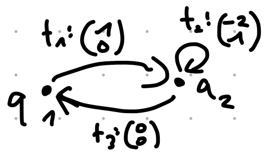
Vector addition systems

We introduce vector addition systems with states (VASS) as an alternative model for Petri nets.
A VASS is a finite automata with a finite number of non-negative counters.

Formally, a VASS $V = (Q, C, T)$ consists of

- a finite set of states Q
- a finite set of counters C
- a finite set of transitions $T \subseteq Q \times \mathbb{Z}^C \times Q$

Example: $V = (\{q_1, q_2\}, \{c_1, c_2\}, \{t_1, t_2, t_3\})$ with:



A configuration $(q, v) \in Q \times \mathbb{N}^C$ consists of a state q and a counter valuation $v \in \mathbb{N}^C$.

A run is a sequence $r = (q_1, v_1)\, t_1\, (q_2\, v_2)\, t_2 \, \dots \, (q_k, v_k)$ of configurations and transitions with $t_i = (q_i, v_{i+1} - v_i, q_{i+1})$ for all $1 \leq i < k$.

We also write $q_1 v_1 \xrightarrow{t_1} q_2 v_2 \xrightarrow{t_2} \dots$, i.e.:

$$r_1 = q_1 \binom{0}{0} \xrightarrow{t_1} q_2 \binom{1}{0} \xrightarrow{t_3} q_1 \binom{1}{0} \xrightarrow{t_1} q_2 \binom{2}{0} \xrightarrow{t_2} q_2 \binom{0}{1}$$

Note that in a run all configuration have non-negative counter values.
We generalize runs to $\mathbb{Z}$-runs that allow for negative counter values, i.e.:

$$r_2 = q_1 \binom{0}{0} \xrightarrow{t_1}_{\mathbb{Z}} q_2 \binom{1}{0} \xrightarrow{t_2}_{\mathbb{Z}} q_2 \binom{-1}{1} \xrightarrow{t_3}_{\mathbb{Z}} q_1 \binom{-1}{1} \xrightarrow{t_1}_{\mathbb{Z}} q_2 \binom{0}{1}$$

The Parikh vector $\Psi(r) \in \mathbb{N}^T$ counts occurrences for each transition
For the above runs we have $\Psi(r_1) = \Psi(r_2) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Lemma (VASS $\leftrightarrow$ PN) Reachability problems can be reduced to each other in linear time:

$$\text{VASS-Reachability} \leq_{lin} \text{PN-Reachability}$$
$$\text{VASS-Reachability} \geq_{lin} \text{PN-Reachability}$$

Proof sketch:

"$\leq_{lin}$": - VASS with a single state can be directly translated into PNs
- Turn VASS $V = (\{q_1, \dots, q_k\}, C, T)$ into single-state VASS:

$$V' = (\{q\}, C \cup Q, T')$$

where we map a configuration $(q_i, v) \in Q \times \mathbb{N}^C$ to

$$q \begin{pmatrix} v \\ e_i \end{pmatrix} \in \{q\} \times \mathbb{N}^{C \cup Q}$$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{ith row}$$

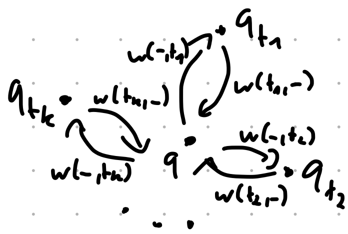
Then T' is defined by

$$\left(q, \begin{pmatrix} z \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ e_i \end{pmatrix} + \begin{pmatrix} 0 \\ e_j \end{pmatrix}, q\right) \in T' \quad \text{if} \quad (q_i, z, q_j) \in T$$

here we must assume that V has no self loops.

"$\geq_{lin}$": Let $N = (P, \{t_1, \dots, t_k\}, W)$

Then we turn N into the following VASS:

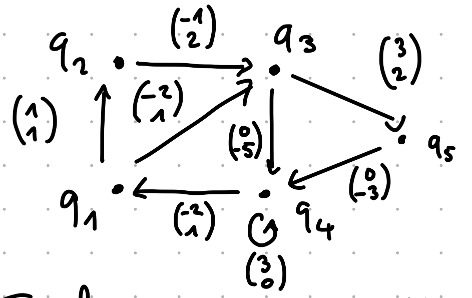


VASS Reachability: Overview

Given: VASS (Q, C, T) and counter valuations $v_1, v_2 \in \mathbb{N}^C$
Question: Is v_2 reachable from v_1?

