Family Name : .... **Exam number :** ....

First name : .... **Reg. number :** ....

Notes on the exam:

Write name and registration number in the corresponding fields. Do **not** use pencils, green or red pens (used in marking). Place name and reg. number on **each sheet**, number sheets **consecutively** and write only on **one side** of the sheets! Memorize or write down the **exam number**.
You are allowed to use a non-programmable pocket calculator and two pages of equations.

Task	1	2	3	4	5	Σ (50)
Mark						

1. Task (16 pts.)

Geometry

a) From the given knot values $\Xi = \{0, 3.5, 0.5, 5\}$ obtain a valid open knot vector of polynomial order $p = 3$. Further, insert knots to this open knot vector to obtain a refined knot vector of maximum continuity and a polynomial order $p = 5$. Write down the new knot vectors $\Xi_{p=3}$ and $\Xi_{p=5}$. What is the total number of basis functions associated with $\Xi_{p=5}$?

b) A straight line between two points $P_0 = (1, 1)$, $P_1 = (5, 2)$ is modeled as a Bézier curve with $p = 1$ and shall be order elevated to $p = 2$. Compute the new control points.

c) The given knot vector $\Xi = \{0, 0, 1, 2, 2\}$ should be refined to obtain an open, uniform knot vector for a B-spline curve of four cubic elements. For refinement, we consider the following two cases:
1) Knot insertion followed by order elevation.
2) Order elevation followed by knot insertion.

Mention the names of these refinement strategies. Write down the resulting knot vectors. Explain your answer with respect to the inter-element continuity of the refined knot vectors.

d) Write down the coordinates of a control point given as P_i with x-coordinate x_i,y-coordinate y_i, z-coordinate z_i, and weights w_i in homogenous coordinates. For NURBS, control points can have only support in a single element, True or False?

e) Is it possible to obtain the meshes as shown in figure 1 and figure 2 with B-Splines or NURBS? Motivate your answer.

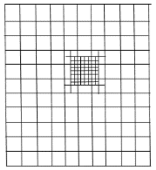


Figure 1: Mesh 1

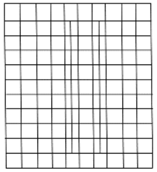


Figure 2: Mesh 2

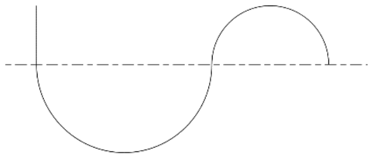
f) Given a B-Spline curve with the open knot vector $\Xi = \{-1, -1, -1, -1, -1, 0.7, 0.7, 0.7, 1, 1, 1, 1, 1\}$. Construct the associated Bézier extraction operator. How can the concept of Bézier extraction be used in the context of Finite Element Analysis codes?

2. Task (8 pts.)

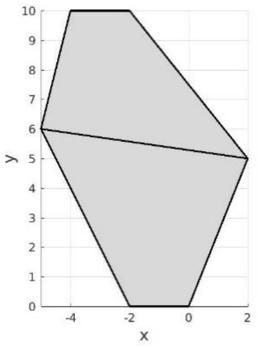
Geometry modeling

The following geometries shall be modeled :

a)



b)



- For the NURBS curve in figure a), work out the minimum required polynomial degree, control point coordinates, weights associated with each control point and knot vector. In addition, sketch the control point grid in the drawing .
- In figure b), the B-spline surface is modeled using the knot vectors $U = \{0, 0, 1, 1\}$ and $V = \{0, 0, 0.5, 1, 1\}$. This surface is further refined to get four elements in the V-direction and two elements in U-direction. Work out the open uniform knot vector of the refined surface and sketch the control point grid in the drawing.

3. Task (9 pts.)

Isogeometric Analysis

We want to perform IGA of a 2D elasticity problem with an alternative ordering of degrees of freedom $\hat{\boldsymbol{u}} = (\hat{u}_{1y}, \hat{u}_{2y}, \hat{u}_{3y}, ..., \hat{u}_{1x}, \hat{u}_{2x}, \hat{u}_{3x}, ...)^T$.

a) Work out the B-matrix for the relation $\boldsymbol{\epsilon} = (u_{y,y}, u_{x,x}, u_{x,y}+u_{y,x})^T = \boldsymbol{B}\hat{\boldsymbol{u}}$.

The following knot vectors $\boldsymbol{\Xi} = \{0, 0, 1, 1\}$ and $\boldsymbol{H} = \{0, 0, 0, 0.5, 1, 1, 1\}$ are used to model a NURBS surface.

b) Calculate the non-zero shape functions for H-direction at $H = 0.25$. Weights of the control points in the H-direction are $\mathbf{w}_H = (1, 0.8535, 0.8535, 1)$. Is the property of partition of unity fulfilled by the calculated shape functions?

For element $[\Xi_2, \Xi_3] \times [H_3, H_4]$,

c) How many integration points are needed to perform numerical integration using Gauss quadrature?

d) Calculate the matrix $\boldsymbol{J}_\phi$ and the associated determinant $|\boldsymbol{J}_\phi|$, considering the affine mapping $\phi : \tilde{\Omega}^e \to \hat{\Omega}^e$ from the parent element to the element in the parametric space.

4. Task (8 pts.)

Collocation

The stationary heat equation is defined as

$$\frac{\partial^2 T}{\partial x^2} + \frac{q(x)}{\lambda} = 0 \qquad \text{on } x \in (0, l),$$

$$\lambda \frac{\partial T}{\partial x} = \bar{q}, \qquad \text{at } x = 0.$$

$$T(x) = 100 + \bar{T}. \qquad \text{at } x = l.$$

The knot vector of the assumed discretization is $\Xi = \{3, 3, 3, 3, 3, 4, 5, 7, 7, 7, 7, 7\}$.

a) Calculate the Greville abscissae for the given knot vector.

b) Write down the stiffness matrix in terms of the discretized version of the stationary heat equation and the corresponding force vector. (Don’t calculate any specific values of the shape functions or their derivatives.)

c) What is the order of the shape function derivatives required for the mapping $x : \hat{\Omega} \to \Omega$ from the parametric space to the physical space?

d) What is the minimum continuity required for the shape functions in case you use a collocation scheme? How is it different from standard Galerkin scheme?

5. Task (9 pts.)

Galerkin

The domain term of the Euler-Bernoulli beam equation is given as

$$EIw_{,xxxx}(x) = f \qquad \text{on } x \in (0, l).$$

A B-Spline basis functions with the knot vector $\boldsymbol{\Xi} = \{0, 0, 0, 0, 0, 0.25, 0.25, 0.5, 0.75, 0.75, 1, 1, 1, 1, 1\}$ is used for the discretization of the trial and test functions of the corresponding weak form.

a) The two boundaries are clamped. How are clamped boundaries modeled?

b) Construct the connectivity matrices ID, IEK, IEN.

c) What is the minimum requirement regarding continuity and polynomial degree to be used for the discretization of the weak form of the Euler-Bernoulli beam equation?

d) Suppose you perform isogeometric analysis of the problem with p = 2,3,4. Sketch the expected error convergence curves for the relative L^2 error of the displacement w vs. number of degrees of freedom in a double logarithmic plot for the different polynomial degrees. How can the orders of convergence be found from the curves?