Characteristic Equations of MGTS

Idea: We know how to solve equations, we do not know how to solve reachability.

Solution: Overapproximate the runs of an MGTS with equations.

The Definition: Let $M = G_0 z_1 G_1 z_2 \dots z_k G_k$ be an MGTS with d counters.
Then the characteristic equation $Char_M(x)$ of M in variables $x \in \mathbb{N}^{Comp}$
(where $Comp = \{(in,0),(out,0),\dots,(in,k),(out,k)\} \cup \{e \in G_i.E \mid i \leq k\}$) is

(shorthand for $(x[e_1],\dots,x[e_\ell])$ where $G_i.E = \{e_1,\dots,e_k\}$)

$$Char_M(x) = \bigwedge_{0 \leq i \leq k} \overset{(1)}{Char_{G_i}(x[in,i], x[out,i], x[E,i])} \wedge \bigwedge_{1 \leq i \leq k} \overset{(2)}{x[out,i] - x[in,i] = z_i}$$

where

$$\overset{(3)}{Char_{G_i}(y_{in}, y_{out}, y_E)} = \overset{(4)}{y_{out} - y_{in} - \Delta_{G_i} \cdot y_E = 0}$$

(MARKING)

$$\bigwedge_{q \in G_i.Q} \sum_{(p,q) \in G_i.E} y_E(v,w) \overset{(5)}{-} \sum_{(q,p) \in G_i.E} y_E(v,w) = \begin{cases} 1, & \text{if } q = q_{out} \\ -1, & \text{if } q = q_{in} \\ 0, & \text{else} \end{cases}$$

(KIRCH)

$$\bigwedge_{\substack{j \leq d \\ G_i.v_{in}[j] \neq \omega}} \overset{(6)}{y_{in}[j] = G_i.v_{in}[j]} \wedge \bigwedge_{\substack{j \leq d \\ G_i.v_{out}[j] \neq \omega}} \overset{(6)}{y_{out}[j] = G_i.v_{out}[j]}.$$

We explain the parts

(1): This overapproximates the runs in each G_i. The system uses three multi-dimensional variables, one for the in-configuration, one for the out-configuration, and one for how often we use the edges.

(2): This ensures that the value at the beginning of the i-th pre cov. graph is obtained by adding the bridge transition z_i to the value at the end of the $i-1$-th precovering graph.

(3): The characteristic equation for the precov. graph. It expects 3 variables given from outside, $y_{in} \in \mathbb{N}^d$ for the entering counter values, $y_{out} \in \mathbb{N}^d$ for the exiting values, and $y_E \in \mathbb{N}^{G_i.E}$ for how often each edge should be taken.

(4): The marking equation

(5): Euler-Kirchoff equation. Note that we do not ask for connectedness.

(6): This ensures that the targets are reached. The $\leq_{spec}$ order allows all values if target is ω, and so do we.

→ See $\mathbb{Z}$-reachability from the previous week for more info on (4) and (5)

Fix MGTS $M = G_0 . z_1 \dots z_k G_k$.

It is easy to see that $Char_M(x)$ overapproximates reaching runs.

Lemma: For every run $r \in Runs_{\mathbb{N}}(M)$, there is a solution $z \in sol(Char_M(x))$ where for all $i \leq k$ and $e \in G_i.E$, r uses e exactly $z[e]$ times.

Proof Sketch: Set $z[e]$ to the number of times e is taken in r, and $z[in,i]$ resp $z[out,i]$ to the respective entering and exiting values of r to G_i.

