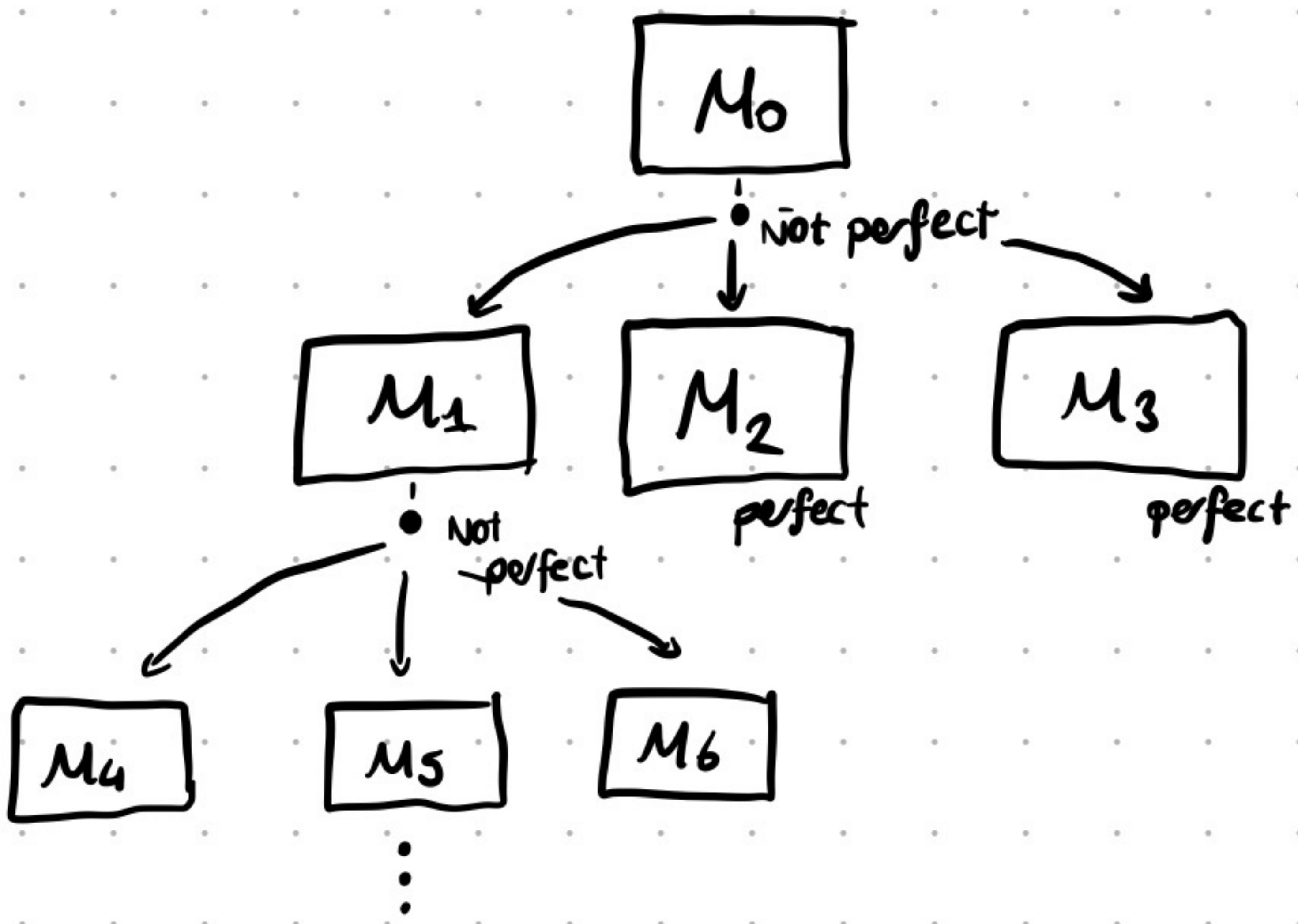
Decomposition and Ranking

So far: If the MGTS M is perfect, we can decide reachability by checking $Sol(Char_M) \neq \emptyset$.

But also: We can reduce the general VASS reachability question to the MGTS reachability question.

<u>However</u>: How do we deal with an MGTS that is not perfect?

Answer: Decompose into lesser MGTS if not perfect. That is: compute a decomposition $dec(M)$ if M is not perfect.



But what if we just keep decomposing infinitely?

Answer: Does not happen: use a well-founded rank and ensure that results of the decomposition are smaller

$\boxed{M_0} \to \boxed{M_1} \to \dots$ and $M_0 \overset{\text{larger in rank}}{>_{rk}} M_1 >_{rk} M_2 >_{rk} \dots \Longrightarrow M_0, M_1, \dots$ finite

Each branch is finite + each node has finite outdegree $\xrightarrow{\text{König's Lemma}}$ The tree is finite

Then, if we have such a decomposition function dec and a fitting $<_{rk}$, we have the following:

<u>Theorem</u>: Given an MGTS M, we can effectively decompose it into a set of MGTS $perf(M)$ that is perfect.

