VAS Reachability by Semi-Linear Invariants

Goal: Show that semi-linear inductive invariants exists if target is not reachable.

Let $V=(C,T)$ be a VAS with initial configuration $m_s \in \mathbb{N}^C$ and final configuration $m_f \in \mathbb{N}^C$

A semi-linear inductive invariant is a set $I \subseteq \mathbb{N}^C$ so that:

(semi-linear) I semi-linear
(initial) $m_s \in I$
(inductive) $post(I) \subseteq I$ with $post(S) = \{ s+t \in \mathbb{N}^C \mid s \in S \wedge t \in T \}$
(safe) $m_f \notin I$

Then reachability set is $post^*(\{m_s\})$ where $post^*(S) = \bigcup_{i \in \mathbb{N}} post^i(S)$

Similarly, we will use $pre(S) = \{ s-t \in \mathbb{N}^C \mid s \in S \wedge t \in T \}$ and $pre^*(S) = \bigcup_{i \in \mathbb{N}} pre^*(S)$

Theorem: There is a semi-linear inductive invariant if $m_f \notin post^*(\{m_s\})$

Then we can solve VAS reachability by running two semi algorithms in parallel:

(1) Enumerate runs from m_s and return YES if m_f is reached

(2) Enumerate Presburger formula $\varphi(x)$ and return NO if it is an inductive invariant:

(initial) $\varphi(m_s)$ is true
(inductive) $\forall x \in \mathbb{N}^C.\ \forall t \in T.$
$\varphi(x) \wedge x+t \geq 0 \rightarrow \varphi(x+t)$ is true
(safe) $\varphi(m_f)$ is false

Correctness of this algorithm follows from $post^*(\{m_s\}) \subseteq I$ and termination by above theorem.

Semi pseudo-linear sets

A set $S \subseteq \mathbb{Z}^d$ is pseudo-linear if there is a linear set $b + P^*$ so that

(OVER) $S \subseteq b + P^*$

(UNDER) $\forall$ finite $I \subseteq \text{interiorCone}(P)$. $\exists b' \in \mathbb{Z}^d$

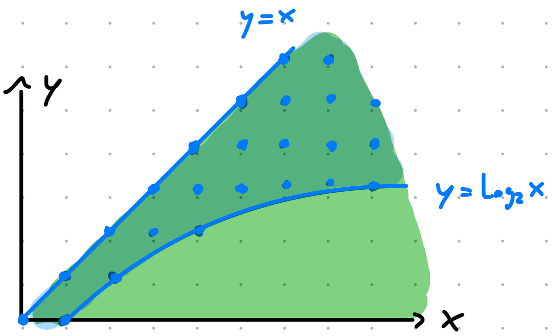
$$b' + I^* \subseteq S$$

where $\text{interiorCone}(P) = P^* \cap (\mathbb{Q}_{>0}\, p_1 + \dots + \mathbb{Q}_{>0}\, p_k)$ where $P = \{p_1, \dots, p_k\}$

We then call $b + P^*$ a linearization of S.

A set is semi pseudo-linear if it is a finite union of pseudo-linear sets.

Example:



$PL = \{(x,y) \in \mathbb{N}^2 \mid \log_2(x) \leq y \leq x\}$ is pseud-linear with linearization

$L = \{(x,y) \in \mathbb{N}^2 \mid y \leq x\}$

Lemma Semi pseudo-linear sets are closed under linear functions.

Proof: Homework

Dimension: We assign a dimension $\dim(S) = \dim(V(S))$ to a set $S \subseteq \mathbb{Z}^d$.

vector space of S

Lemma $\dim(PL) = \dim(L)$ for pseudo-linear set PL with linearization L

Proof: (OVER) implies $\dim(PL) \leq \dim(L)$

For converse let $L = b + P^*$ and $x \in \text{interiorCone}(P)$

Then, $x + P \subseteq \text{interiorCone}(P)$ and (UNDER) implies $\exists y \in PL$.

$$y + (x+P)^* \subseteq PL$$

Thus $\dim(L) = \dim(y + (x+P)^*) \leq \dim(PL)$.