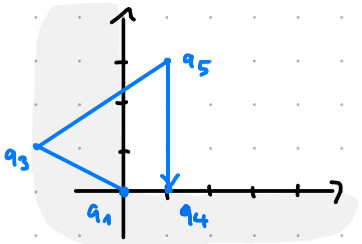
- one of the biggest problems in Track B
- open for 50 years and solved recently: $\mathbb{F}_\omega$ - complete
  - ↳ upper bound 2019 Leroux & Schmitz
  - ↳ lower bound 2021 Leroux and Czerwiński & Orlikowski

Idea of algorithm: Consider the following VASS:



Question: $q_1\binom{0}{0} \longrightarrow^* q_4\binom{1}{0}$?

1) Find $\mathbb{Z}$-run $\rho_\mathbb{Z}$: allow for negative counter values
   ↳ decidable by solving a linear equation system

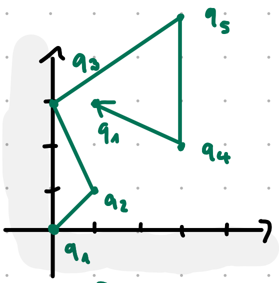


$\rho_\mathbb{Z}$

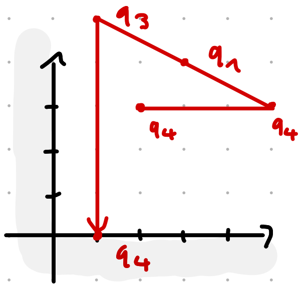
2) Find situations where positivity is irrelevant:

Assume there are $\mathbb{N}$-runs ρ_{up} and ρ_{dn} so that
- ρ_{up} is a cycle from $q_1\binom{0}{0}$ that increases all counters
- ρ_{dn} is a cycle ending in $q_4\binom{1}{0}$ that decreases all counters
- $\text{effect}(\rho_{up}) + \text{effect}(\rho_{dn}) = 0$

In the example ρ_{up} and ρ_{dn} could be:



ρ_{up}

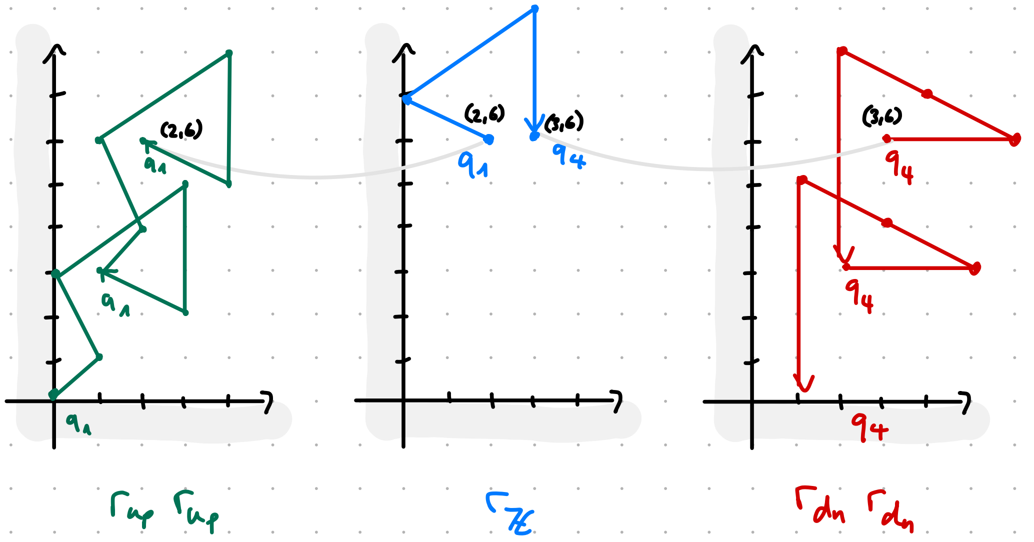


ρ_{dn}

Then we can turn $\rho_\mathbb{Z}$ into an $\mathbb{N}$-run by repeating ρ_{up} and ρ_{dn}.

In the example we need to repeat ρ_{up} and ρ_{dn} twice.

The $\mathbb{N}$-run then is $\rho_{up}\,\rho_{up}\,\rho_\mathbb{Z}\,\rho_{dn}\,\rho_{dn}$.