In the following two lectures we will show this theorem, which shows that VASS-REACH is decidable.

Cycles and Ranks

The rank should capture complexity and the decomposition should reduce complexity.

The idea of the rank: Look at what the cycles of an MGTS may do, and assign a number to their complexity.

Tool: Vector Spaces from linear algebra.

Let G be a precovering graph. We say that $\sigma = t_0 \dots t_k$ is a **cycle** if it is a cycle in the graph of G.

We define **effects** for (sequences of) transitions: We write $eff(q,u,p) = u \in \mathbb{Z}^d$ for one transition and $eff(t_0 \dots t_k) = \sum_{i \leq k} eff(t_i) \in \mathbb{Z}^d$.

The cycle space $\mathbb{V}(G)$ of the precovering graph G is the vector space spanned by all cycle effects.

$$\mathbb{V}(G) = \operatorname{span}\{\mathit{eff}(\sigma) \mid \sigma \text{ is a cycle in } G\}$$
$$= \{r_1 \cdot \mathit{eff}(\sigma_1) + r_2 \cdot \mathit{eff}(\sigma_2) + \dots + r_k \cdot \mathit{eff}(\sigma_k) \mid r_1, \dots, r_k \in \mathbb{R};\ \sigma_1, \dots \sigma_k \text{ cycles in } G\}$$

As we recall from linear algebra, $\dim(\mathbb{V}(G))$, the dimension of $\mathbb{V}(G)$ is the least $m \in \mathbb{N}$ such that $\mathbb{V}(G) = \operatorname{span}\{u_1, \dots, u_m\}$ for some $u_1, \dots, u_m \in \mathbb{R}^d$.

We can now define rank. The rank $rk(M) \in \mathbb{N}^{d+2}$ of an MGTS $M = G_0 z_1 \dots z_k G_k$ is the sum of the ranks of the precovering graphs $rk(M) = \sum_{i \leq k} rk(G_i)$, and the rank $rk(G)$ of a precovering graph G is $rk(G) = |G_i.T| \cdot \mathbb{1}_{\dim(\mathbb{V}(G))+1} + (|\Omega_{in}(G)| + |\Omega_{out}(G)| + |\Omega(G)|) \cdot \mathbb{1}_0$

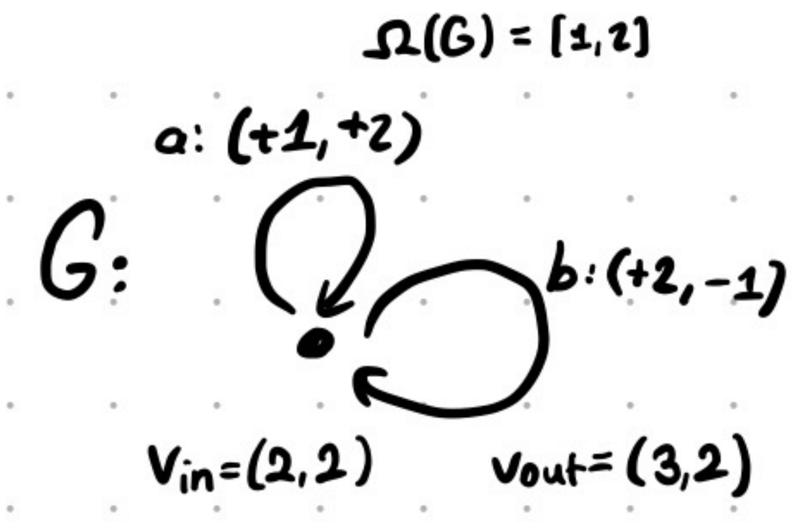
We write $M <_{rk} M'$ if $rk(M) <_{rlex} rk(M')$, where $<_{rlex} \subseteq \mathbb{N}^{d+2} \times \mathbb{N}^{d+2}$ the reverse lexicographical order:

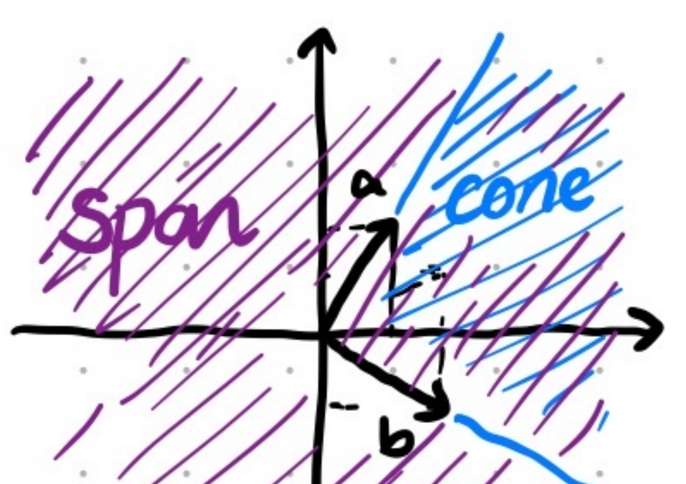
$\mathbb{1}_i$: Vector in $\mathbb{N}^{d+2}$ with the i-th component (here indexing from 0) is 1, and all others 0.