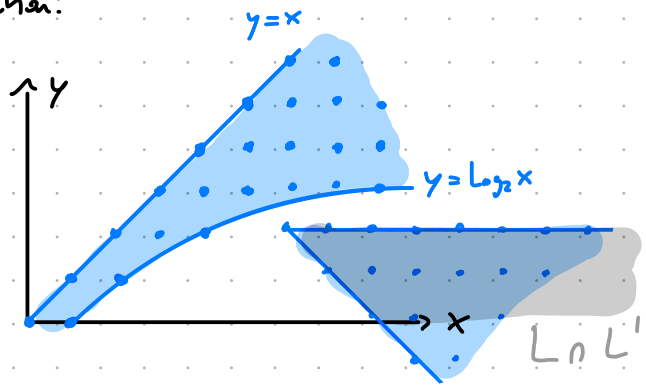
Later to construct semi-linear inductive invariants we will use the following result shown in "The general vector addition system reachability problem by Presburger inductive invariants" by Leroux.

<u>Lemma</u> Let PL and PL' be pseudo-linear set with linearizations L and L'. Then $L \cap L'$ is a semi-linear set $\bigcup_{i=1..k} b_i + P_i$ and

$PL \cap PL' = \emptyset$ implies $\dim(P_i) < \dim(PL \cup PL')$

for all $i = 1, \dots, k$.

Illustration:



$PL \cap PL' = \emptyset$

$L = \{(x,y) \in \mathbb{N}^2 \mid y \leq x\}$

$L' = PL'$

$L \cap L' = \{\binom{6}{2}, \binom{7}{1}, \binom{8}{0}\} + \{\binom{1}{0}\}^*$

$\dim(\{\binom{1}{0}\}^*) = 1 < 2 \quad \dim(PL \cup PL')$

VAS reachability set is semi pseudo-linear

Recall the Iteration Lemma:

> Let $\mathcal{M} = G_0 z_1 \ldots z_k G_k$ be a perfect MGTS with feasible $Char_\mathcal{M}$.
>
> Let $u_i \in Pump_{up}(G_i)$ and $v_i \in Pump_{dn}(G_i)$ for all $1 \le i \le k$.
>
> There are $r_i, w_i \in G_i T^*$ and $n_0 \in \mathbb{N}$ so that
>
> $$u_0^n w_0^n r_0 v_0^n z_1 \ldots\ldots z_k u_k^n w_k^n r_k v_k^n \quad \text{is an } \mathbb{N}\text{-run}$$
>
> for all $n \ge n_0$.

Let m_i be the configuration reached by $u_0^n w_0^n r_0 v_0^n z_1 \ldots z_i u_i^n$

The proof of the iteration lemma also shows that:

$$(*)\ m_i[j] \ge n \quad \text{for all } 1 \le i \le k \text{ and all } j \in \Omega(G_i)$$

Indeed: If $j \in \Omega_{up}(G_i)$: $m_i[j] \ge \underset{\text{input configs of } G_i \text{ in counter } j}{S_n[in, i, j]} - t \cdot n \cdot \underset{\text{max neg. effect of } u_i}{|u_i|} \ge n$

If $j \in \Omega_{up}(G_i) \setminus \Omega_{up}(G_i)$: $m_i[j] \ge S_n[in, i, j] + n \cdot |u_i| \ge n$

We will expoit this to show that VAS runs have semi pseudo-linear Parikh image.

<u>Lemma</u> Let $V = (C, T)$ be a VAS and $m_s, m_f \in \mathbb{N}^C$.
Then $\Psi(Runs_{\mathbb{N}}(V, m_s, m_f))$ is semi-pseudo-linear.

<u>Corollar</u> $post^*(S)$ is semi pseudo-linear for semi-linear $S \subseteq \mathbb{N}^C$.

The corollar follows because $post^*(S)$ is a linear function applied to $\Psi(Runs_{\mathbb{N}}(V))$.

<u>Proof for Lemma:</u>

We show that $\Psi(Runs_{\mathbb{N}}(\mathcal{M}))$ is pseudo-linear for a perfect MGTS $\mathcal{M} = G_0 z_1 \ldots z_k G_k$ with feasible $Char_\mathcal{M}$.
Then, the lemma follows by decomposition.

We choose $P := \min\{x \in sol(HChar_\mathcal{M}) \mid x \ne 0\}$ and show (OVER) and (UNDER):

(OVER) $\exists b \in \mathbb{Z}^C.\ \Psi(\mathrm{Runs}_{\mathbb{N}}(\mathcal{M})) \subseteq b + P^*$:

Since $\mathcal{M}$ is perfect and $\mathrm{Char}_{\mathcal{M}}$ is feasible there is

- a solution s of $\mathrm{Char}_{\mathcal{M}}$ and
- a full support solution h of $H\mathrm{Char}_{\mathcal{M}}$.

