1. Elastizitätstheorie

1.1 Dehnung

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$2\varepsilon_{xy} = \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad ; \quad 2\varepsilon_{yz} = \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad ; \quad 2\varepsilon_{xz} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(i) Transformation

$$\varepsilon_\xi = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\varphi + \tfrac{1}{2}\gamma_{xy}\sin 2\varphi$$

$$\varepsilon_\eta = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\varphi - \tfrac{1}{2}\gamma_{xy}\sin 2\varphi$$

$$\tfrac{1}{2}\gamma_{\xi\eta} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\varphi + \tfrac{1}{2}\gamma_{xy}\cos 2\varphi$$

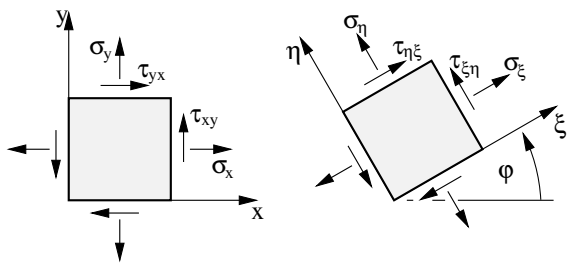
(ii) Hauptdehnungen

<u>ε extremal:</u>

$$\tan 2\varphi^* = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad ; \quad \varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\tfrac{1}{2}\gamma_{xy}\right)^2}$$

1.2 Spannung

(i) Transformation

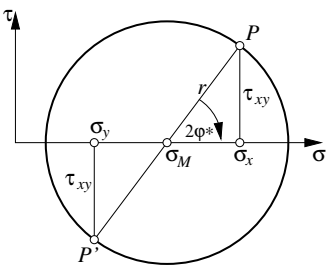


$$\sigma_\xi = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi$$

$$\sigma_\eta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi - \tau_{xy}\sin 2\varphi$$

$$\tau_{\xi\eta} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi$$

(ii) Hauptspannungen



<u>σ extremal:</u>

$$\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad ; \quad \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

<u>τ extremal:</u>

$$\tan 2\varphi^{**} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad ; \quad \tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\tfrac{1}{2}(\sigma_1 - \sigma_2)$$

1.3 Elastizitätsgesetz (mit Lamé-Konstanten bzw. E,ν)

- 3D Spannung aus Dehnung

$$\begin{aligned}
\sigma_x &= 2\mu\varepsilon_x + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) \quad ; \quad \sigma_y = 2\mu\varepsilon_y + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) \quad ; \\
\sigma_z &= 2\mu\varepsilon_z + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) \\
\tau_{xy} &= \mu\gamma_{xy} \quad ; \quad \tau_{yz} = \mu\gamma_{yz} \quad ; \quad \tau_{zx} = \mu\gamma_{zx}
\end{aligned}$$

$$\begin{aligned}
\sigma_x &= \tfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y+\varepsilon_z)\right] \quad ; \quad \sigma_y = \tfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_x+\varepsilon_z)\right]; \\
\sigma_z &= \tfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x+\varepsilon_y)\right] \\
\tau_{xy} &= G\gamma_{xy} \quad ; \quad \tau_{yz} = G\gamma_{yz} \quad ; \quad \tau_{zx} = G\gamma_{zx}
\end{aligned}$$

- 3D Dehnung aus Spannung

$$\begin{aligned}
\varepsilon_x &= \tfrac{1}{E}\left[\sigma_x - \nu(\sigma_y+\sigma_z)\right] \quad ; \quad \varepsilon_y = \tfrac{1}{E}\left[\sigma_y - \nu(\sigma_z+\sigma_x)\right] \quad ; \quad \varepsilon_z = \tfrac{1}{E}\left[\sigma_z - \nu(\sigma_x+\sigma_y)\right] \\
\gamma_{xy} &= \tfrac{1}{G}\tau_{xy} \quad ; \quad \gamma_{yz} = \tfrac{1}{G}\tau_{yz} \quad ; \quad \gamma_{zx} = \tfrac{1}{G}\tau_{zx}
\end{aligned}$$