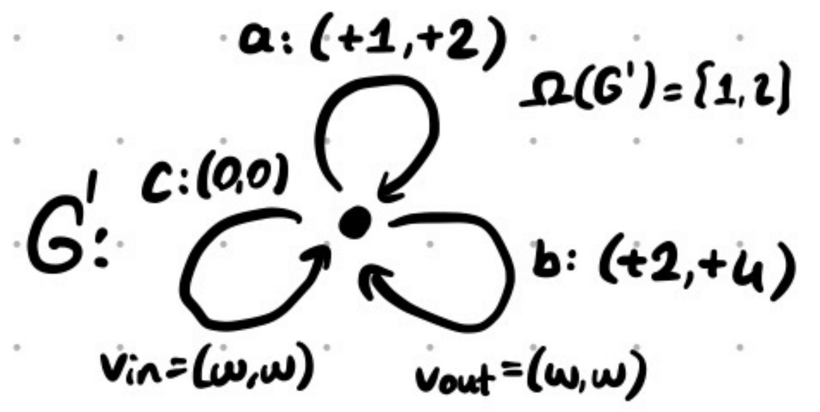
$v <_{rlex} w$ iff $v[i] < w[i]$ for some $i \in [0, \dots, d+1]$ and $v[j] = w[j]$ for all $j \in [0, \dots, d+1]$ with $j > i$.

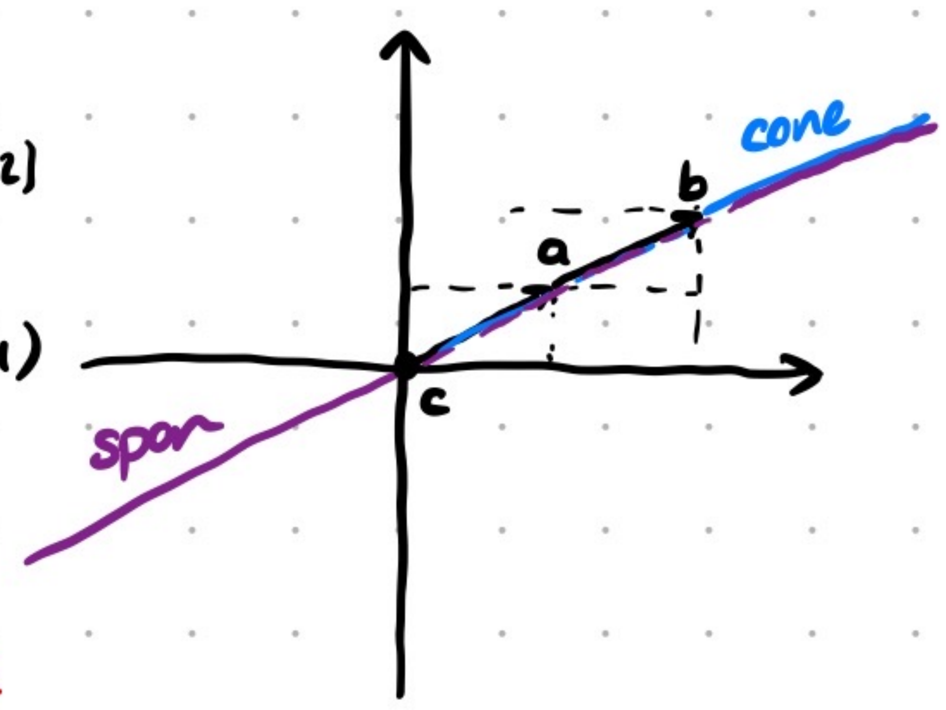
Idea: Dimension counts the number of "actual counters" and the transitions of precovering graphs with larger dimensions count more.

