Decomposition for (CSUP)

Let (CSUP) not hold for $M = G_0 z_1 .. z_k G_k$.

Then there is an $i \le k$ and $j \in [1,d]$ such that $(in, i, j) \notin sup(M)$ or $(out, i, j) \notin sup(M)$, but $G_i.v_{in}[j] = \omega$
resp. $G_i.v_{out}[j] = \omega$.

Let $(in, i, j) \notin sup(M)$, the out case is similar.

The key insight is that there is a computable bound $K_{bd} \in \mathbb{N}$ such that for all $s \in sol(Char_M)$, $s[in, i, j] \le K_{bd}$.

This follows from our arguments on Linear Integer Systems and Presburger Arithmatic.

<u>Lemma</u>: Let $\mathcal{S}$ be a Linear Integer System in variables $x \in \mathbb{N}^{Vars}$, and S_{hom} the corresponding homogeneous system. If $\sigma \in Vars \setminus sup(S_{hom})$, then, there is a bound $K_{bd} \in \mathbb{N}$, computable given S, such that for all $s \in sol(\mathcal{S})$, $s[\sigma] \le K_{bd}$.

<u>Proof</u>: Such a $K_{bd} \in \mathbb{N}$ exists: By our arguments on LIS, we know that $sol(S) = B + P^*$ for some finite $B, P \subseteq \mathbb{N}^{Vars}$.
In our proof for this statement, we constructed P such that $sol(S_{hom}) = P^*$.
We assume this. Since $\sigma \notin sup(S_{hom})$, $h[\sigma] = 0$ for all $h \in sol(S_{hom})$.
Then, for all $p \in P$, $p[\sigma] = 0$.
This means that for all $s \in sol(S_{hom})$, $s[\sigma] = b[\sigma]$ for some $b \in B$, since all P vectors have 0 for this entry.

We can compute this K_{bd}: We can express K_{bd} by a Presburger Formula, and apply QE.
(Like in Homework 3)

$\square$

For each $a \le K_{bd}$, we construct M_a by replacing the j'th counter of $G_i.v_{in}$ by a.

We return $dec(M) = \{ M_a \mid a \le K_{bd} \}$.

<u>Proof of dec-Lemma in case (CSUP):</u>

$M_a <_{rk} M$: Let $a < K_{bd}$.

The MGTS M_a and M only differ in the i'th precovering graph.
Let M_a's i-th precov. graph $G_{i,a}$. We show $G_{i,a} <_{rk} G_i$.
We have $G_{i,a}.T = G_i.T$ so $\mathbb{V}(G_{i,a}) = \mathbb{V}(G_i)$ and thus $rk(G_{i,a})[c] = rk(G_i)[c]$ for all $c \ge 1$.
We have $\Omega_{in}(G_{i,a}) \subset \Omega_{in}(G_i)$, $\Omega_{out}(G_{i,a}) = \Omega_{out}(G_i)$, and $\Omega(G_{i,a}) = \Omega(G_i)$.
Then $rk(G_{i,a})[0] < rk(G_i)[0]$ and thus $G_{i,a} <_{rk} G_i$.

$\bigcup_{a \le K_{bd}} Reach_{\mathbb{N}}(M_a) = Reach_{\mathbb{N}}(M)$: Clearly, $Reach_{\mathbb{N}}(M_a) \subseteq Reach_{\mathbb{N}}(M)$ for all $a \le K_{bd}$ since we added a constraint.
For the other direction, let $(x, y) \in Reach_{\mathbb{N}}(M)$. Then there is a run $\rho \in Runs_{\mathbb{N}}(M)$ that reaches y from x. As we saw, we can construct a solution $z \in sol(Char_M)$ that has the same entry/exit values for the precov. graphs. Then $z[in, i, j] \le K_{bd}$ and ρ enters G_i with this value. So we can imitate this run in $M_{z[in,i,j]}$.
Thus, $(x, y) \in Reach_{\mathbb{N}}(M_{z[in,i,j]})$.