Note: we use multiple coordinate systems for better readability

Problem: Unclear how to find ρ_{up} and ρ_{dn}

More precisely: the condition $\text{effect}(\rho_{up}) + \text{effect}(\rho_{dn}) = 0$

Solution: Replace 2) by two simpler conditions.

2.1) Find ρ'_{up} / ρ'_{dn} that increase / decrease counter values and <u>must not</u> satisfy $\text{effect}(\rho'_{up}) + \text{effect}(\rho'_{dn}) = 0$

↳ this is equivalent to unboundedness/coverability problem.

↳ I.e. ρ'_{up} exists <u>iff</u> $q_1\binom{1}{1}$ is coverable from $q_1\binom{0}{0}$

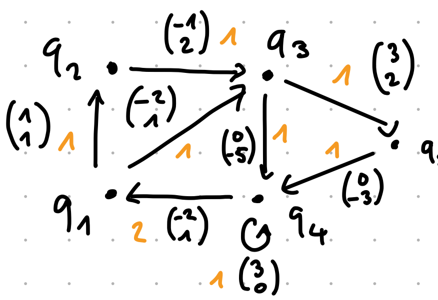
2.2) Find 0-effect cycle ρ_0 ($\mathbb{Z}$-run):

- ρ_0 is a cycle
- $\text{effect}(\rho_0) = 0$
- $\Psi(\rho_0) \geq 1$ // ρ_0 uses all transitions

↳ again decidable by solving linear equation system

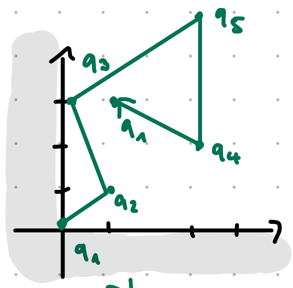
We show how to use 2.1) + 2.2) to construct ρ_{up} and ρ_{dn} from 2)

Assume we found ρ_0 in 2.2) with $\Psi(\rho_0)$ as depicted below:

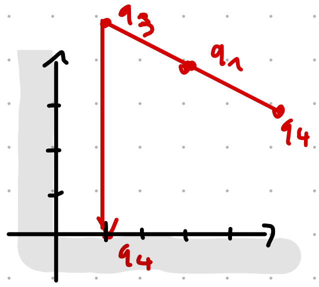


Note that
- $\Psi(\rho_0)$ can be realized by a cycle
- $\text{effect}(\rho_0) = 0$
- $\Psi(\rho_0) \geq 1$

Further assume we found in 2.1) the following $\mathbb{N}$-runs:



r'_{up}



r'_{dn}

Note that

- $\text{effect}(r'_{up}) + \text{effect}(r'_{dn}) \neq 0$

To construct r_{up} and r_{dn} we embed r'_{up} and r'_{dn} in r_0:

- We construct r'_0 that satisfies the conditions from 2.2) and contains all transitions of r'_{up} and r'_{dn}:

  $$\Psi(r'_0) - \Psi(r'_{up}) - \Psi(r'_{dn}) \geq 1$$

  Note that we can always construct r'_0 by repeating r_0.

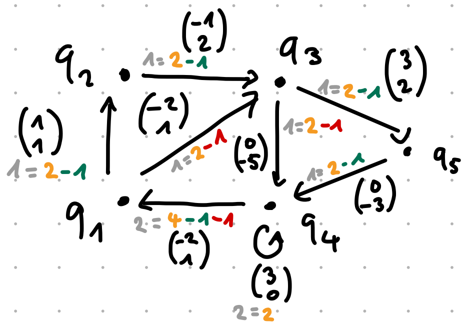
  In the example we can choose $r'_0 = r_0 r_0$.

- The remaining transitions form a cycle r_{rest} with

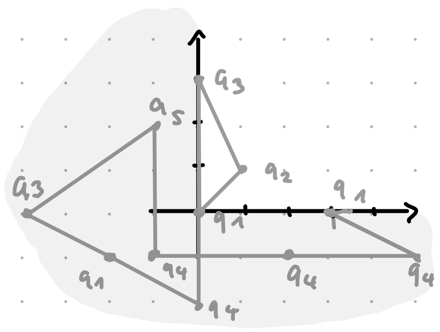
  $$\Psi(r_{rest}) = \Psi(r'_0) - \Psi(r'_{up}) - \Psi(r'_{dn})$$

