# Decomposition for (UP)/(DOWN)

The decomposition for (DOWN) is the same as (UP) but we track runs in reverse. So we only do (UP).

Let there be an $i \leq k$ such that in $M = G_0 z_1 \dots z_k G_k$, $CSup(G_i) = \emptyset$, that is, there is no up-pumping sequence in $G_i$.

We make a case distinction based on whether $\Omega_{up}(G_i)$ contains a fixed counter.

We say that the counter $j \in [1,d]$ is fixed in $G_i$, if no cycle changes its value; formally, for all cycles $\sigma$ in $G_i$, $eff(\sigma)[j] = 0$.

## Case 1: There is no fixed $j \in \Omega_{up}(G_i)$.

We first observe that $CSup(G_i) = \emptyset$ implies that all runs in $G_i$ have a counter that remains bounded.

**Lemma**: We can compute a $Bd \in \mathbb{N}$ such that for all $\rho \in Runs_{IN}(G_i)$, there is a counter $j \in \Omega_{up}(G_i)$ that remains in $[0, Bd]$ at each step.

**Proof Sketch**: We compute the Karp-Miller Graph $KM(G_i)$ for $G_i$. If there is a marking $v$ in $G_i$ with all $\Omega(G)$ counters $\omega$, then we would have a pumping sequence.
We have $CSup(G_i) = \emptyset$, so there is no such marking: one counter is always concrete.
Let $Bd$ be the largest concrete number in $KM(G_i)$.
Then all runs from some $w \leq_{spec} G_i.v_{in}$ have a counter that remains below $Bd$.

It is possible to do a bit better.

**Lemma**: A bound $Bd \in \mathbb{N}$ as given above can be computed in EXPSPACE.

**Proof**: This bound is obtained by a Rackoff-style argument.
We won't do this here, check out "Reachability in Vector Addition Systems is Primitive-Recursive in Fixed Dimension" for the details (Appendix A)

Let $Bd \in \mathbb{N}$ be such a bound.

For each $j \in \Omega_{up}(G_i)$, we construct a VASS $V_j$ that tracks counter $j$ concretely up to the bound $Bd$.
This captures all reaching runs, by the lemma above.

Formally, let $V_j = (Q \times [0, Bd], [1,d], T_j)$ with

$$((q,a), u, (p,b)) \in T_j \quad \text{if} \quad (q,u,p) \in G_i.T, \; a,b \in [0,Bd], \text{ and } b = a + u[j].$$

For every counter $j \in [1,d]$, and basic (cycleless) path $\rho$ in $V_j$ from $(q_{in}, G_i.v_{in}[j])$ to $(q_{out}, a)$ we construct the MGTS $\mathcal{U}_\rho = H_0 y_1 \dots y_{|\rho|} H_{|\rho|}$ using the same principle as in the (ESUP) case:

$y_1, \dots, y_{|\rho|}$ updates along $\rho$,
$H_\ell$ the strongly connected component of $\rho[\ell]$ in $V_j$ for $\ell \leq |\rho|$, where $H_\ell.q_{in} = H_\ell.q_{out} = \rho[\ell]$ with

$$H_j.\lambda(q,v) = G_i.\lambda(q) \qquad H_j.v_{in} = \begin{cases} G_i.v_{in}, & j = 0 \\ H_j.\lambda(H_j.q_{in}), & \text{else} \end{cases} \qquad H_j.v_{out} = \begin{cases} G_i.v_{out}, & j = |\rho| \\ H_j.\lambda(H_j.q_{out}), & \text{else} \end{cases}$$

Let $M_{p,j}$ be a version of $M$ where $G_i$ is replaced with $\mathcal{U}_{p,j}$.

We return $R = \{M_{p,j} \mid j \in \Omega_{up}(G), p \text{ a basic path in } V_j \text{ from } (q_{in}, 0) \text{ to } (q_{out}, a) \text{ with } a \leq_{spec} G_i.v_{out}[j]\}$.

