Exercises to the lecture
Concurrency Theory
Sheet 3

Prof. Dr. Roland Meyer
Eren Keskin & Jan Grünke Delivery until 11.06.2026 at 15:00

Exercise 3.1 (Linearly Independent Periods)

Write the linear sets $L_i \subseteq \mathbb{N}^3$ below as a union of linear sets $L_i = \bigcup_j b_{i,j} + P_{i,j}$ whose periods $P_{i,j} \subseteq \mathbb{N}^3$ are linearly independent.

a) $L_1 = (2,3,0) + \{(2,2,2),\ (1,2,1),\ (2,3,2)\}^*$

b) $L_2 = (2,1,0) + \{(2,1,0),\ (0,1,1),\ (1,2,3),\ (1,2,0)\}^*$

Exercise 3.2 (Primeful Feasability)

Develop an algorithm to decide the following problem. Show that your algorithm is correct.

ALL-PRIMES-FEASABILITY

Given: $A \in \mathbb{Z}^{\ell \times k}, b \in \mathbb{Z}^{\ell}$

Decide: For all prime $n \in \mathbb{N},$ there is a $y \in \mathbb{N}^d$ with $A \cdot y \geq b$ and $y[1] = n$.

Use the Dirichlet Prime Number Theorem:

Theorem 1 *Let $n \in \mathbb{N}_{\geq 1}$. For all $0 < a < n$ that is co-prime to n, there are infinitely many prime numbers $p_1, p_2, \ldots$ with $a \equiv p_i \mod n$ for all $i \in \mathbb{N}$.*

Your algorithm may assume that given $A \in \mathbb{Z}^{\ell \times k}$ and $b \in \mathbb{Z}^{\ell}$, the semilinear representation $B, P \subseteq \mathbb{N}^d$ of the solution space $B + P^* = \mathsf{sol}(A \cdot x \geq b)$ can be computed.

Exercise 3.3 (Understanding Semilinear Sets)

Prove or disprove the following statements.

a) All semilinear $S \subseteq \mathbb{N}$ can be written as

$$S = F \cup (B + \{p\}^*)$$

for some $p \in \mathbb{N}$ and finite $B, F \subseteq \mathbb{N}$.

b) Let $p, r \in \mathbb{N}$ be co-prime. Then

$$\{p, r\}^* = \mathbb{N} \setminus F$$

for finite $F \subseteq \mathbb{N}$.

c) All semilinear $S \subseteq \mathbb{N}^2$ can be written as

$$S = F \cup (B + P^*)$$

for finite $B, F, P \subseteq \mathbb{N}^2$.

d) If $S \subseteq \mathbb{N}^d$ is semilinear, then so is S^*.

Exercise 3.4 (Presburger Logic)

For each question, construct the first order Presburger logic formula with the given properties. Let $\Delta(x)$ be a Presburger formula with free variables $x \in \mathbb{N}^d$.

a) Ψ *with no free variables:* Ψ holds iff there are infinitely many $y \in \mathbb{N}^d$ with $\Delta(y)$.

b) $\Phi_c(x)$ *with d free variables, $x \in \mathbb{N}^d$:* Let $c \in \mathbb{N}^d$. For all $y \in \mathbb{N}^d$, $\Phi_c(y)$ holds iff $\Delta(y)$ holds and y maximizes the inner product $\langle y, c\rangle$ among the vectors that fulfill $\Delta(x)$, i.e.

$$\langle y, c\rangle = \max\{\langle u, c\rangle \mid u \in \mathbb{N}^d,\ \Delta(u)\}$$

c) Γ_A *with one free variable, $x \in \mathbb{N}$:* Let $A \in \mathbb{Z}^{\ell \times d}$. For all $s \in \mathbb{N}$, $\Gamma_A(s)$ holds iff

$$\max_{b \in B} ||b||_1 = s$$

where $B = \min\{y \in \mathbb{N}^d \mid \Delta(y)\}$, is the set of minimal elements that fulfill $\Delta(x)$ wrt. the $\leq_A$-order.

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