1.4 Ebene Zustände

– Ebener Spannungszustand: $\sigma_z = \tau_{yz} = \tau_{zx} = 0.$

– Ebener Verzerrungszustand: $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0.$

2. Stäbe

- Normalspannung

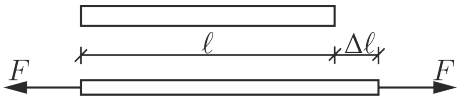
$$\sigma = \frac{N}{A}$$

- DGL der Verschiebung

$$EAu' = N(x),$$
$$(EAu')' = -n(x)$$

- Längenänderung

$$\Delta\ell = F \cdot \ell / EA, \qquad EA \text{ und } F \text{ konstant.}$$



3. Balken

3.1 Geometrische Eigenschaften des Querschnittes

- Statisches Moment: $S_y = \int_A z\,\mathrm{d}A\,, \quad S_z = \int_A y\,\mathrm{d}A$
- Flächenträgheitsmoment: $I_y = \int_A z^2 \mathrm{d}A$

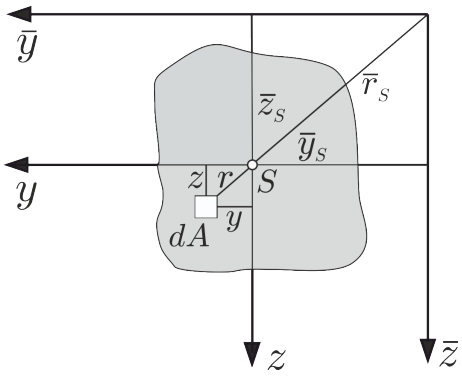
$$I_z = \int_A y^2 \mathrm{d}A$$
$$I_{yz} = -\int_A y\,z\,\mathrm{d}A$$

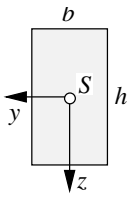
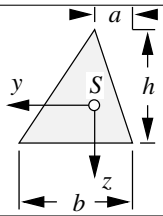
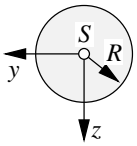
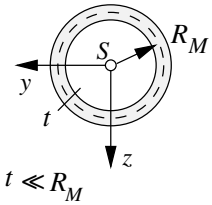
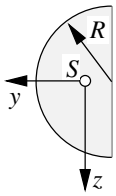
- Polares Moment: $I_S = \int_A r^2 dA$

$$= I_y + I_z$$

- Satz von Steiner: $I_{\bar{y}} = I_y + \bar{z}_S^2 A$

$$I_{\bar{z}} = I_z + \bar{y}_S^2 A$$
$$I_{\bar{y}\bar{z}} = I_{yz} - \bar{y}_S \bar{z}_S A$$
$$I_{\bar{S}} = I_S + \bar{r}_S^2 A$$



	Rechteck	Dreieck	Kreis	dünner Kreisring	Halbkreis
					
I_y	$\frac{bh^3}{12}$	$\frac{bh^3}{36}$	$\frac{\pi R^4}{4}$	$\pi R_M^3 t$	$\frac{\pi R^4}{8}$
I_z	$\frac{hb^3}{12}$	$\frac{bh}{36}(b^2 - ba + a^2)$	$\frac{\pi R^4}{4}$	$\pi R_M^3 t$	$\frac{R^4}{72\pi}(9\pi^2 - 64)$
I_{yz}	0	$\frac{bh^2}{72}(2a - b)$	0	0	0

- Transformation: Siehe 1.2 (mit $\sigma_x = I_y, \quad \sigma_y = I_z, \quad \tau_{xy} = I_{yz}$)

3.2 Balken unter Biegung

- Normalspannung

  (i) Gerade Biegung: $\sigma = \frac{M}{I_y} z$

  (ii) Schiefe Biegung + Normalkraft:

  $$\sigma = \frac{N}{A} + \frac{1}{\Delta}\left[(-M_z I_y + M_y I_{yz})y + (M_y I_z - M_z I_{yz})z\right],$$