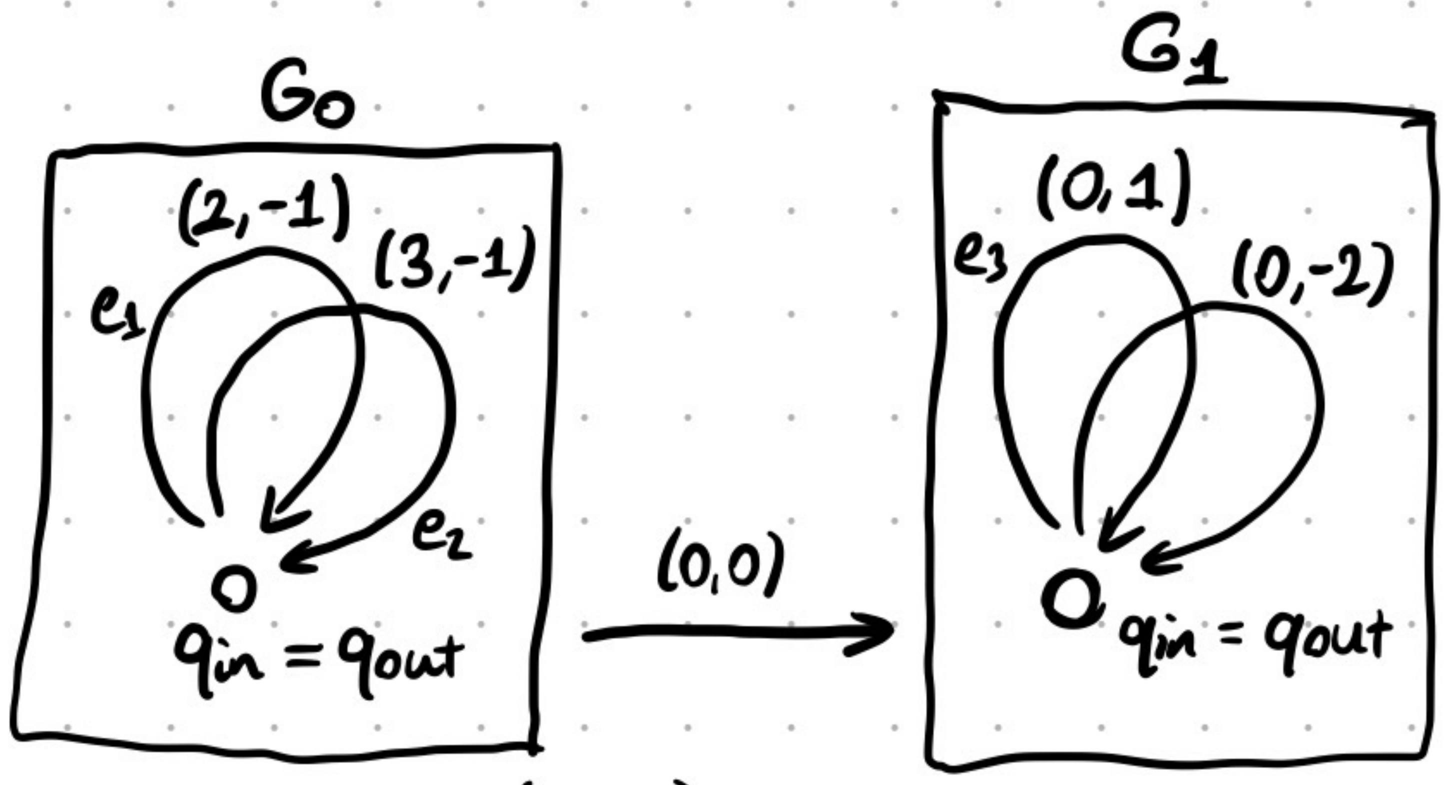
Since r is an $\mathbb{N}$-run, the edge counts fulfill Kirchoff, the Marking equations hold, and all targets $G_0.v_{in}, G_0.v_{out}, \dots, G_k.v_{out}$ are reached.

$\square$

Corollary: If $sol(Char_M(x)) = \emptyset$, then $Runs_{\mathbb{N}}(M) = \emptyset$.

Example: For

$M =$



$v_{in} = (1,3)$, $v_{out} = (\omega, \omega)$ $v_{in} = (\omega, \omega)$, $v_{out} = (19, 0)$

$Char_M(x)$ is infeasible:

- $x[e_1], x[e_2] \leq 3$: This is because $3 - x[e_1] - x[e_2] = 0$ (MARKING) + targets must hold.
- $x[out,0][0]$ is at most 10, achieved by taking e_1 3 times.
- $x[in,1] = x[out,1]$ also holds.
- $x[out,1][0] = x[in,1][0]$, since the second precov graph edges all have 0 effect on the 0'th counter.

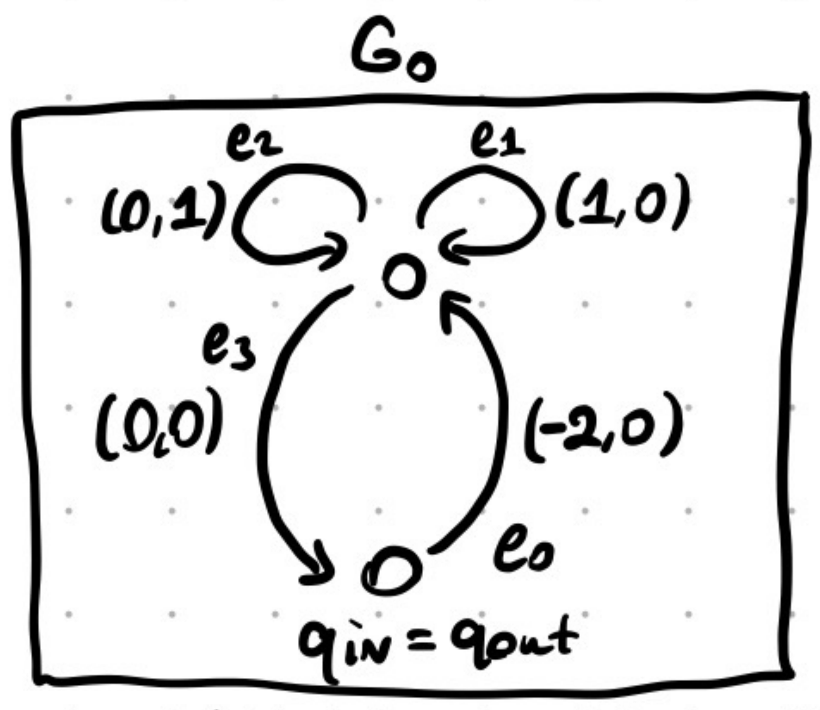
$\Rightarrow$ "Simple" proof for unreachability.

