\|\mathbf{d}_1\|_\infty \leq \delta + \rho \|\mathbf{d}_2\|_\infty. \tag{5.16}$$

In the context of digitalization, we have $\mathbf{d}_2 \equiv 0$ and can therefore always choose $\rho = 0$. Now, we can use embedding and obtain the following core result:

Theorem 5.25 (Equivalency of stability under digitalization).
Consider a nonlinear control system (5.8) *with feedback* $\mathbf{u}_T : \mathcal{X} \to \mathcal{U}$ *and its respective zero order hold. Suppose a bounded neighborhood* $\mathcal{N}(\mathbf{x}^\star)$ *of the operating point* $\mathbf{x}^\star = 0$ *to be given. Then the following statements are equivalent:*

- *The feedback controlled system* (5.8) *is asymptotically stable for all* $\mathbf{x} \in \mathcal{N}(\mathbf{x}^\star)$.
- *The family of systems* $\mathbf{x}_{Tj}$ *is practically asymptotically stable for sufficiently large* j *and the comparison function* $\beta \in \mathcal{KL}$ *is independent from* j.

From Theorem 5.25 we see that asymptotic stability of the continuously controlled system is transferred to the digitally controlled system in the semiglobal practical sense. Here, the sampling time T shows various effects on the quality of the digital closed loop:

- The constants ε, $\hat{\varepsilon}$ and η^j are in general larger if the sampling time T is increased.
- The neighborhood $\mathcal{N}(\mathbf{x}^\star)$ within which practically asymptotic stability can be shown is in general smaller if the sampling time T is increased.
- The performance of solutions is in general degrading if the sampling time T is increased.

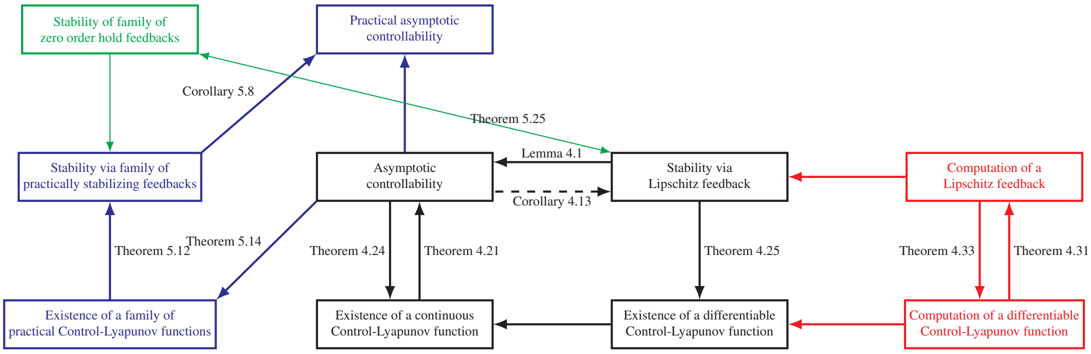


Figure 5.2.: Schematic connection of stability results to derive digital controls

Again, we summarized the results in a schematic sketch given in Figure 5.2.
Here, we see that Theorem 5.25 links the continuous/analog and zero order hold/digital worlds. Hence, we now have the argument to simply apply zero order hold or higher order digitalization methods to feedbacks, which are originally designed for continuous/analog application, and retain stability. Note that this also applies to all control laws derived via classical methods such as PID and all other methods we discussed so far such as cascade or disturbance control.

Table 5.1.: Advantages and disadvantages of digital control

Advantage	**Disadvantage**
✓ Simple zero hold hold derivation	✗ Requires bounded initial state
✓ Allows usage of continuous control methods	✗ Performance may degrade at operating point
✓ Higher orders possible	✗ Performance may degrade far from operating point
✓ Applies to all digitalizations	✗ Range of initial values may be limited

Appendices

APPENDIX A

LAPLACE TRANSFORM

The following Table A.1 recites some of the main properties and laws of computation for Laplace transformed functions.

Table A.1.: Properties of Laplace transformed functions

Property	**Time domain**	**Frequency domain**
Linearity	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 \hat{f}_1(s) + c_2 \hat{f}_2(s)$
Time scaling	$f(at)$	$\frac{1}{a}\hat{f}\left(\frac{s}{a}\right)$
Frequency derivative	$tf(t)$	$-\hat{f}'(s)$
Frequency general derivative	$t^n f(t)$	$(-1)^n \hat{f}^{(n)}(s)$
Time Derivative	$f'(t)$	$s\hat{f}(s) - f(0^+)$
Time general derivative	$f^{(n)}(t)$	$s^n \hat{f}(s) - \sum\limits_{k=1}^{n} s^{n-k} f^{k-1}(0^+)$
Frequency integration	$\frac{1}{t} f(t)$	$\int\limits_s^\infty \hat{f}(\sigma) d\sigma$
Time integration	$\int\limits_0^t f(\tau) d\tau = (\eta * f)(t)$	$\frac{1}{s}\hat{f}(s)$
Frequency shifting	$\exp(at) \cdot f(t)$	$\hat{f}(s-a)$
Time shifting	$f(t-a) \cdot \eta(t-a)$	$\exp(-as) \cdot \hat{f}(s)$