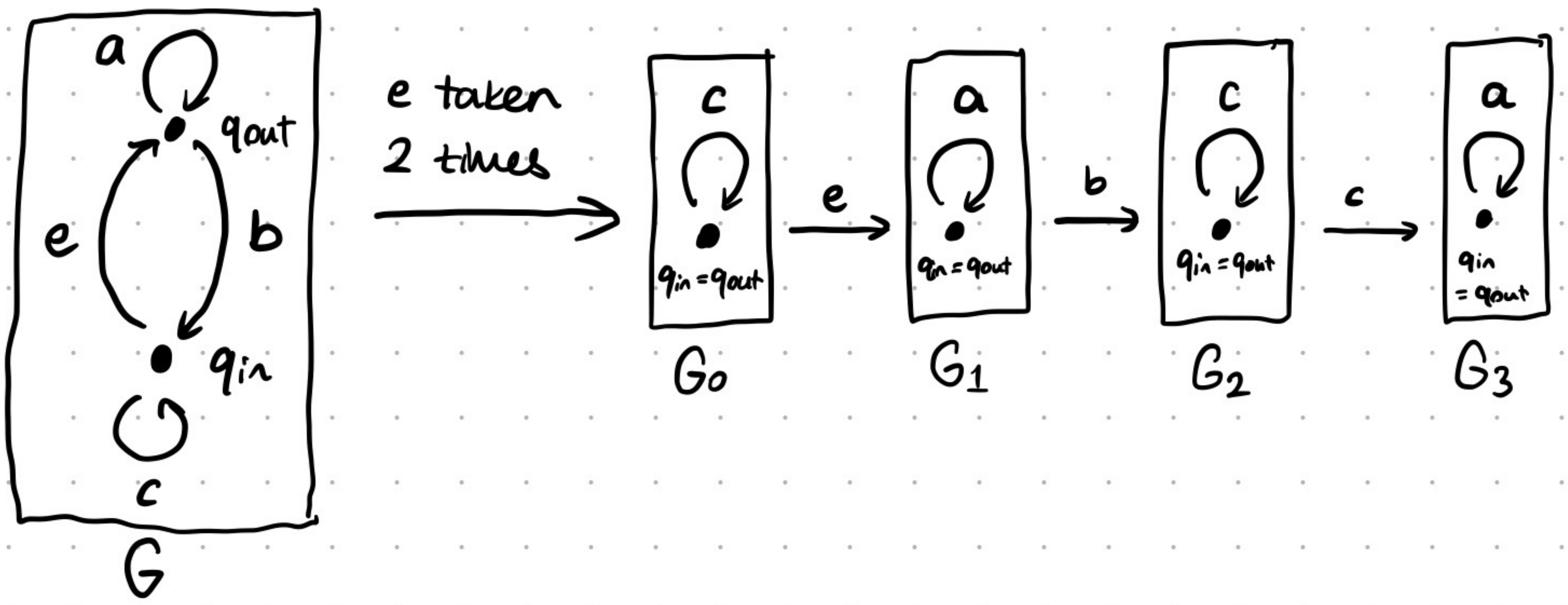
Decomposition for (ESUP)

Let there be an $i \le k$ such that for $\mathcal{M} = G_0 z_1 \dots z_k G_k$, we have $e \in G_i.T$ with $e \notin \sup(\mathcal{M})$.

Then, we can compute a bound $K_{bd} \in \mathbb{N}$ such that <u>for all</u> $e \in G_i.T \setminus \sup(\mathcal{M})$, all solutions $s \in sol(Char_{\mathcal{M}})$ have $s[e] \le K_{bd}$, same as in (CSUP).

$\Rightarrow$ Any reaching run takes e at most K_{bd} times.

We restrict the number of times such an edge can be taken.



<u>Problem</u>: We produce copies of the edges when we do this

$\downarrow$

If the dimension of the new precov. graphs are the same as the old ones, the rank increases

$\downarrow$

The dimension of the new precov. graphs must be lower.

How do we ensure this?

If we restrict <u>all</u> bounded edges in a precovering graph, the dimension does decrease.
(not at all obvious)

For the rest of the case, let $i \le k$ such that $G_i.T \setminus \sup(\mathcal{M}) \neq \emptyset$ and let $T_{bd} = G_i.T \setminus \sup(\mathcal{M})$, and $T_\infty = G_i.T \cap \sup(\mathcal{M})$.

First we construct the MGTS formally.

We introduce a counter that counts the number of taken T_{bd} edges and make a VASS

Then we extract the SCC's from this VASS, make them into precov. graphs. V.

Let $V_{bd} = (Q', [d], T')$ be a VASS with

- $\rightarrow Q' = G_i.Q \times [0, K_{bd}]^{T_{bd}}$
- $\rightarrow T'$ the set with

  $((q,c), u, (q',c))$ if $(q,u,q') \in T_\infty$

  $((q,c), u, (q',c'))$ if $t = (q,u,q') \in T_{bd}$, $c' = c + 1_t$, $c' \in [0, K_{bd}]^{T_{bd}}$.