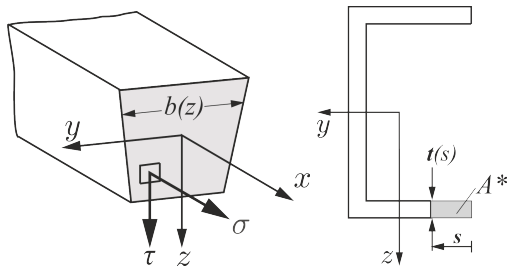
mit $\quad \Delta = I_y I_z - I_{yz}^2 \cdot$

- DGL der Biegelinie

  $$EIw'' = -M(x); \qquad (EIw'')' = -Q(x); \qquad (EIw'')'' = q(x).$$

- Schubspannung infolge Querkraft

  $$\tau(z) = \frac{Q_z S_y^*(z)}{I_y b(z)}\ ; \quad S_y^*(z) = \int_{A^*} z\, dA \cdot$$



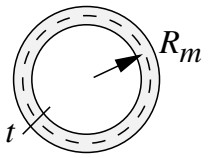
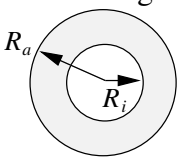
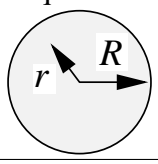
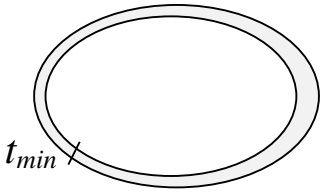
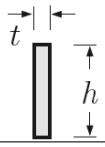
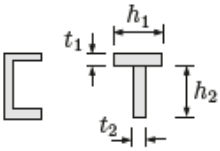
- Dünnwandige Querschnitte

  $$\tau(s) = \frac{Q_z S_y^*(s)}{I_y t(s)} \cdot$$

3.3 Torsion

- Schubspannung: $\tau_{max} = \frac{M_T}{W_T};$
- DGL der Verdrehung

$$\begin{aligned} GI_T \vartheta' &= M_T; \\ (GI_T \vartheta')' &= -m_T \cdot \end{aligned}$$

Querschnitt	W_T	I_T
dünnwandiges Kreisrohr 	$2\pi R_m^2 t$	$2\pi R_m^3 t$
dickwandiges Kreisrohr  $\alpha = \frac{R_i}{R_a}$	$\frac{\pi R_a^3}{2}(1-\alpha^4)$	$\frac{\pi R_a^4}{2}(1-\alpha^4)$
Vollkreisquerschnitt 	$\frac{\pi R^3}{2}$	$\frac{\pi R^4}{2}$
dünnwandige geschlossene Hohlquerschnitte 	$2A_m \cdot t_{min}$ A_m: die von der Profilmittellinie eingeschlossene Fläche.	$\frac{(2A_m)^2}{\oint \frac{ds}{t}}$
schmales Rechteck 	$\frac{1}{3}ht^2$	$\frac{1}{3}ht^3$
zusammengesetzte Profile 	$\approx \frac{1}{3}\frac{\sum h_i t_i^3}{t_{max}}$	$\approx \frac{1}{3}\sum h_i t_i^3$

3.4 Knickung

- DGL

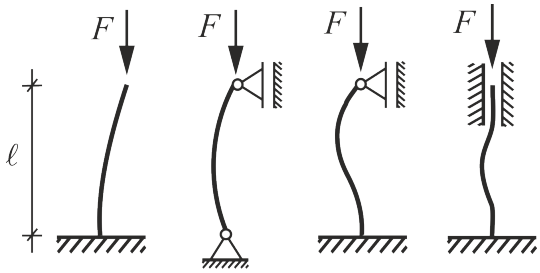
  $w^{IV} + \lambda^2 w'' = 0,$

  mit $\lambda^2 = F/EI,$ EI Konstant

Ansatz

$$w = A\cos\lambda x + B\sin\lambda x + C\lambda x + D$$

- $F_{krit} = \pi^2 \frac{EI}{\ell_k^2}$



4. Energiemethoden

4.1 Äußere Arbeit und gespeicherte Energie

- Äußere Arbeit

$$W = \int F \mathrm{d}u = \frac{1}{2} F u \qquad ; \qquad W = \int M \mathrm{d}\varphi = \frac{1}{2} M \varphi$$

