Perfectness and Iteration Lemma

Goal: Construct $\mathbb{N}$-Runs using solutions to characteristic equations for perfect MGTS.

First: Define perfectness

Perfectness

An MGTS $M = G_0 z_1 G_1 \dots G_k$ is called **perfect** if the following conditions hold

(CSUP) For all $i \leq k$, and $j \leq d$, if $G_i.\mathrm{vin}[j] = \omega$, $(in, i, j) \in \mathrm{sup}(M)$
, if $G_i.\mathrm{vout}[j] = \omega$, $(out, i, j) \in \mathrm{sup}(M)$

(ESUP) For all $i \leq k$, and $e \in G_i.T$, $e \in \mathrm{sup}(M)$

(UP) For all $i \leq k$, $\mathrm{Pump}_{up}(G_i) \neq \emptyset$

(DOWN) For all $i \leq k$, $\mathrm{Pump}_{dn}(G_i) \neq \emptyset$

Shortly: All variables that are not explicitly set to 0 in the char. eq. are in the support, and all precovering graphs have up and down pumping sequences.

(we drop the x)

We show that for MGTS with feasible Char_M, we can construct an $\mathbb{N}$-Run by enriching a solution with iterations of a full-support-solution.

Iteration Lemma

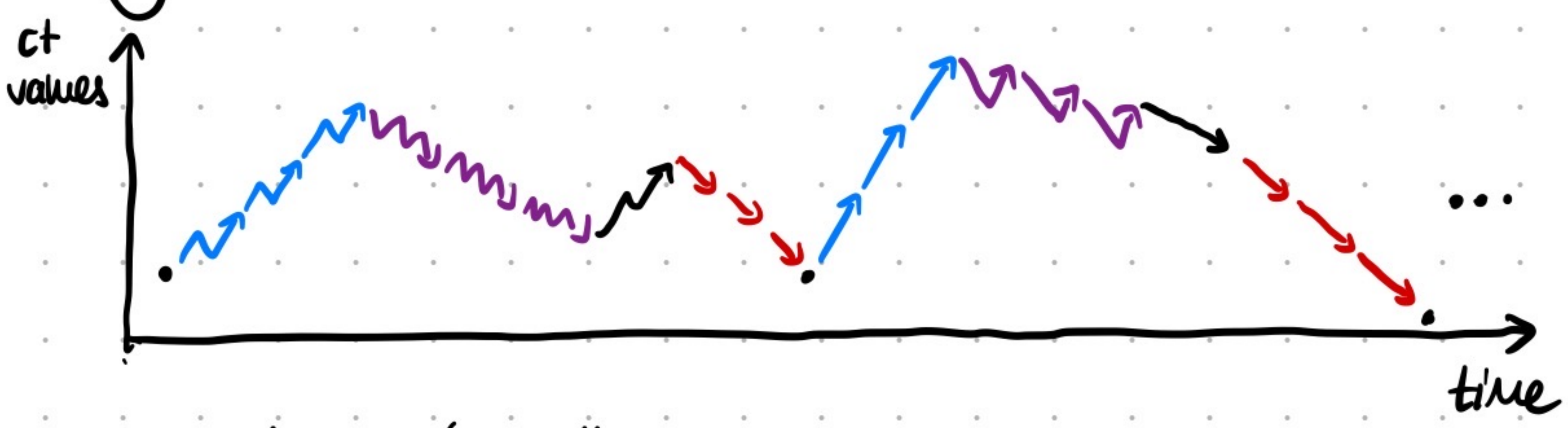
We show the following lemma, which not only shows the existence of an $\mathbb{N}$-run, but also how it can be constructed from any fitting ingredients.

Lemma
(Iteration Lemma): Let $M = G_0 . z_1 \dots z_k G_k$ be a perfect MGTS.
Let $s \in \mathrm{sol}(\mathrm{Char}_M)$ and $h \in \mathrm{sol}(\mathrm{HChar}_M)$ a full-support solution.
Let $u_0, \dots, u_k$ and $v_0, \dots, v_k$ be seqs where for all $j \leq k$, $u_j \in \mathrm{Pump}_{up}(G_j)$ and $v_j \in \mathrm{Pump}_{dn}(G_j)$.
Then, we can compute
$n_0 \in \mathbb{N}$ $w_0, \dots, w_k$ with $w_i \in G_i.T^*$ for all $i \leq k$
and $r_0, \dots, r_k$ with $r_i \in G_i.T^*$ for all $i \leq k$