Example:









$\mathbb{V}(G) = \mathbb{R}^2$

$rk(G) = (\overset{\omega\text{'s}}{2}, \overset{\text{dim.0 trans.}}{0}, \underset{\text{dim.1 trans.}}{0}, \overset{\text{dim.2 trans.}}{2})$

$|\Omega(G)| = 2, |\Omega_{in}(G)| = |\Omega_{out}(G)| = 0$

I can move in 2 indep. directions

$\mathbb{V}(G') = \{(x, 2x) \mid x \in \mathbb{R}\}$

$rk(G') = (\overset{\omega\text{'s}}{6}, \overset{\text{dim.0 trans.}}{0}, \underset{\text{dim.1 trans.}}{3}, \overset{\text{dim.2 trans.}}{0})$

$|\Omega(G')| = |\Omega_{in}(G')| = |\Omega_{out}(G')| = 2$

I can move in one direction

$G >_{rk} G'$ because $rk(G')[3] = 0$, but $rk(G)[3] = 2$

(cone: Multiplying with only $\mathbb{R}_{\geq 0}$ coefficients
span: Multiplying with $\mathbb{R}$ coefficients)

The spirit of the decomposition: Bound these "actual" counters and reduce the cycle spaces.

The General Structure of the Decomposition

The full decomposition $\mathit{perf}(M)$ returns perfect MGTS that capture all reachable counter values.

We define $\mathit{Reach}_{\mathbb{N}}(M) = \{(x, y) \in \mathbb{N}^d \times \mathbb{N}^d \mid u_0\, t_0\, u_1\, t_1 \dots u_n \in \mathit{Runs}_{\mathbb{N}}(M), u_0 = x, u_n = y\}$.

Then we will show:

Lemma: There is a computable function decomp such that for an MGTS M, $\mathit{perf}(M)$ is a set of perfect MGTS with $\mathit{Reach}_{\mathbb{N}}(M) = \mathit{Reach}_{\mathbb{N}}(\mathit{perf}(M))$

Recall that we apply functions meant for elements to sets, $\mathit{Reach}_{\mathbb{N}}(\mathcal{U})$ is $\bigcup_{M' \in \mathcal{U}} \mathit{Reach}_{\mathbb{N}}(M')$

We can further simplify our task to finding a decomposition step. Assume the following:

Lemma: There is a computable function dec such that for an MGTS M, $dec(M)$ is a set of MGTS with

$$Reach_{N}(M) = Reach_{N}(dec(M))$$

$dec(M) = \{M\}$ if M is perfect, and if M is not pefect, $M' <_{rk} M$ for all $M' \in dec(M)$.

As we discussed, we let $perf(M)$ applies the decomposition step $dec(M)$ until stabilization.

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perf(M):
| U = {M}
| while U not stable
| | U = ⋃_{M' ∈ U} dec(M')
| return U
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Then we can prove the main theorem.

Proof Sketch of Theorem (assuming dec): Let U_i be the set of MGTS perf constructs at iteration $i \in \mathbb{N}$. Then $Reach_{N}(U_i) = Reach_{N}(U_{i+1})$ for all $i \in \mathbb{N}$. Then, if perf terminates, U_ℓ is a set of pefect MGTS for some $\ell \in \mathbb{N}$, and $Reach_{N}(M) = Reach_{N}(U_0)$
$= Reach_{N}(U_\ell)$.

Suppose decomp(M) does not terminate. Then, the decomposition tree is infinite. Since $dec(M')$ is finite for all M', there must be an infinite sequence of imperfect $(M_i)_{i \in \mathbb{N}}$ with $M_{i+1} \in dec(M_i)$ for all $i \in \mathbb{N}$.

Then $(M_i)_{i \in \mathbb{N}}$ is decreasing wrt. $<_{rk}$. But $<_{rk}$ is well-founded, so this is a contradiction. $\lightning$ $\square$

Then, we need to develop the decomposition step.

The Decomposition Step

The decomposition step $dec(M)$ first checks whether M is pefect and returns $\{M\}$ if M is pefect. This check is decidable.

Lemma: Given an MGTS M, we can check in elementary time ($\text{const height} \left\{ 2^{2^{\cdot^{\cdot^{2^{O(n)}}}}} \right.$-time) whether M is perfect

Proof: For (ESUP) and (CSUP) we can check whether an element $\sigma \in \{(in,0,1),\dots,(out,k,d)\} \cup \bigcup_{i \leq k} G_i.T$ is in support by checking the feasibility of $HChar_M \wedge (x[\sigma] \geq 1)$.

For (UP) and (DOWN) we can use Abdulla's Backward Search and decide coverability of the pumped v_{in}/v_{out} values in EXPSPACE $\subseteq$ 2-EXPTIME. $\leftarrow 2^{2^{O(n^c)}}$-time $\square$

If M is not perfect, then we check whether $sol(Char_M) = \emptyset$, and return $\emptyset$ if so. (Because then $Runs_N(M) = \emptyset$ as well)

If M is imperfect with $sol(Char_M) \neq \emptyset$, then we apply a decomposition, depending on the violated condition.

We have four cases: (CSUP), (ESUP), (UP), (DOWN).