  Why? r'_0, r'_{up}, r'_{dn} are cycles and removing cycles from a cycle yields cycles. These cycles are connected due to $\Psi(r_{rest}) \geq 1$

In the example $\Psi(r_{rest})$ and r_{rest} could be:



$\Psi(r_{rest})$



r_{rest}

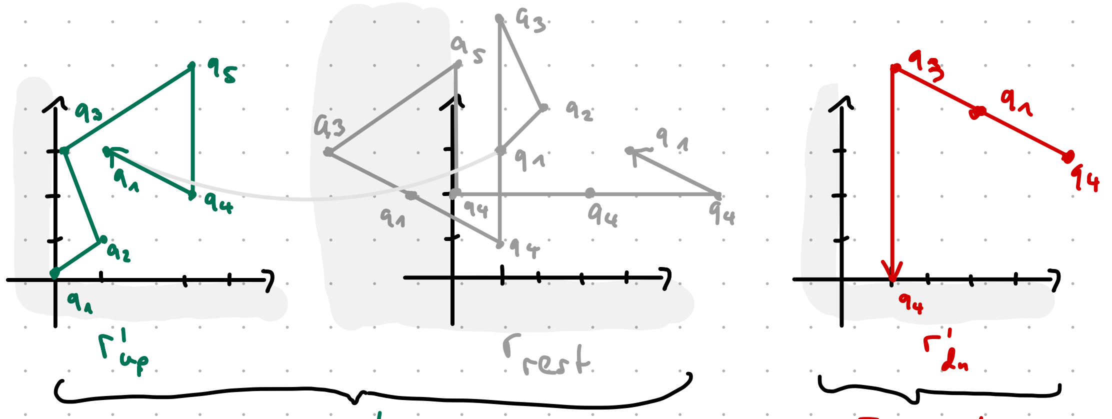
To satisfy $\text{effect}(r_{up}) + \text{effect}(r_{dn}) = 0$ we can choose

$r_{up} = r'_{up} \cdot r_{rest}$

$r_{dn} = r'_{dn}$

$$\begin{aligned} &\text{effect}(r_{up}) + \text{effect}(r_{dn}) \\ &= \text{effect}(r'_{up}\, r_{rest}) + \text{effect}(r'_{dn}) \\ &= \text{effect}(r'_0) \qquad \text{by def. } r_{rest} \\ &= 0 \end{aligned}$$

Illustration:



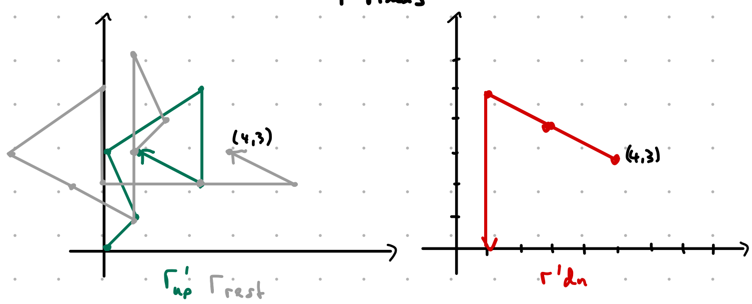
$r_{up} = r'_{up} \cdot r_{rest}$ $\qquad$ $r_{dn} = r'_{dn}$

But then r_{up} has still negative counter values.
However, one can show that r_u will become positive if we repeat r'_{up}, r'_{dn}, and r_{rest} as follows for sufficiently large $i \in \mathbb{N}$:

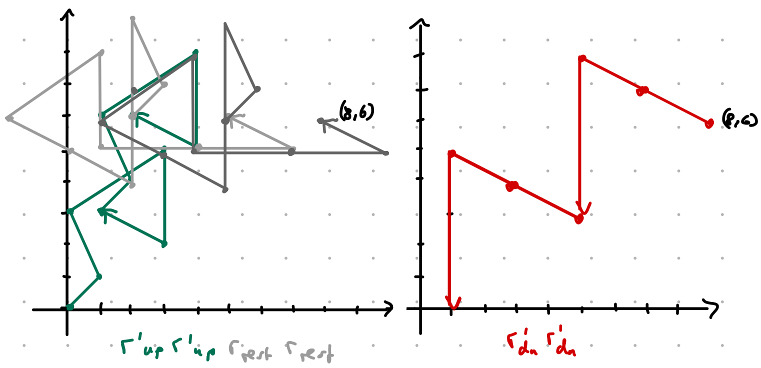
$$r_{up} = \underbrace{r'_{up} \cdots r'_{up}}_{i \text{ times}} \cdot \underbrace{r_{rest} \cdots r_{rest}}_{i \text{ times}} \qquad r_{dn} = \underbrace{r'_{dn} \cdots r'_{dn}}_{i \text{ times}}$$