Definition
$\mathrm{Edges}(\mathrm{Runs}_{\mathbb{N}}(M))$
removes the counter values from the run, only leaving edges

where (1) for all $n \geq n_0$, $u_0^n . w_0^n r_0 v_0^n . z_1 . u_1^n w_1^n r_1 v_1^n \dots z_k u_k^n w_k^n r_k v_k^n \in \mathrm{Edges}(\mathrm{Runs}_{\mathbb{N}}(M))$
(2) there is a $c \in \mathbb{N}$ where for all $i \leq k$, $\Psi(u_i) + \Psi(v_i) + \Psi(w_i) = c \cdot h[G_i.T]$ and
(3) $\Psi(r_i) = s[G_i.T] + h[G_i.T]$

Intuitively, $u_0^n \cdot w_0^n r_0 v_0^n u_1^n w_1^n r_1 v_1^n \ldots u_k^n w_k^n r_k v_k^n$ looks like



u, w, v "cancel out" when taken together and we use them to make buffer for r.

↳ here, cancelling out means doing what h does, which also pumps counters

r is the realization of the solution s, but we add edges in h to ensure connectedness.

Proof: Let $s, h, (u_i)_{i \le k}$, and $(v_i)_{i \le k}$ be as given in the lemma.

Constructing the $\mathbb{Z}$-reaching runs $r_{i \le k}$

Note that $h[e] \ge 1$ for all transitions e.

Then, $s' = s + h \in sol(Chor_M)$ where $s'[e] \ge 1$ for all transitions e.

So, by using Euler-Kirchoff on each precovering-graph, for all $i \le k$, we can construct the path r_i in G_i that takes $e \in G_i.T$, $s'[e]$ times.

This ensures (3).

Constructing the remainder runs $w_{i \le k}$

Let $t \in \mathbb{N}$ be $t = \max_{i \le k} \max_{(p,u,q) \in G_i.T} \|u\|_1$.

We choose a $c \in \mathbb{N}$ that has the following properties

(a) for all $i \le k$, and $e \in G_i.T$, $c \cdot h[e] - \Psi(u_i)[e] - \Psi(v_i)[e] \ge 1$

Reason: After substracting the pumps, the result must be connected so that it becomes a run.

Need 2 (b) for all $i \le k$, and $j \le d$ with $G_i.V_{in}[j] = \omega$, or $G_i.V_{out}[j] = \omega$,

$$c - t \cdot |u_i| - t \cdot |v_i| \ge 1$$

Reason: Executing the pumps once should not consume all tokens given by one iteration

By (CSUP) and (ESUP), the coefficient of c is at least 1 in (a).

So, all the conditions impose lower bounds on c: a high enough c suffices.

Note that for all $i \le k$, u_i and v_i are cycles, so they fulfill the hom. Euler-Kirchoff equation.

Then, $c \cdot h[G_i.T] - \Psi(u_i) - \Psi(v_i)$ also fulfills the hom. Euler-Kirchoff equation for all $i \le k$.

By (a), this vector has a non-zero value for each edge in G_i.

Thus, for all $i \le k$, we construct w_i by realizing the edge applications of $c \cdot h[G_i.T] - \Psi(u_i) - \Psi(v_i)$ via Euler-Kirchoff, just as we did it for r_i. We choose q_{in} as the starting and end state.

By construction, $\Psi(u_i) + \Psi(v_i) + \Psi(w_i) = c \cdot h[G_i.T]$ for all $i \le k$.

This ensures (2). It remains to show (1).

Showing (1): The sequence is a path in M

We have that u_i and v_i are cycles on $G_i.q_{in}$, r_i goes from $G_i.q_{in}$ to $G_i.q_{out}$, and v_i is a cycle on $G_i.q_{out}$ for all $i \le k$. Then, $u_i^n w_i^n r_i v_i^n$ is a path from $G_i.q_{in}$ to $G_i.q_{out}$ in G_i.

Showing (1): The correct values are reached

Let $s_n = s + h + n \cdot c \cdot h \in sol(Char_M)$.