Then for every solution s' of $\mathrm{Char}_{\mathcal{M}}$ there is $c \in \mathbb{N}$ so that

$$s' + c \cdot h \geq s$$

Since $\min(\mathrm{sol}(\mathrm{Char}_{\mathcal{M}}))$ is finite, by scaling h we can assume

$$s' + h \geq s \qquad \text{for all } s' \in \mathrm{sol}(\mathrm{Char}_{\mathcal{M}})$$

Then $\Psi(\mathrm{Runs}_{\mathbb{N}}(\mathcal{M})) \subseteq (s - h) + P^*$

(UNDER) $\forall\ h_1, \dots, h_e \in \mathrm{interiorCone}(P)\ \exists b' \in \mathbb{Z}^C.\ b' + \{h_1, \dots h_e\}^* \subseteq \Psi(\mathrm{Runs}_{\mathbb{N}}(\mathcal{M}))$:

Let $h_1, \dots, h_e \subseteq \mathrm{interiorCone}(P) = P^* \cap (\mathbb{Q}_{>0}\, p_1 + \dots + \mathbb{Q}_{>0}\, p_n)$

Note that $h_1, \dots, h_e$ are support solutions by $\mathrm{interiorCone}(P) \subseteq P^*$ and have full support by $\mathrm{interiorCone}(P) \subseteq \mathbb{Q}_{>0}\, p_1 + \dots + \mathbb{Q}_{>0}\, p_n$.

By Euler-Kirchhoff there are cycles $r_{j,i}$ in G_i for all $i \leq k$ so that

$$h_j[\mathrm{out}, i] = h_j[\mathrm{in}, i] + \mathrm{eff}(r_{j,i}).$$

We use the Iteration lemma to construct a run $r = u_0^n w_0^n r_0 v_0^n z_1 \dots z_k u_k^n w_k^n r_k v_k^n$ and choose $n \geq n_0$ sufficiently large so that

see (*) above

$$r_{j,i} \text{ is enabled in } m_i \quad \forall j = 1, \dots, e\ \forall i = 1, \dots, k$$

where m_i is the configuration reached by $v_0^n w_0^n r_0 v_0^n z_1 \dots z_i u_i^n$

We choose $b' = s_r$ where s_r is the solution of $\mathrm{Char}_{\mathcal{M}}$ corresponding to r.

Since, the transition relation is monotone and $h_j[\mathrm{in}, i] \geq 0$ we can insert $r_{j,i}$ into s:

$$m_i + h_j[\mathrm{in}, i] \xrightarrow{r_{j,i}} m_i + h_j[\mathrm{out}, i]$$

As $h_j[\mathrm{out}, i] \geq 0$ a straightforward induction shows

$$m_i + \sum_{j=1,\dots,e} n_j \cdot h_j[\mathrm{in}, i] \xrightarrow{r_{1,i}^{n_1} \cdot r_{2,i}^{n_2} \cdot \dots \cdot r_{e,i}^{n_e}} m_i + \sum_{j=1,\dots,e} n_j \cdot h_j[\mathrm{out}, i]$$

for any linear combination $n_1 \cdot h_1 + \dots + n_e \cdot h_e \in \{h_1, \dots, h_e\}^*$

Then, any solution in $s_r + \{h_1, \dots, h_e\}^*$ has a realization and thus $s_r + \{h_1, \dots, h_e\}^* \subseteq \Psi(\mathrm{Runs}_{\mathbb{N}}(\mathcal{M}))$.

While semi pseudo-linear sets are <u>not closed</u> under intersections with semi-linear sets, we can show that $post^*(S) \cap S'$ is semi-pseudo-linear for semi-linear sets S and S':

<u>Lemma</u> $post^*(S) \cap S'$ and $S \cap pre^*(S')$ are semi pseudo-linear for semilinear $S, S' \subseteq \mathbb{N}^d$.

proof: Let $S = b + P^*$ and $S' = b' + P'^*$

Implement $post^*(S) \cap S'$ into VASS $V_{S,S'}$

$$\begin{tikzcd} q_0 \arrow[r, "b"] & q_S \arrow[loop above, "P"] \arrow[r] & q_V \arrow[loop above, "V.T"] \arrow[r] & q_{S'} \arrow[loop above, "-P'"] \arrow[r, "-b'"] & q_f \end{tikzcd}$$

Then, $V_{S,S'}$ can be turned into a VAS $W_{S,S'}$ and $post^*(S) \cap S'$ and $S \cap pre^*(S')$ are by a linear function applied to $\Psi(Runs_{\mathbb{N}}(W_{S,S'}))$, which is semi-pseudo-linear.