Illustration:

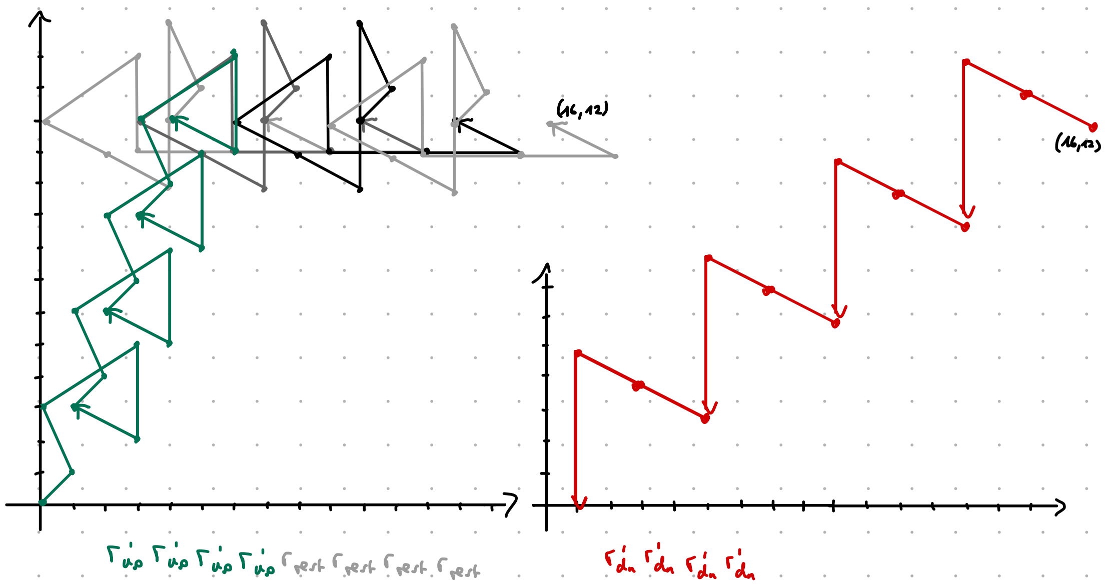
$i = 1$



$i = 2$



$i = 4$



Recap: To find an $\mathbb{N}$-run we search for

1) $\mathbb{Z}$-run $r_{\mathbb{Z}}$

2.1) $\mathbb{N}$-runs r'_{up} and r'_{dn} to increase / decrease counters

2.2) 0-effect cycle r_0 with $\psi(r_0) \geq 1$

$\Big\}$ $\exists$ $\mathbb{N}$-runs r_{up}, r_{dn} with $effect(r_{up}) + effect(r_{dn}) = 0$

We can decide 1) and 2.2) by solving linear equation systems and 2.1) by coverability techniques.

The algorithm checks 1), 2.1), 2.2) and returns YES if all conditions hold.

Problem: What if a condition does not hold?

Solution: Then the $\mathbb{N}$-runs of the VASS are bounded in some way.

How are $\mathbb{N}$-runs bounded?

1) If $r_{\mathbb{Z}}$ does not exist: no $\mathbb{N}$-runs exists because any $\mathbb{N}$-run is a $\mathbb{Z}$-run

→ The algorithm returns NO.

2.1) If r_{up} / r_{dn} does not exist: some counter has bounded values in all $\mathbb{N}$-runs

2.2) If r_0 does not exist: some transition occurs only boundedly many times in $\mathbb{N}$-runs.

Decomposition: If a counter value (2.1) or transition (2.2) is bounded we

- construct a new VASS that tracks bounded behaviour in finite control state explicitly
- The newly constructed VASS is smaller in a well-founded rank.
- The algorithm iteratively checks 1), 2.1), and 2.2) and either returns YES / NO or decomposes.
  Decomposition occurs only finitely often because it reduces a well-founded rank.

Marked Graph Transition sequences (MGTS):

In the decomposition we actually cannot track boundedness of a counter in the control state.

Therefore, we will introduce MGTS which are a generalization of VASS.

Intuitively MGTS are VASS + counter boundedness information

↳ annotate states with the counter bounds