We show that executing $u_0^n w_0^n r_0 v_0^n \dots u_{j-1}^n w_{j-1}^n r_{j-1} v_{j-1}^n z_j$ from $s_n[in,0]$ reaches $s_n[in,j]$.
and executing $u_0^n w_0^n r_0 v_0^n \dots u_j^n w_j^n r_j^n v_j^n$ from $s_n[in,0]$ reaches $s_n[out,j]$

Induction on j:

Base Case $j=0$: The in-case is trivial, since we start from $s[in,0]$ and do nothing.
For the out-case, we have $s_n[out,0] = s_n[in,0] + \Delta_{G_0} \cdot s_n[G_0.T]$ by the marking equation.
We have $s_n[G_0.T] = (s+h+n \cdot c \cdot h)[G_0.T]$.
Since $\Psi(u_0) + \Psi(w_0) + \Psi(v_0) = c \cdot h[G_0.T]$ and $\Psi(r_0) = (s+h)[G_0.T]$,
$\Psi(u_0^n w_0^n r_0 v_0^n) = n \cdot \Psi(u_0) + n \cdot \Psi(w_0) + \Psi(r_0) + n \cdot \Psi(v_0) = s_n[G_0.T]$.
Then $s_n[out,0] = s_n[in,0] + \Delta_{G_0} \cdot \Psi(u_0^n w_0^n r_0 v_0^n)$, so $u_0^n w_0^n . r_0 . v_0^n$ has the correct effect.

Inductive Case $j-1 \to j$: Let $\rho = u_0^n w_0^n r_0^n v_0^n \dots u_{j-1}^n w_{j-1}^n r_{j-1} v_{j-1}^n$.
For the in-case, we need to show executing $\rho . z_j$ reaches $s_n[in,j]$.
By the Char. eq. we have $s_n[out,j-1] + z_j = s_n[in,j]$.
By IH, $s_n[in,0] + eff(\rho) = s_n[out,j-1]$ where $eff(\rho)$ is the effect of ρ.
(see next lecture for the concrete definition)
For the out-case, we use the in-case to get
$s_n[in,0] + eff(\rho) + z_j = s_n[in,j]$ and argue as in the base case.

Showing (1): The run is an $\mathbb{N}$-Run

Let $\rho_i^{(n)} = u_0^n w_0^n r_0 v_0^n \dots u_i^n w_i^n r_i^n v_i^n$. We show that there is an $n_0 \in \mathbb{N}$ where for all $n \ge n_0$ the counter valuations remain in $\mathbb{N}^d$ while executing $\rho_k^{(n)}$ from $s_n[in,0]$.
Since we had shown that $\rho_k^{(n)}$ reaches the correct values, $\rho_k^{(n)} \in Edges(Runs_{\mathbb{N}}(M))$ follows from this.
Actually, we only need to do something simpler, since we have shown that each $\rho_i^{(n)}$ reaches the values in s_n.
→ Show that for all $i \le k$, there is an $n_{0,i} \in \mathbb{N}$ such that executing $u_i^n w_i^n r_i v_i^n$ from $s_n[in,i]$ remains in $\mathbb{N}^d$ for all $n \ge n_{0,i}$.
Since $\rho_i^{(n)}$ and $\rho_i^{(n)} . z_{i+1}$ reach $s_n[out,i]$ and $s_n[in,i+1]$ for all $n \in \mathbb{N}$ and applicable $i \le k$, choosing $n_0 = \max_{i \le k} n_{0,i}$ ensures that $\rho_k^{(n)}$ remains in $\mathbb{N}^d$ for all $n \ge n_0$ in each snippet.

Showing (1): Snippet remains in $\mathbb{N}^d$

We choose $n_{0,i} = t \cdot |w_i| + t \cdot |r_i| + 1$. Let $n \ge n_{0,i}$.

In $\mathbb{N}^d$ while executing u_i^n from $s_n[in,i]$: Let $j \le d$. If $j \in \Omega_{up}(G_i)$, since u_i is a pumping sequence, u_i already has a positive effect on j, and u_i never falls below 0 on this counter. Then neither does u_i^n.

If $j \notin \Omega_{up}(G_i)$, then $j \notin \Omega(G_i)$ or $G_i.v_{in}[j]=\omega$.

In the former case, j value is concretely tracked and it can neither become negative nor changed by a cycle.

In the latter case, we have $s_n[in,i,j] \geq n\cdot c\cdot h[in,i,j]$ and $c\cdot h[in,i,j]-\mathbf{t}\cdot|u_i| \geq 1$.