By the lemma we showed, we already get $Reach_N(M) = Reach_N(R)$.

We need to show that $rk(M_{p,j}) < rk(M)$ for all $M_{p,j} \in R$. This is where we use un-fixedness.

We show $rk(\mathcal{U}_{p,j}) < rk(G_i)$. We show the lemma below.

Same as in (ESUP), this implies $\dim(\mathbb{V}(H_e)) < \dim(\mathbb{V}(G_i))$ for each precov. graph $H_e$ in $\mathcal{U}_{p,j}$ and thus $rk(\mathcal{U}_{p,j}) < rk(G_i)$.

<u>Lemma:</u> For all precov. graphs $H_e$ in $\mathcal{U}_{p,j}$, $\mathbb{V}(H_e) \subset \mathbb{V}(G_i)$.

<u>Proof:</u> Let $H_e$ be a precov. graph in $\mathcal{U}_{p,j}$. Since cycles in $H_e$ are cycles in $G_i$, $\mathbb{V}(H_e) \subseteq \mathbb{V}(G_i)$.

We argue $\mathbb{V}(G_i) \nsubseteq \mathbb{V}(H_e)$.

Since $j$ is not fixed in $G_i$, there is a cycle $\sigma$ in $G_i$ with $eff(\sigma)[j] \neq 0$. We have $eff(\sigma) \in \mathbb{V}(G_i)$.

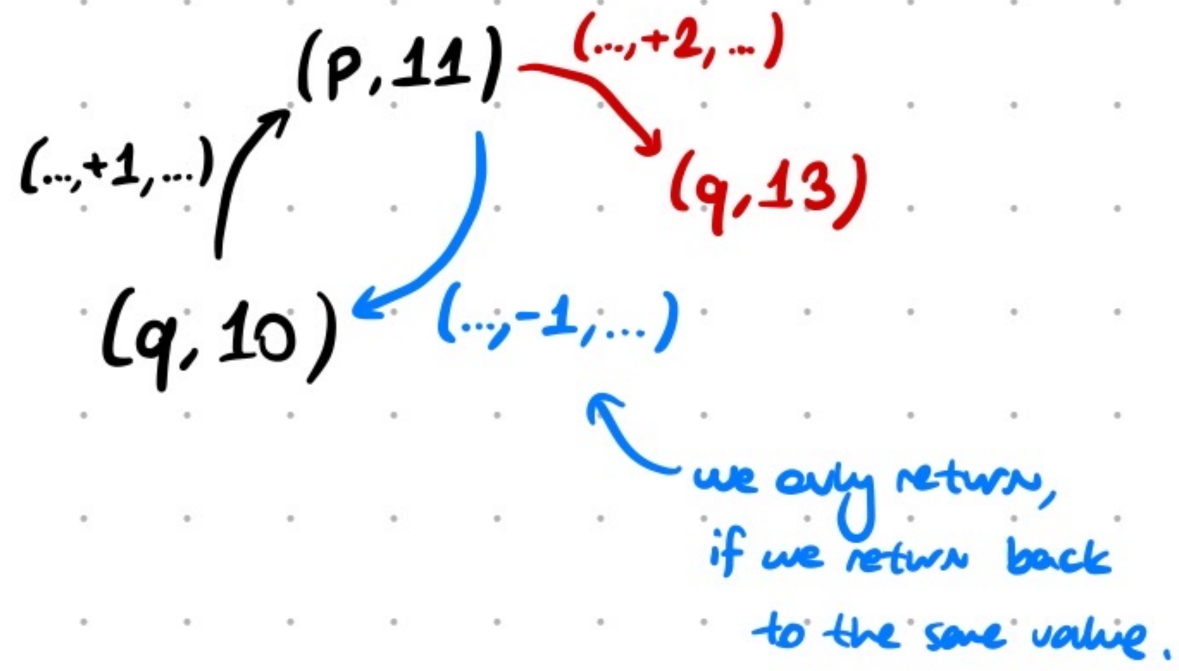
Since $V_j$ tracks counter $j$ concretely, all cycles in $V_j$, and thus $H_e$, have $0$ effect.

