Last Name : ....

Exam Number : ....

First Name : ....

Matriculation No. : ....

General examination rules:

Write your name and matriculation number on **all** sheets.

Please remember or note your **exam number**.

Please do **NOT** write with a lead pencil or red or green colors.

New task = **new sheet**. Write only on **one side** of each sheet.

All calculations must be **comprehensible**.

Allowed tools: a scientific calculator and a double-sided hand written A4 sheet for formulae etc.

Task	1	2	3	4	5	Σ60
Points						

Problem 1 (11 Points)

A state of stress is given by:

$$\sigma_{11} = \sigma_{22} = 10 \text{ MPa}, \ \sigma_{12} = -6 \text{ MPa},$$
$$\sigma_{13} = \sigma_{23} = \ 0 \text{ MPa}, \ \sigma_{33} = \ 1 \text{ MPa}.$$

a) Decompose the stress tensor into hydrostatic and deviatoric parts.

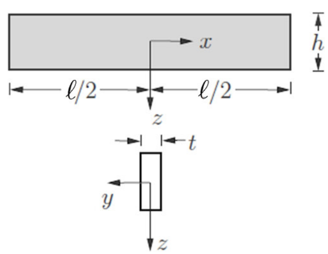
b) Determine the principal stresses of the deviatoric part.

c) Calculate the principal directions of the stress tensor σ_{ij} .

Problem 2 (13 Points)

A beam of rectangular cross-section undergoes a deformation; the dimensions of the beam are as shown in the figure. Assuming a state of plane stress, the following stresses are given:

$$\tau_{xz} = -\frac{6B}{h^3}\left(\frac{h^2}{4} - z^2\right) x,$$
$$\sigma_z = -\frac{6B}{h^3}\left(\frac{z^3}{3} - \frac{h^2}{4} z + \frac{h^3}{12}\right),$$
$$\sigma_x = \frac{12}{h^3}\left(A + \frac{B\ell^2}{8} - \frac{Bh^2}{20}\right) z - \frac{6B}{h^3} x^2 z + \frac{4B}{h^3} z^3$$



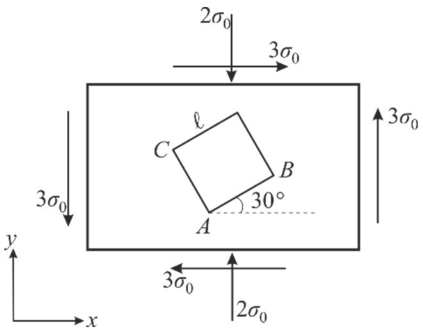
a) Check if the balance equations are satisfied.

b) Calculate and describe the boundary conditions the beam was subjected to which produced the given stress state. What meanings do the constants A and B have?

Given: ℓ, h, t.

Problem 3 (13 Points)

On a thin elastic plate, Young's modulus E and Poisson's ratio ν, a square of length ℓ is marked. The sides of the plate are under constant stresses as shown in the figure.



a) Calculate the deformed length ℓ^* of the side AB.

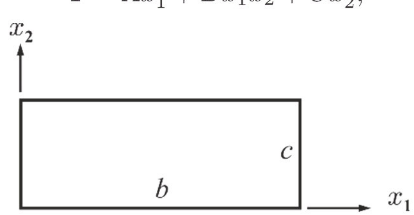
b) Compute the deformed angle between the sides AB and AC of the square.

Given: σ_0, ℓ, E, $\nu = \frac{1}{3}$.

Problem 4 (9 Points)

Given a rectangular plate of length b and width c, as shown. For an Airy's stress function

$$\Phi = Ax_1^2 + Bx_1x_2 + Cx_2^2,$$



a) determine the in-plane stresses, compute and sketch the tractions on all the four edges.

b) As a *plane-strain* solution, determine the remaining stresses and all the strain components.

Problem 5 (14 Points)

1. What are the definitions of the scalar invariants of a tensor and why are they called invariants?
2. How many parameters we need to describe a linear elastic material? And a linear elastic isotropic material? What is an "isotropic material"?
3. Give the definition of principal strains and directions. Describe the possible configurations of the principal directions, accounting for the possible values that the principal strains can have, and state how it is possible to obtain a system of reference.
4. Given the Lamé constants λ and G define the elastic potential and the elasticity tensor $\mathbb{C}$. Describe also the relations between σ_{dev} and ε_{dev} and σ_{hyd} and ε_{vol}.
5. While coupling mechanical and thermal boundary values problems, which equations need to be modified and how?
6. Give the physically meaningful (or practically most interesting) limit values for the Poisson's ratio ν. Illustrate also the behaviors related to those values.
7. Give the weak form of a mechanical boundary value problem and explain its physical meaning.