- Formänderungsenergie pro Längeneinheit

$$\mathrm{d}\Pi = \frac{1}{2}\frac{N^2}{EA} \quad \text{(Stab unter Zug/Druck)} \qquad \mathrm{d}\Pi = \frac{1}{2}\frac{M^2}{EI} \quad \text{(Balken unter Biegung)}$$

$$\mathrm{d}\Pi = \frac{1}{2}\frac{Q^2}{GA_S} \quad \text{(Balken unter Querkraft)} \qquad \mathrm{d}\Pi = \frac{1}{2}\frac{M_T^2}{GI_T} \quad \text{(Stab in Torsion)}$$

4.2 Satz von Castigliano

$$\frac{\partial \Pi}{\partial F_i} = u_i \qquad ; \qquad \frac{\partial \Pi}{\partial u_i} = F_i \qquad ; \qquad \frac{\partial \Pi}{\partial M_i} = \varphi_i \qquad ; \qquad \frac{\partial \Pi}{\partial \varphi_i} = M_i$$

4.3 Satz von Betti

$$F_j u_{ji} = F_i u_{ij}$$

u_{ij}: Verschiebung an Stelle i aufgrund einer Kraft an Stelle j

Entsprechendes für Momente und Verdrehungen.

4.4 Arbeit virtueller Kräfte

$$\mathbf{1} u = \int \frac{N\bar{N}}{EA}\mathrm{d}x + \int \frac{M\bar{M}}{EI}\mathrm{d}x + \int \frac{Q\bar{Q}}{GA_S}\mathrm{d}x + \int \frac{M_T\bar{M}_T}{GI_T}\mathrm{d}x$$

N, M, Q, M_T: Schnittgrößen aufgrund äußerer Lasten

$\bar{N}, \bar{M}, \bar{Q}, \bar{M}_T$: Schnittgrößen aufgrund virtueller Kraft **1**

Entsprechendes für virtuelle Momente und Verdrehungen.

4.5 Koppeltafel zu 4.4

	M	k, s	k, s	k, s	k_1, k_2, s
	i, s	sik	$\frac{1}{2}sik$	$\frac{1}{2}sik$	$\frac{1}{2}si(k_1+k_2)$
	i, s	$\frac{1}{2}sik$	$\frac{1}{3}sik$	$\frac{1}{6}sik$	$\frac{1}{6}si(k_1+2k_2)$
	i_1, i_2, s	$\frac{1}{2}s(i_1+i_2)k$	$\frac{1}{6}s(i_1+2i_2)k$	$\frac{1}{6}s(2i_1+i_2)k$	$\frac{1}{6}s(2i_1k_1+2i_2k_2$ $+i_1k_2+i_2k_1)$
	quadratisch i, s	$\frac{2}{3}sik$	$\frac{1}{3}sik$	$\frac{1}{3}sik$	$\frac{1}{3}si(k_1+k_2)$
	quadratisch i, s	$\frac{2}{3}sik$	$\frac{5}{12}sik$	$\frac{1}{4}sik$	$\frac{1}{12}si(3k_1+5k_2)$
	quadratisch i, s	$\frac{1}{3}sik$	$\frac{1}{4}sik$	$\frac{1}{12}sik$	$\frac{1}{12}si(k_1+3k_2)$
	kubisch i, s	$\frac{1}{4}sik$	$\frac{1}{5}sik$	$\frac{1}{20}sik$	$\frac{1}{20}si(k_1+4k_2)$
	kubisch i, s	$\frac{3}{8}sik$	$\frac{11}{40}sik$	$\frac{1}{10}sik$	$\frac{1}{40}si(4k_1+11k_2)$
	kubisch i, s	$\frac{1}{4}sik$	$\frac{2}{15}sik$	$\frac{7}{60}sik$	$\frac{1}{60}si(7k_1+8k_2)$

∘ Kennzeichnet den Nullpunkt der Dreieckslast