Since $\mathbb{V}(H_e)$ consists of lineer combinations of $H_e$ cycles, $v[j] = 0$ for all $v \in \mathbb{V}(H_e)$.

Thus, $eff(\sigma) \notin \mathbb{V}(H_e)$.

<u>Example Cycle in $V_{bd}$</u>

For $(q, u, p), (p, u', q), (p, u'', q) \in G_i.T$ with $u[j] = 1, u'[j] = 2, u''[j] = -1$:




<u>Case 2: There is $j \in \Omega_{up}(G_i)$ that is fixed:</u>

We handle this case by tracking the counter concretely in $\lambda: Q \to \mathbb{N}_\omega^d$.

First, we argue that there is an assignment to $\mathbb{Z}$:

<u>Lemma:</u> If $j \in \Omega_{up}(G_i)$ is fixed in $G_i$, then we can compute an assignment $\ell: G_i.Q \to \mathbb{Z}$ in polynomial time with $\ell(q) = \ell(p) + u$ for all $(p, u, q) \in G_i.T$.

<u>Proof:</u> We argue that sequences between the same states have the same effect on the $j$-th counter.

If this is the case, we can let $\ell(q) = G_i.v_{in}[j] + eff(\rho)[j]$ for all $q \in G_i.Q$ and some $\rho = t_0 \dots t_k \in G_i.T^*$ from $q_{in}$ to $q$. This is well defined under the assumption: all such $\rho$ have the same effect.

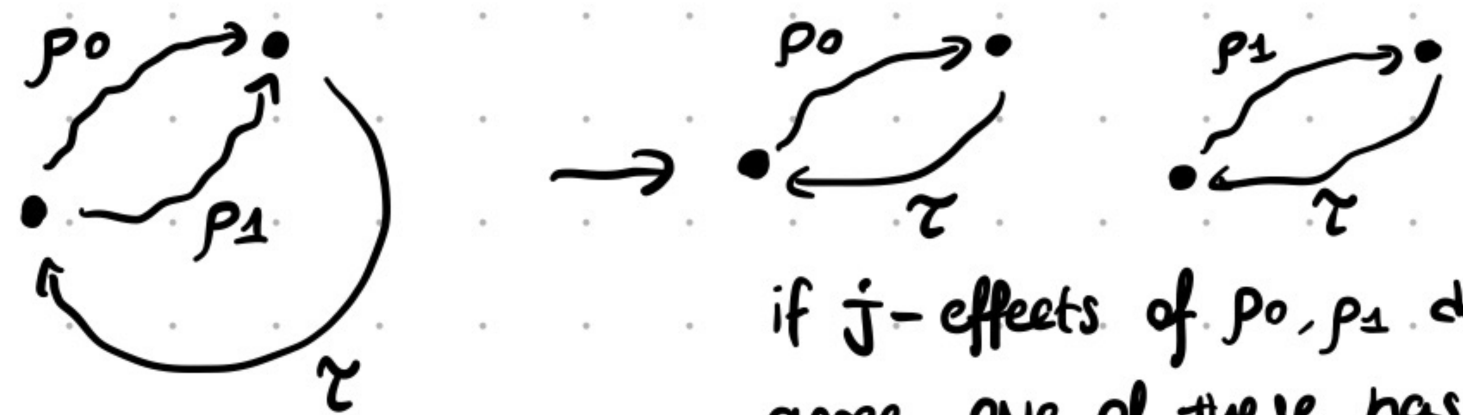
Now let $q, p \in G_i.Q$ and suppose there are two $\rho_0, \rho_1 \in G_i.T^*$ from $q$ to $p$ with $eff(\rho_0)[j] \neq eff(\rho_1)[j]$.