1

$$\sigma_m = \frac{1}{3}(10+10+1) = 7$$

$$\sigma_h \begin{pmatrix} 7 & & \\ & 7 & \\ & & 7 \end{pmatrix} ; \quad \sigma_d = \begin{pmatrix} 3 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

b) $$|\sigma_d - \lambda I| = 0$$

$$\lambda_1 = -6; \quad \lambda_2 = -3; \quad \lambda_3 = 9$$

$$\underline{n}(\lambda_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \quad \underline{n}(\lambda_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{n}(\lambda_3) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

②

a) Balance Equations ✓

b) • Boundary $z = \frac{h}{2}$

$\sigma(x) = 0$; $\tau_{xz}(x) = 0$

→ Bottom boundary stress free.

• $z = -\frac{h}{2}$

$\sigma_z(x) = -B$; $\tau_{xz} = 0$

⇒ Top boundary has constant

stress in z-direction. (say) $P_z = -\sigma_z = B$

• $x = \frac{\ell}{2}$
• $x = -\frac{\ell}{2}$

similar to above.

→ Both of these (left & right)

boundaries have shear & normal

stresses.

The resultant forces & moments on the left & right boundaries are

↳

Normal : $N\left(\frac{l}{2}\right) = N\left(-\frac{l}{2}\right) = t\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz$

$= 0$

Shear : $V\left(\frac{l}{2}\right) = V\left(-\frac{l}{2}\right) =$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz$$

$$= \frac{1}{2} t B l$$

Moment :

$$M\left(\frac{l}{2}\right) = M\left(-\frac{l}{2}\right) = t\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,\sigma_x \, dz$$

$$= At.$$

3

$$\sigma_x = 0; \quad \sigma_y = -2\sigma_0 \quad ; \quad \sigma_{xy} = \tau_{xy} = 3\sigma_0$$

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\left[\underline{\underline{\sigma}} - \frac{\nu}{1+\nu}\,\mathrm{tr}(\underline{\underline{\sigma}})\,\underline{\underline{I}}\right]$$

$$= \frac{2\sigma_0}{3E}\begin{pmatrix} 1 & 6 & 0 \\ 6 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unit vectors in $\overline{AB}$ & $\overline{AC}$ directions

$$\underline{e}_1 = \begin{bmatrix} \cos 30° \\ \sin 30° \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \underline{e}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

Strain in $\overline{AB}$ direction

$$\varepsilon_{11} = \underline{e}_1^T\,\underline{\underline{\varepsilon}}\,\underline{e}_1 = 2\frac{\sigma_0}{E}\sqrt{3}$$

Also $\quad \varepsilon_{11} = \dfrac{\ell^* - \ell}{\ell} \quad \Rightarrow \quad \ell^* = \ell\left(1 + 2\sqrt{3}\dfrac{\sigma_0}{E}\right)$

Deformed angle between $\overline{AB}$ & $\overline{AC}$

$$\alpha = \frac{\pi}{2} - \gamma_{12} \quad ; \quad \text{where } \gamma_{12} = 2\,\underline{e}_1^T\,\underline{\underline{\varepsilon}}\,\underline{e}_2$$

$$\Rightarrow \quad \gamma_{12} = 1{,}69\,\frac{\sigma_0}{E}$$

④

Φ is Bioharmonic

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} = 2C \quad ; \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = -B$$

$$\sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} = 2A$$

at $x_2 = 0$; $\underline{t}_1 = \underline{\underline{\sigma}}\,\underline{n} = \begin{pmatrix} B \\ -2A \end{pmatrix}$; with $\underline{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

at $x_2 = c$; $\underline{t}_2 = \begin{pmatrix} -B \\ 2A \end{pmatrix}$

at $x_1 = 0$; $\underline{t}_3 = \begin{pmatrix} -2C \\ B \end{pmatrix}$; at $x_1 = b$; $\underline{t}_4 = \begin{pmatrix} 2C \\ -B \end{pmatrix}$

b) $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) = 2\nu(C+A)$

$$\sigma_{13} = \sigma_{23} = 0$$

$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$ ($\because$ plain strain)

$$\varepsilon_{11} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{11} - \nu\sigma_{22}\right] = \frac{2(1+\nu)}{E}\left[C(1-\nu) - \nu A\right]$$

similarly ε_{22}, ε_{12}.