VASS $\mathbb{Z}$ - Reachability

Goal: Characterize $\mathbb{Z}$-runs by a Linear equation system

More precisely: Characterize Ψ($\mathbb{Z}$-runs)

Approach: To find a $\mathbb{Z}$-run from (q_1, v_1) to (q_2, v_2) check if
- there is a path through the state space from q_1 to q_2
- the path has $\underbrace{\text{overall effect}}_{\text{only depends on Parikh image}}$ $v_2 - v_1$

Lemma Let $V = (Q, C, T)$ be a VASS.
There is a $\mathbb{Z}$-run r from (q_s, v_s) to (q_t, v_t)

iff $\exists\, x \in \mathbb{N}^T$. so that

(PATH) $\exists\, \sigma = (q_1, z_1, q_2).(q_2, z_2, q_3) \dots (q_{k-1}, z_{k-1}, q_k) \in T^*$.
$q_1 = q_s \wedge q_k = q_t \wedge \Psi(\sigma) = x$

(MARKING) $v_2 = v_1 + \Delta \cdot x$
where
$\Delta \in \mathbb{Z}^{C \cup T}$ with $\Delta[c, t] = z[c]$
for all $c \in C$ and $t = (q, z, q') \in T$.

Proof:

"<=" Note that a run is well-defined by a start configuration and a transition sequence $\sigma \in T^*$, which is given by (PATH).

"=>" (PATH): Let $x := \Psi(r)$ and σ be the projection of r to T.
Then $q_1 = q_s$, $q_k = q_t$, and $\Psi(\sigma) = \Psi(r) = x$.

(MARKING): Induction on the length of r.
Intuitively, the order of the transitions in r does not affect the overall effect on the counters.

(MARKING) is called the <u>marking equation</u>.
Note that (MARKING) is already a linear equation system.

(PATH) cannot be expressed by linear equations in general but we can use linear equations if we assume that all transitions in V with $x[t] > 0$ are connected.
(in a graph theoretic sense)

Let $G = (V, E)$ be a graph and $x \in \mathbb{N}^E$.
Then $G[x] := (V[x], E[x])$ is the induced subgraph with
$E[x] := \{ e \in E \mid x[e] > 0 \}$ $V[x] := \{ v \in V \mid \exists w.\ (w,v) \in E[x] \lor (w,v) \in E[x] \}$

<u>Lemma (Euler - Kirchhoff)</u>

Let $G = (V, E)$ be a directed, $v_s, v_f \in V$ and $x \in \mathbb{N}^E$.
Then there is a path p from v_s to v_f with $\Psi(p) = x$

<u>iff</u> (CON) induced subgraph $G[x]$ is connected

(KIRCH) $\forall v \in V.\ \sum_{e=(w,v)} x[e] - \sum_{e=(v,w)} x[e] = \begin{cases} -1 & , v = v_s \neq v_f \\ 1 & , v = v_f \neq v_s \\ 0 & , \text{else} \end{cases}$

<u>Proof</u>:

"$\Rightarrow$" (CON) is satisfied by connectedness of p.
We show (KIRCH) by induction on the number of $\underbrace{\text{simple}}_{\text{no vertex repeats}}$ cycles that p goes through (can be removed).

<u>$\# = 0$</u>: p is a simple path and clearly satisfies (KIRCH)

<u>$\# = n+1$</u>: Then p has the form
$$p = p_1 \cdot c \cdot p_2$$
where c is a simple cycle.
A simple cycle clearly satisfies the $\underbrace{\text{homogenous variant}}_{\text{rhs is set to zero}}$ of (KIRCH)

Moreover, $p_1 \cdot p_2$ goes through less simple cycles.
Thus, $\Psi(p_1 \cdot p_2)$ satisfies (KIRCH) by the IH.
Then $\Psi(p_1 \cdot p_2) + \Psi(c)$ satisfies (KIRCH) and since
$$\Psi(p) = \Psi(p_1 \cdot c \cdot p_2) = \Psi(p_1 p_2) + \Psi(c)$$
also $\Psi(p)$ satisfies (KIRCH).

"$\Leftarrow$": We do induction on the number of simple cycles in $G[x]$

#=0: Then $G[x]$ is acyclic. Together with (KIRCH) this implies that $G[x]$ is a simple path.

#>0: Pick a simple cycle c' in $G[x]$
Choose $x' = x - \psi(c')$. Then

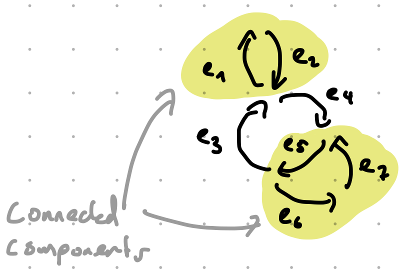
- $x' \geq 0$ because $\psi(c') \leq x$
- x' satisfies (KIRCH) because $\psi(c')$ satisfies homogenous variant of (KIRCH)

We repeat subtracting $\psi(c')$ until $x'[e] = 0$ for some $e \in E$ with $x[e] \neq 0$.
Then $G[x']$ has less simple cycles than $G[x]$

Let $G_1, \dots, G_k$ be the connected components of $G[x']$.
Since x' satisfies (KIRCH) and each edge contributes $+1$ and -1 to its connected component there is one connected component that contains v_s and v_t.
Wlog. let $v_s, v_t \in G_1$.

We invoke the IH on $G_1, \dots G_k$ and obtain a path p_1 and cycles $c_2, \dots, c_k$.

We construct p from p_1 by inserting the following cycle:
We repeat c' (the number of time we removed $\psi(c')$ from x) and insert $c_2, \dots, c_k$ as illustrated below:



$x = (2,2,1,1,2,1,1)$

Primitive cycle $c' = e_3 . e_4 . e_5$

$x' = x - \psi(c') = (2,2,0,0,1,1,1)$

IH yields $c_1 = e_1 e_2 e_1 e_2$
and $c_2 = e_5 e_6 e_7$

$\Rightarrow c = e_3 . c_1 . e_4 . c_2 . e_5$