After executing the ℓ-th edge in u_i^n, the value must be at least $s_n[in,i,j]-\mathbf{t}\cdot\ell$. Since u_i^n has $n\cdot|u_i|$ edges,

$$s_n[in,i,j]-\mathbf{t}\cdot\ell \geq s_n[in,i,j]-n\cdot\mathbf{t}\cdot|u_i| \geq n\cdot c\cdot h[in,i,j]-n\cdot\mathbf{t}\cdot|u_i|$$
$$= n\cdot(\underbrace{c\cdot h[in,i,j]-\mathbf{t}\cdot|u_i|}_{\geq 1 \text{ by (b) and (CSUP)}}) \geq n$$

So it never becomes negative.

<u>In $\mathbb{N}^d$ while executing w_i^n from $s_n[in,i]+n\cdot eff(u_i)$:</u> Unlike for u_i, we have no guarantees on the shape of w_i.

For the proof to work, we need a different argument. Let $j\in\Omega(G_i)$. The case $j\notin\Omega(G_i)$ is similar to above.

We know that w_i^n goes from $s_n[in,i]+n\cdot eff(u_i)$ to $s_n[out,i]-n\cdot eff(v_i)-eff(r_i)$.

We show that in the j-th counter, both of these values are at least n.

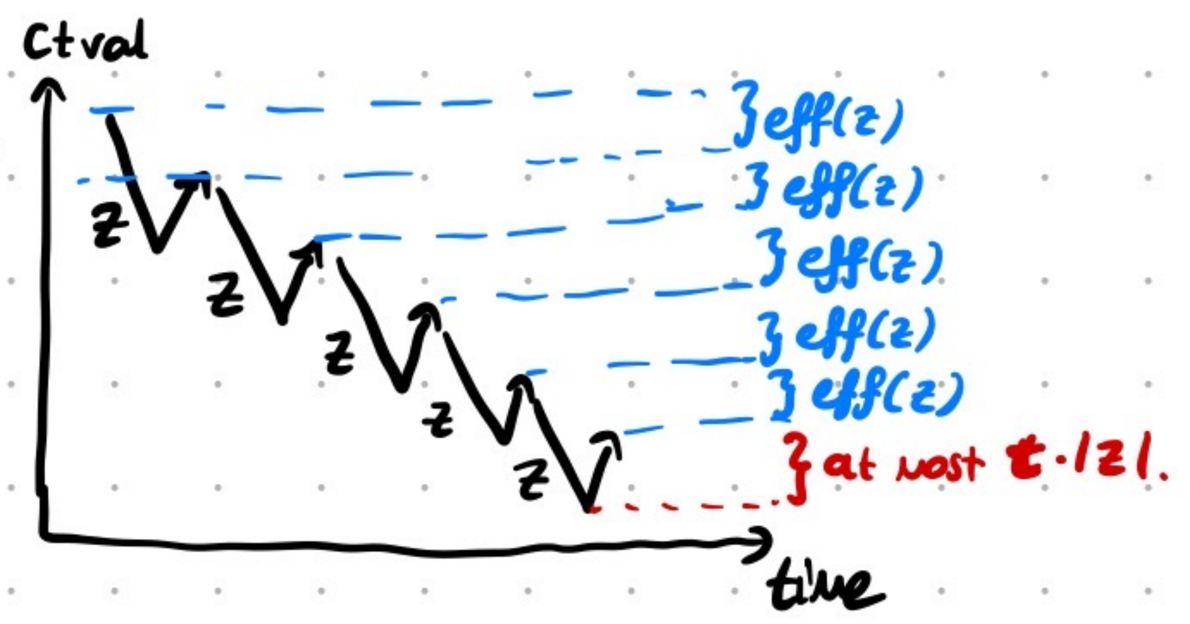
If $j\in\Omega_{up}(G_i)$, then u_i pumps at least one token per execution, thus $s_n[in,i]+n\cdot eff(u_i)\geq n$.

If $j\notin\Omega_{up}(G_i)$, then $G_i.v_{in}[j]=\omega$ so $h[in,i,j]\geq 1$ and $c-\mathbf{t}|u_i|-\mathbf{t}|v_i|\geq 1$, so $s_n[in,i]+n\cdot eff(u_i)\geq n$.

It may seem like in order to execute some edge sequence z n-times, we need $|z|\cdot\mathbf{t}\cdot n$ tokens in the worst case.

But, a finer analysis shows that as long as we start and end at $\mathbf{t}\cdot|z|$, we also remain in $\mathbb{N}$ along the way.