Let $\tau \in G_i.T^*$ be a sequence from $p$ to $q$ (strong connectedness). Then $\rho_0.\tau$ and $\rho_1.\tau$ are cycles.

We have $eff(\rho_\ell.\tau)[j] = eff(\rho_\ell)[j] + eff(\tau)[j]$ for $\ell \in \{0,1\}$.

Then $eff(\rho_0.\tau)[j] \neq eff(\rho_1.\tau)[j]$. So, $eff(\rho_\ell.\tau)[j] \neq 0$ for some $\ell \in \{0,1\}$. This contradicts $j$ being fixed.

<u>Picture:</u>




if $j$-effects of $\rho_0, \rho_1$ do not agree, one of these has $\neq 0$ effect.

If $\ell$ from above only maps to $\mathbb{N}$, we merge it into $G_i.\lambda$ and return the result.
Formally, we construct $G_i'$ from $G_i$ by only changing the $j$-th component of $\lambda$ such that $G_i.\lambda(q)[j] = \ell(q)[j]$ for all $q \in G_i.Q$.
We construct $M'$ by replacing $G_i$ with $G_i'$. Since $\ell$ tracks all runs from $G_i.v_{in}[j]$, $Reach_{\mathbb{N}}(M) = Reach_{\mathbb{N}}(M')$
We also have $rk(G_i')[\ell] = rk(G_i)[\ell]$ for all $\ell \geq 1$ since we only changed $\lambda$, and $rk(G_i')[0] < rk(G_i)[0]$ since $\Omega(G_i') \subseteq \Omega(G_i)$ and we removed $j \in \Omega(G_i)$ from $\Omega(G_i')$. It follows that $rk(M') < rk(M)$.

Let $\ell$ not map to $\mathbb{N}$. Let $q \in G_i.Q$ with $\ell(q) < 0$.
Then, no $\rho \in Runs_{\mathbb{N}}(M)$ can visit $q$, since the $j$-th counter becomes $\ell(q) < 0$.
We remove this state and all in- and out-edges, and break down to SCC's.
The number of edges along an MGTS goes down, and thus the rank.
Let $V_{-q}$ be the VASS of $G_i$ where we remove the state $q$ and all connected edges.
We define the MGTS $\mathcal{U}_p$ for all paths $p$ from $G_i.q_{in}$ to $G_i.q_{out}$ in $V_{-q}$.
We cannot guarantee reducing the dimension in this case.
To make the rank go down, we need to avoid having multiple copies of an edge in $G_i$.
We adjust our construction.
Let $p$ be such a path, and let $in_\ell, out_\ell \in [0, |p|]$ be the indices in $p$ where the $\ell$-th visited SCC is entered/exited.

We let $\mathcal{U}_p = H_1 y_2 \cdots y_L H_L$ where $p$ visits $L$ SCC's,

$y_\ell$ the update from index $out_\ell$ to $in_{\ell+1}$ in $p$ for all $\ell \leq L$