Continued on next page

Table A.1 – continued from previous page

Property	**Time domain**	**Frequency domain**
Multiplication	$f(t) \cdot g(t)$	$\frac{1}{2\pi i} \lim_{T\to\infty} \int_{r-iT}^{r+iT} \hat{f}(\sigma) \cdot \hat{g}(\sigma) d\sigma$
Convolution	$f(t) * g(t) = \int_{0}^{T} f(\tau) \cdot g(t-\tau) d\tau$	$\hat{f}(s) \cdot \hat{g}(s)$
⋮	⋮	⋮

The Laplace transform and of its inverse are typically applied using equivalence tables. Table B.1 summarizes a few of these equivalencies.

APPENDIX B

TRANSFER FUNCTION AND PROPERNESS

Table B.1.: Equivalence table for Laplace transformations

Time domain	**Frequency domain**
$\delta(t)$	1
$\eta(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\exp(at)$	$\frac{1}{s-a}$
$t^n \exp(at)$	$\frac{n}{(s-a)^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\exp(at)\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$\exp(at)\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\vdots$	$\vdots$

Definition B.1 (Controllable normal form).
Consider a transfer function

$$G(s) = \frac{z(s)}{n(s)} \tag{B.1}$$

with coprime polynoms $z(s)$ and $n(s)$. Then the minimal realization

$$s \cdot \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n_x-1} \\ \hat{x}_{n_x} \end{pmatrix}(s) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n_x-2} & -a_{n_x-1} \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n_x-1} \\ \hat{x}_{n_x} \end{pmatrix}(s) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \hat{\mathbf{u}}(s) \tag{B.2a}$$

$$\hat{y} = \begin{pmatrix} b_0 & b_1 & \cdots & b_{n_x-2} & b_{n_x-1} \end{pmatrix} \cdot \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n_x-1} \\ \hat{x}_{n_x} \end{pmatrix}(s) + b_{n_x} \hat{\mathbf{u}}(s) \tag{B.2b}$$

is called controllable normal form or first standard form.

Definition B.2 (Observable normal form).
Consider a transfer function

$$G(s) = \frac{z(s)}{n(s)} \tag{B.3}$$

with coprime polynoms $z(s)$ and $n(s)$. Then the minimal realization

$$s \cdot \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n_x-1} \\ \hat{x}_{n_x} \end{pmatrix}(s) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & 1 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_x-2} \\ 0 & 0 & \cdots & 1 & -a_{n_x-1} \end{bmatrix} \cdot \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n_x-1} \\ \hat{x}_{n_x} \end{pmatrix}(s) + \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n_x-2} \\ b_{n_x-1} \end{pmatrix} \hat{\mathbf{u}}(s) \tag{B.4a}$$

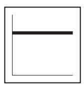
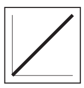
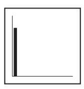
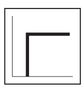
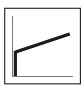
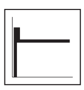
$$\hat{y} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n_x-1} \\ \hat{x}_{n_x} \end{pmatrix} (s) + b_{n_x} \hat{\mathbf{u}}(s) \tag{B.4b}$$

is called observable normal form or second standard form.

APPENDIX C

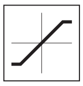
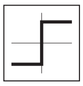
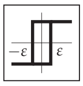
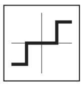
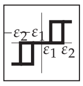
BLOCK DIAGRAM

Table C.1.: List of block symbols

Block symbol	**Meaning**
K_P 	P controller
K_I 	I controller
K_D 	D controller
K_P, K_T 	Latency
K_P, K_I 	PI controller
K_P, K_D 	PD controller

Continued on next page

Table C.1 – continued from previous page

Block symbol	Meaning
K_P, K_D 	PDT1 controller
y_{min}, y_{max} 	Saturation
K_{DT} 	Decay
	Limit
y_{min}, y_{max} 	Bang-bang
y_{min}, y_{max} 	Bang-bang with hysteresis
y_{min}, y_0, y_{max} 	Double-setpoint
y_{min}, y_0, y_{max} 	Double-setpoint with hysteresis
	Nonlinear
	Triangle

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During summer term 2024 I give the lecture to the module *Control Engineering 2 (Regelungstechnik 2)* at the Technical University of Braunschweig. To structure the lecture and support my students in their learning process, I prepared these lecture notes. The aim of the lecture notes is to provide participating students with knowledge of terms of system theory and control engineering. Moreover, students shall be enabled to understand complex control structures, apply control schemes and analyze control systems. After successfully completing the module, students shall additionally be able to apply the discussed methods within real life applications and be able to assess results.