<u>Proof sketch as a picture:</u>



By applying the argument for u_i in reverse, $s_n[out,i]-n\cdot eff(v_i)\geq n$ and thus for the end valuation of w_i^n,

$$s_n[in,i]+n\cdot eff(u_i)+n\cdot eff(w_i) = s_n[out,i]-n\cdot eff(v_i)-eff(r_i)$$
$$\geq n-eff(r_i)\geq n-\mathbf{t}\cdot|r_i| \geq \mathbf{t}\cdot|w_i|.$$

Then, w_i^n is executable.

<u>In $\mathbb{N}^d$ while executing r_i from $s_n[in,i]+n\cdot eff(u_i)+n\cdot eff(w_i)$:</u>

The $j\notin\Omega(G_i)$ case is similar to u_i. Let $j\in\Omega(G_i)$.

As we have established,

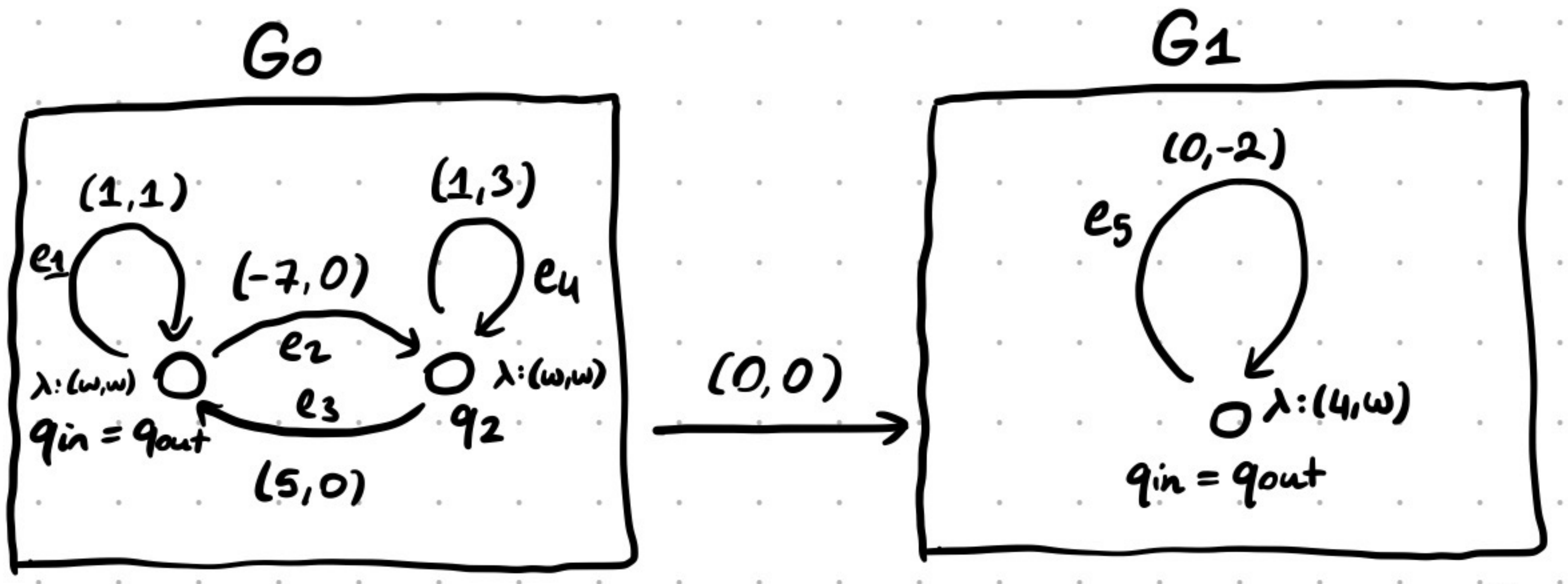
$$s[out,i]-n\cdot eff(r_i)-n\cdot eff(v_i) = s_n[in,i]+n\cdot eff(u_i)+n\cdot eff(w_i)\geq n \text{ and thus } \geq \mathbf{t}\cdot|r_i| \text{ if } j\in\Omega(G_i).$$

Similarly to the argument for u_i^n, r_i cannot consume enough tokens and fall below 0.

<u>In $\mathbb{N}^d$ while executing v_i^n from $s_n[in,i]+n\cdot eff(u_i)+n\cdot eff(w_i)+eff(r_i)$:</u> We know that $u_i^n w_i^n r_i v_i^n$ reaches $s_n[out,i]$ from $s_n[in,i]$ in reverse. Then we can apply the same argument as u_i^n and track the transitions from $s_n[out,i]$ in reverse using (b).