Unfortunately, the overapproximation is not complete.

Lemma: There are MGTS M where $sol(Char_M(x)) \neq \emptyset$ but $Runs_{\mathbb{N}}(M) = \emptyset$

Proof:

$M =$



$v_{in} = (1,0)$ $v_{out} = (20, 20)$

All ≥ 0 effects are possible in G_0, so $sol(Char_M(x)) \neq \emptyset$.
But in an $\mathbb{N}$-run, $(-2,0)$ cannot ever be taken starting from $(1,0)$, so only $(1,0)$ can be reached.

$\Rightarrow Runs_{\mathbb{N}}(M) = \emptyset$.

Tomorrow, we will see that for special MGTS, named perfect, feasible $Char_M(x)$ does imply a reaching run.

<u>Lemma:</u> For perfect MGTS M, if $sol(Char_M(x)) \neq \emptyset$ then $Runs_{IN}(M) \neq \emptyset$.
(Coming soon)

In order to prepare for this result, we need to talk about the homogenous version of the characteristic equations, $HChar_M(x)$.

As we discussed in Linear Integer Systems, the homogenous system is obtained by replacing all all constants by 0.

For the sake of completeness:

$$HChar_M(x) = \bigwedge_{0 \le i \le k} HChar_{G_i}(x[in,i], x[out,i], x[E,i]) \wedge \bigwedge_{1 \le i \le k} x[out,i] - x[in,i] = 0$$

where

$$HChar_{G_i}(y_{in}, y_{out}, y_E) = \quad y_{out} - y_{in} - \Delta_{G_i} \cdot y_E = 0$$

$$\bigwedge_{q \in G_i.Q} \sum_{(p,q) \in G_i.E} y_E(v,w) - \sum_{(q,p) \in G_i.E} y_E(v,w) = 0$$

$$\bigwedge_{\substack{j \le d \\ G_i.v_{in}[j] \neq \omega}} y_{in}[j] = 0 \wedge \bigwedge_{\substack{j \le d \\ G_i.v_{out}[j] \neq \omega}} y_{out}[j] = 0$$

The homogenous system tells us the cycles we can repeat without breaking a solution.