For any cycleless path p in V_{bd} that reaches (q_{out}, v) from $(q_{in}, 0)$, we construct the MGTS $\mathcal{M}_p$ where we replace G_i with the MGTS $\mathcal{U}_p = H_0 y_1 H_1 \dots H_{|p|}$ where

→ $y_1 \dots y_{|p|}$ are the updates along p

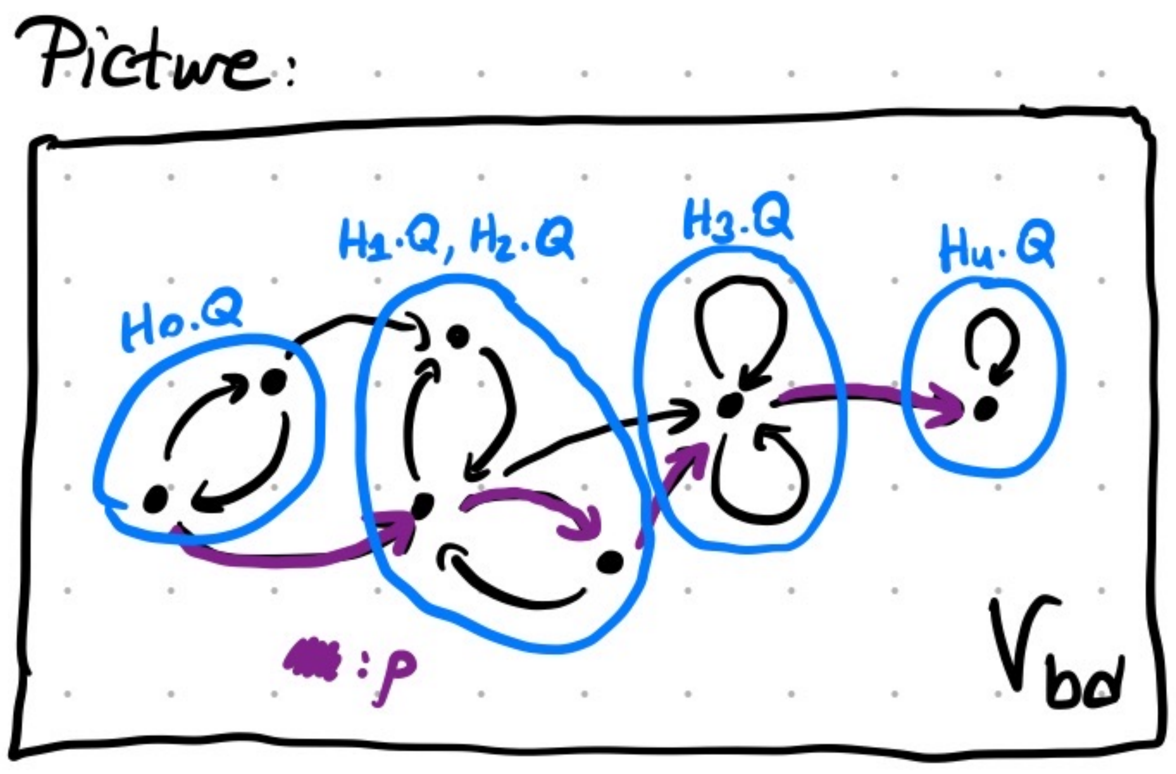
→ H_j the strongly connected component of the j-th state in V_{bd} with $q_{in} = q_{out}$ being this state,

$$H_j.\lambda(q,v) = G_i.\lambda(q)$$

$$H_j.v_{in} = \begin{cases} G_i.v_{in}, & j=0 \\ H_j.\lambda(H.q_{in}), & \text{else} \end{cases}$$

$$H_j.v_{out} = \begin{cases} G_i.v_{out}, & j=|p| \\ H_j.\lambda(H.q_{out}), & \text{else} \end{cases}$$

Picture:



By a similar argument to (CSUP), all reaching runs may use $e \in T_{bd}$ at most K_{bd} times.

It is easy to see that all such runs are captured in $\{M_p \mid p$ a cycleless path in V_{bd} from $(q_{in},0)$ to $(q_{out},v)\}$

Lemma: $Reach_{IN}(M) = Reach_{IN}(\{M_p \mid p$ a cycleless path in V_{bd} from $(q_{in},0)$ to $(q_{out},v)\})$

We still need to show that rank decreases. Fix a cycleless path p in V_{bd} from $(q_{in},0)$ to (q_{out},v).

Let $\mathcal{U}_p = H_0 y_1 \dots y_{|p|} H_{|p|}$.

We show that $\mathcal{U}_p <_{rk} G_i$ and thus $M_p <_{rk} M$ since other precov. graphs are untouched.

The edges in a strongly connected component of V_{bd} can be repeated indefinitely, so they do not increase the edge counter, which means they come from T_∞.