$\square$

The Iteration Lemma principle on an example



$V_{in} = (2,0)$ $\quad$ $V_{out} = (4,\omega)$ $\qquad$ $V_{in} = (4,\omega)$ $\quad$ $V_{out} = (4,0)$

A solution: $s[e_1] = s[e_4] = 1 \quad s[e_5] = 2$

$s[in,0] = (2,0)$, $s[out,0] = (4,4)$, $s[in,1] = (4,4)$, $s[out,1] = (4,0)$.

Pumping Sequences: $\overset{\text{up}}{u_0} = e_1$ ef: $(1,1)$ $\qquad$ $\overset{\text{down}}{v_0} = e_2 e_3$ ef: $(-2,0)$

$u_1 = \varepsilon$ ef: $(0,0)$ $\qquad$ $v_1 = e_5$ ef: $(-2,0)$

A full-support hom. solution:

$h[e_1] = 1 \quad h[e_2] = 1 \quad h[e_3] = 1 \quad h[e_4] = 1$ $\qquad$ effect: $(0,4)$

$h[e_5] = 2$ $\qquad$ effect: $(0,-4)$

Reaching run: $s' \begin{cases} e_1 \to 2 \\ e_2 \to 1 \\ e_3 \to 1 \\ e_4 \to 2 \\ e_5 \to 4 \end{cases}$ $\qquad$ $r_0 : e_1 e_1 e_2 e_4 e_4 e_3$

effect: $(2,8)$

$r_1 : e_5 e_5 e_5 e_5$

effect: $(0,-8)$

The rest run: $t = 7$ $\qquad$ $c \cdot h[G_i.T] - \Psi(u_i) - \Psi(v_i) \geq 1$ $\quad$ (at least one of each edge remains after taking out)

$\Leftrightarrow c \geq 2$

$c - t|u_i| - t \cdot |v_i| \geq 1$

$\Leftarrow c - 7 - 14 \geq 1$

$\Leftrightarrow c \geq 22$

$\Psi(w_0) = \begin{cases} e_1 : 21 \\ e_2 : 21 \\ e_3 : 21 \\ e_4 : 22 \end{cases}$ $\qquad$ $\Psi(w_1) = \{ e_5 : 43$

$w_0 = (e_2 e_3)^{20} e_2 e_4^{22} e_3 e_1^{21}$ $\qquad$ $w_1 = e_5^{43}$

effect: $(1,87)$ $\qquad$ effect: $(0,-86)$

Recap of ingredients

$r_0 : e_1 e_1 e_2 e_4 e_4 e_3$

effect: $(2,8)$

Pumping Sequences: up: $u_0 = e_1$ ef: $(1,1)$; $u_1 = \varepsilon$ ef: $(0,0)$

down: $v_0 = e_2 e_3$ ef: $(-2,0)$; $v_1 = e_5$ ef: $(-2,0)$

$r_1 : e_5 e_5 e_5 e_5$

effect: $(0,-8)$

$w_0 = (e_2 e_3)^{20} e_2 e_4^{22} e_3 e_1^{21}$

effect: $(1,87)$

at worst: $(-47,0)$

$w_1 = e_5^{43}$

effect: $(0,-86)$

at worst: $(0,-86)$

$n_{0,0} = 100 \cdot 7 + 10 \cdot 7$

$n_{0,1} = 100 \cdot 7$

Choose $n_0 = 1000$. Let $n = n_0 = 1000$.

We look at the execution of $u_0^n w_0^n r_0 v_0^n z_1 u_1^n w_1^n r_1 v_1^n$

$u_0^n w_0^n r_0 v_0^n$

$$e_1^{1000} \left((e_2 e_3)^{20} e_2 e_4^{22} e_3 e_1^{21} \right)^{1000} e_1 e_1 e_2 e_4 e_4 e_3 \, (e_2 e_3)^{1000}$$

$(2,0)$ $(1002,1000)$ $(2002,88000)$ $(2004,88008)$ $(4,88008)$

first copy of w_0: lowest eff from start: $(-47,1000)$

so always at least $(955,1000)$

starting pt. after first copy: $(1001,1087)$

starting pt. after second copy: $(1002,1174)$ growing

⋮

$u_1^n w_1^n r_1^n v_1^n$

$$e_5^{43000} \cdot e_5^{4} \cdot e_5^{1000}$$

$(4,88008)$ $(4,2008)$ $(4,2000)$ $(4,0)$