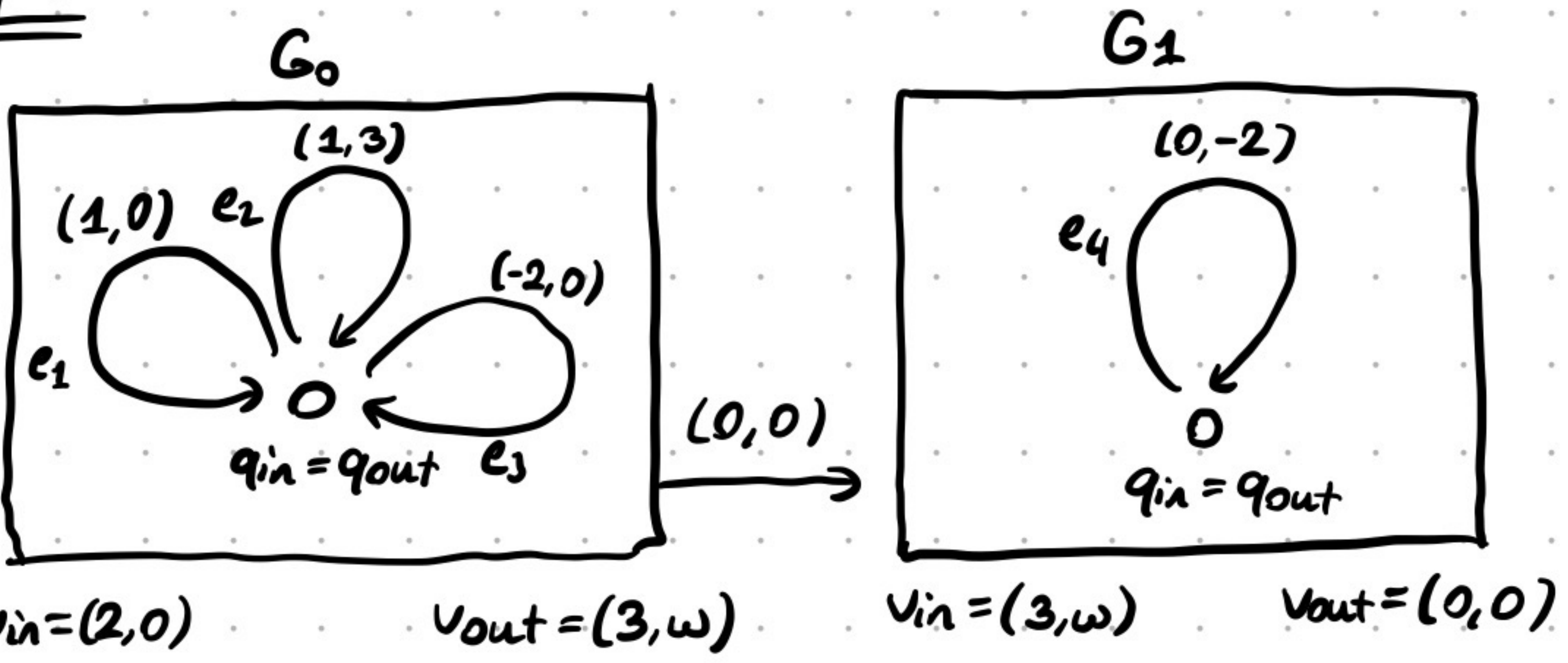
The effects of the bridge transitions are ignored.

The Euler-Kirchoff equation no longer fixes a start/end point: it only asks for cycles.

Instead of reaching the concrete targets, our solution must have 0 effect where a target is concrete.

Intuition: If you add this to a solution, the value for that point remains stable.

<u>Example:</u>



for $x[in,0] = x[out,1] = 0$:

→ $x[e_1] = 2$, $x[e_3] = 1$ is a homogenous solution (0 effect, we can repeat)

→ $x[e_1] = 1$ is <u>not</u>, the out target in G_1 not stable.

→ $x[e_2] = 2$, $x[e_3] = 1$, $x[e_4] = 3$ <u>is</u> a homogenous solution. Effect at the exit of G_0 is $(2,6) + (-2,0) = (0,6) \neq (0,0)$, but $(0,6) \le_{spec} (\omega, 0)$ holds, and this effect is cancelled out by 3 e_4's, exiting with $(0,0)$. Without $x[e_4] = 3$, it is <u>not</u> a hom. sol.

As we saw with Linear Integer Systems, we can add homogeneous solutions to get another hom. sol. and we can add them to solutions, to get other solutions.

Lemma: Let $h, g \in sol(HChar_M(x))$ and $s \in sol(Char_M(x))$.

Then $h+g \in sol(HChar_M(x))$ and $s+h \in sol(Char_M(x))$.

The Support

An important object we will use is the support of the equation system.

We write $\sigma \in sup(HChar_M(x))$, if there is a solution $s \in sol(HChar_M(x))$ with $s[\sigma] \neq 0$.

(Here σ is a component name e.g. $(in,0), (out,0), \dots, (in,k), (out,k)$, or an edge.)

We shorten $sup(HChar_M(x))$ to $sup(M)$.

We call $s \in HChar_M(x)$ a full-support solution, if $s[\sigma] \neq 0$ for all $\sigma \in sup(M)$.

A very important property (which we will use tomorrow) is the following:

Lemma: There is always a full support solution $h \in sol(HChar_M(x))$.

Proof: For each $\sigma \in sup(M)$, let $h_\sigma \in HChar_M(x)$ be the hom. solution with $h_\sigma[\sigma] \neq 0$.

Since $h_\sigma \geq 0$ must hold, $h_\sigma[\sigma] \geq 1$.

Let $h = \sum_\sigma h_\sigma$. Since hom. solutions are closed under addition, $h \in sol(HChar_M(x))$.

Let $\sigma \in sup(M)$. Then $h[\sigma] \geq h_\sigma[\sigma] \geq 1$.

Example: In the prev. example, $(out,0), e_1, e_2, e_3, e_4 \in sup(M)$.

Setting $x[out,1] = 6$, $x[e_1] = 2$, $x[e_2] = 2$, $x[e_3] = 4$, $x[e_4] = 3$ and all else 0 gives a full-support solution.

(Summed all valid examples)