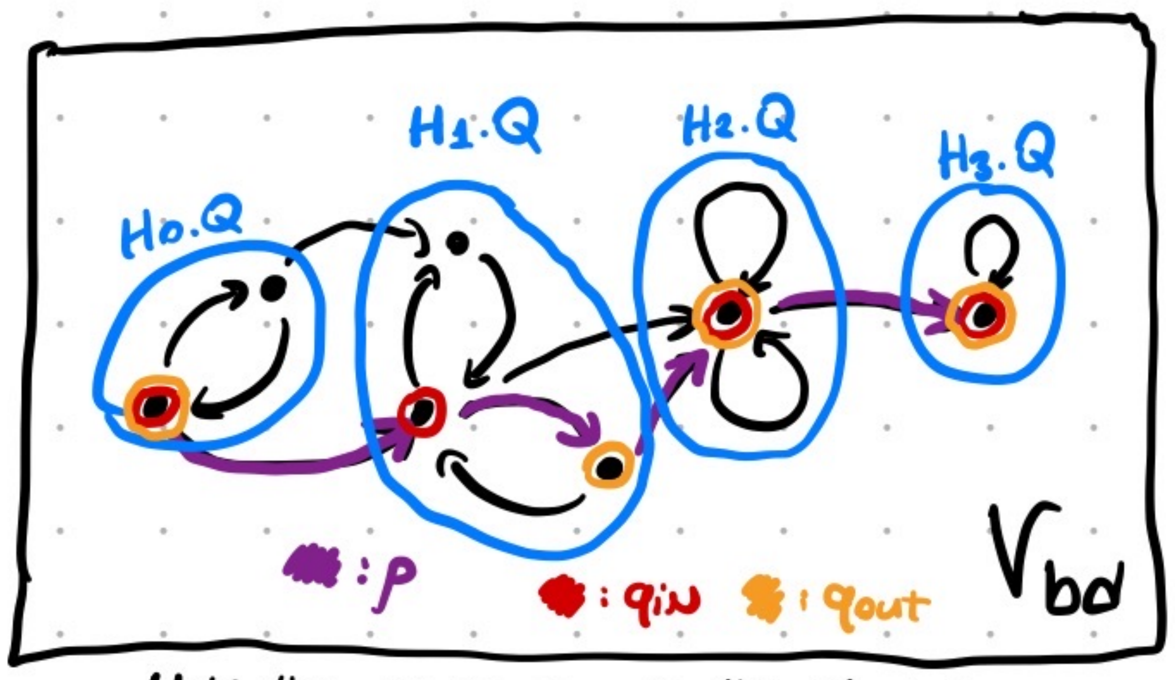
$H_i$ the strongly connected component of $p[in_\ell]$ in $V_{-q}$ with $H_\ell.q_{in} = p[in_\ell]$, $H_\ell.q_{out} = p[out_\ell]$, and:

$$H_i.\lambda(p) = G_i.\lambda(p) \quad \text{for all } p \in H_\ell.Q$$

$$H_\ell.v_{in} = \begin{cases} G_i.v_{in}, & \text{if } \ell = 0 \\ G_i.\lambda(H_\ell.q_{in}), & \text{else} \end{cases}$$

$$H_\ell.v_{out} = \begin{cases} G_i.v_{out}, & \text{if } \ell = L \\ G_i.\lambda(H_\ell.q_{out}), & \text{else.} \end{cases}$$

Picture:




Note the contrasts to the simpler construction.

Let $M_p$ be $M$ where $G_i$ is replaced with $\mathcal{U}_p$. We return

$$R = [M_p \mid p \text{ is a path from } G_i.q_{in} \text{ to } G_i.q_{out} \text{ in } V_{-q}].$$

It is easy to see that this captures all runs that do not visit $q$.
But, we have already argued no $\tau \in Runs_{\mathbb{N}}(M)$ does this, so $Reach_{\mathbb{N}}(M) = Reach_{\mathbb{N}}(R)$.
We argue that the rank goes down.
Let $\mathcal{U}_p = H_1 y_2 \cdots y_L H_L$. Same as the previous cases, $\mathbf{V}(H_\ell) \subseteq \mathbf{V}(G_i)$ for all $\ell \leq L$.
We argue $\sum_{\ell \leq L} |H_\ell.T| < |G_i.T|$ which shows $rk(\mathcal{U}_p) < rk(G_i)$ when applied with the inclusion above.
Note that $|V_{-q}.T| < |G_i.T|$ since we removed edges leading to/stemming from $q$.
By construction $\mathcal{U}_p$ contains at most 1 copy of each SCC in $V_{-q}$, so each edge appears at most once.
Then, $\sum_{\ell \leq L} |H_\ell.T| \leq |V_{-q}.T| < |G_i.T|$. This concludes the proof.