We show

Lemma: For all $j \le |p|$, $\mathbb{V}(H_j) \subset \mathbb{V}(G_i)$.

This shows $\mathcal{U}_p <_{rk} G_i$: $\mathbb{V}(H_j) \subset \mathbb{V}(G_i)$ implies $\dim(\mathbb{V}(H_j)) < \dim(\mathbb{V}(G_i))$.

Then, $rk(\mathcal{U}_p)[\ell] = 0$ for all $\ell \ge \dim(\mathbb{V}(G_i))+1$ but

$$rk(G_i)[\dim(\mathbb{V}(G_i))+1] = |G_i.T|$$

Proof: Let $j \le |p|$.

Since the cycles in H_j are also cycles in G_i, $\mathbb{V}(H_j) \subseteq \mathbb{V}(G_i)$.

Suppose $\mathbb{V}(H_j) = \mathbb{V}(G_i)$. We show $T_\infty = G_i.T$, which contradicts $T_{bd} \neq \emptyset$.

We use two ideas: Cycles in H_j, if we could take them negatively, could cancel out any effect in G_i (by $\mathbb{V}(H_j) = \mathbb{V}(G_i)$).

Start from a hom. solution that takes each H_j edge enough such that by removing cycles, we get the effect of "taking cycles negatively".

→ we can construct a hom. solution that uses all edges

Let τ be a cycle in G_i that takes each edge at least once.

Then, $eff(\tau) \in \mathbb{V}(G_i) = \mathbb{V}(H_j)$.

Thus there are cycles $\sigma_1, \dots, \sigma_\ell$ in H_j with $r_1, \dots, r_\ell \in \mathbb{R}$ where $\sum_{b \le \ell} r \cdot eff(\sigma_b) = eff(\tau)$.

By linear algebra, we can assume $r_1, \dots, r_\ell \in \mathbb{Q}$. Then there is a $K \in \mathbb{N}$ with $K \cdot r_1, \dots, K \cdot r_\ell \in \mathbb{Z}$.

Then:
$$\sum_{b \le \ell} K \cdot r_b \cdot eff(\sigma_b) = K \cdot eff(\tau).$$

Let h be a full-support sol. to HChar_M.

Let $u_b \in \mathbb{N}^{G_i.T}$ be the count of edges in G_i, that correspond to the edges σ_b takes.

Let $a \in \mathbb{N}$ with $a = \sum_{b \le \ell} |K \cdot r_b| + \|u_b\|_1 + 1$ for all $b \le \ell$.

Let $h' = a \cdot h - \sum_{b \le \ell} (0, \dots, 0, \underbrace{K \cdot r_b \cdot u_b}_{\substack{G_i.T \\ \text{coordinates}}}, 0, \dots, 0) + (0, \dots, 0, \underbrace{K \cdot \Psi(\tau)}_{\substack{G_i.T \\ \text{coordinates}}}, 0, \dots, 0)$

We know that $h' \in \mathbb{N}^{\mathrm{Vars}}$, since h takes each T_0 edge (and thus $H_j.T$ edge) at least once and a is larger than the absolute value of the largest decrement.

We claim that h' is a homogenous solution.

We added/substacted G_i cycles to h, so the only problem may be the marking equation for G_i.

Note that $(*)\ \Delta_{G_i} \cdot \sum_{b \le \ell} K \cdot r_b \cdot u_b = \Delta_{G_i} \cdot K \cdot \Psi(\tau)$.

We have

$$
\begin{aligned}
h'[out,i] - h'[in,i] &= a \cdot h[out,i] - a \cdot h[in,i] \\
&\overset{\mathrm{MarkEq}}{=} a \cdot \Delta_{G_i} \cdot h[G_i.T] \\
&= \Delta_{G_i} \cdot (a \cdot h[G_i.T]) \\
&= \Delta_{G_i} \cdot a \cdot h[G_i.T] - \Delta_{G_i} \cdot \sum_{b \le \ell} K \cdot r_b \cdot u_b + \Delta_{G_i} \cdot \sum_{b \le \ell} K \cdot r_b \cdot u_b \\
&\overset{(*)}{=} \Delta_{G_i} \cdot a \cdot h[G_i.T] - \Delta_{G_i} \cdot \sum_{b \le \ell} K \cdot r_b \cdot u_b + \Delta_{G_i} \cdot K \cdot \Psi(\tau) \\
&= \Delta_{G_i} \cdot \Big(a \cdot h[G_i.T] - \sum_{b \le \ell} K \cdot r_b \cdot u_b + K \cdot \Psi(\tau)\Big) \\
&= \Delta_{G_i} \cdot h'[G_i.T].
\end{aligned}
$$

$\square$