

Characteristic Equations of MGTs

Idea: We know how to solve equations, we do not know how to solve reachability.

Solution: Overapproximate the runs of an MGTs with equations.

The Definition: Let $M = G_0 z_1 G_1 z_2 \dots z_k G_k$ be an MGTs with d counters.

Then the characteristic equation $\text{Char}_M(x)$ of M in variables $x \in \mathbb{N}^{\text{Comp}}$ (where $\text{Comp} = \{(\text{in}, 0), (\text{out}, 0), \dots, (\text{in}, k), (\text{out}, k)\} \cup \{e \in G_i.T \mid i \leq k\}$) is

$$\text{Char}_M(x) = \bigwedge_{0 \leq i \leq k} \text{Char}_{G_i}(x[\text{in}, i], x[\text{out}, i], x[G_i.T]) \wedge \bigwedge_{1 \leq i \leq k} x[\text{out}, i] - x[\text{in}, i] = z_i$$

(1) (2)

(short-hand for $(x[e_1], \dots, x[e_l])$ where $G_i.T = \{e_1, \dots, e_l\}$)

where

$$\text{Char}_{G_i}(y_{\text{in}}, y_{\text{out}}, y_T) = y_{\text{out}} - y_{\text{in}} - \Delta_{G_i} \cdot y_T = 0$$

(3) (4)

(MARKING)

(KIRCH)

$$\bigwedge_{q \in G_i.Q} \sum_{(p, u, q) \in G_i.T} y_T(p, u, q) - \sum_{(q, u, p) \in G_i.T} y_T(q, u, p) = \begin{cases} 1, & \text{if } q = q_{\text{out}} \neq q_{\text{in}} \\ -1, & \text{if } q = q_{\text{in}} \neq q_{\text{out}} \\ 0, & \text{else} \end{cases}$$

(5)

$$\bigwedge_{\substack{j \leq d \\ G_i.v_{\text{in}}[j] \neq \omega}} y_{\text{in}}[j] = G_i.v_{\text{in}}[j] \wedge \bigwedge_{\substack{j \leq d \\ G_i.v_{\text{out}}[j] \neq \omega}} y_{\text{out}}[j] = G_i.v_{\text{out}}[j].$$

(6) (6)

We explain the parts

(1): This overapproximates the runs in each G_i . The system uses three multi-dimensional variables, one for the in-configuration, one for the out-configuration, and one for how often we use the edges.

(2): This ensures that the value at the beginning of the i -th pre cov. graph is obtained by adding the bridge transition z_i to the value at the end of the $i-1$ -th precovering graph.

(3): The characteristic equation for the pre cov. graph. It expects 3 variables given from outside, $y_{\text{in}} \in \mathbb{N}^d$ for the entering counter values, $y_{\text{out}} \in \mathbb{N}^d$ for the exiting values, and $y_T \in \mathbb{N}^{G_i.T}$ for how often each edge should be taken.

(4): The marking equation

(5): Euler-Kirchoff equation. Note that we do not ask for connectedness.

(6): This ensures that the targets are reached. The $\leq_{\text{spec}}$ order allows all values if target is ω , and so do we.

→ See $\mathbb{Z}$ -reachability from the previous week for more info on (4) and (5)

Fix MGTS $M = G_0 \cdot z_1 \dots z_k G_k$.

It is easy to see that $\text{Char}_M(x)$ overapproximates reaching runs.

Lemma: For every run $r \in \text{Runs}_N(M)$, there is a solution $z \in \text{sol}(\text{Char}_M(x))$ where for all $i \leq k$ and $e \in G_i.T$, r uses e exactly $z[e]$ times.

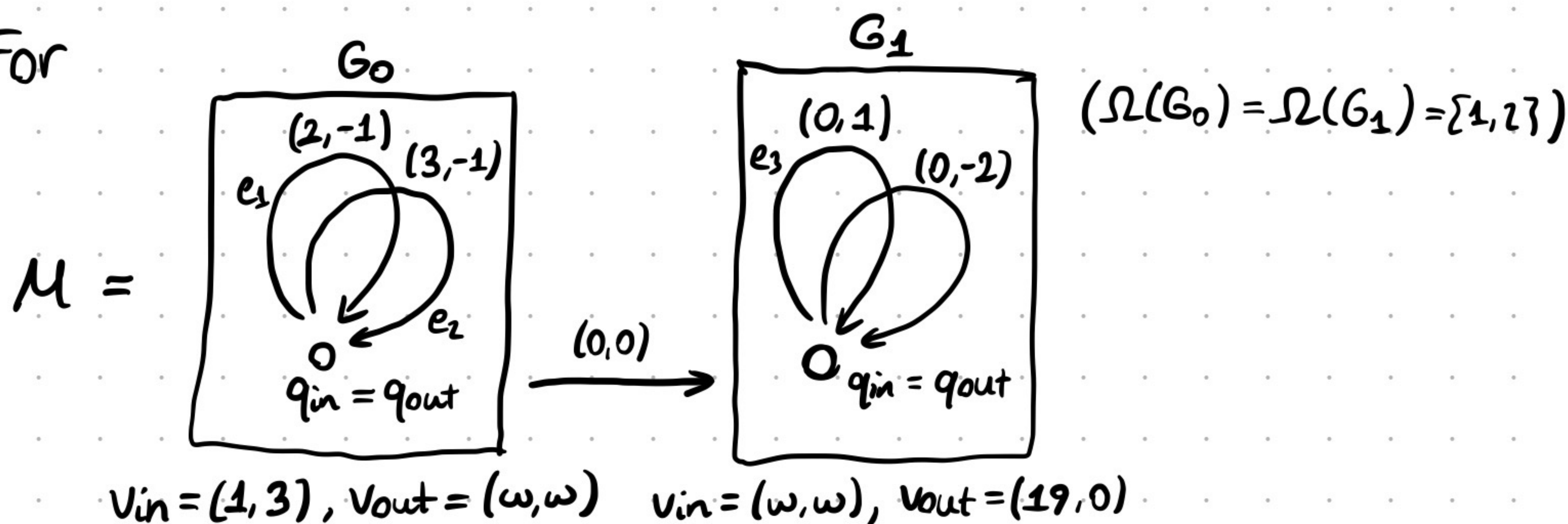
Proof Sketch: Set $z[e]$ to the number of times e is taken in r , and $z[\text{in}, i]$ resp $z[\text{out}, i]$ to the respective entering and exiting values of r to G_i .

Since r is an N -run, the edge counts fulfill Kirchhoff, the Marking equations hold, and all targets $G_0.v_{\text{in}}, G_0.v_{\text{out}}, \dots, G_k.v_{\text{out}}$ are reached.

□

Corollary: If $\text{sol}(\text{Char}_M(x)) = \emptyset$, then $\text{Runs}_N(M) = \emptyset$.

Example: For



$\text{Char}_M(x)$ is infeasible:

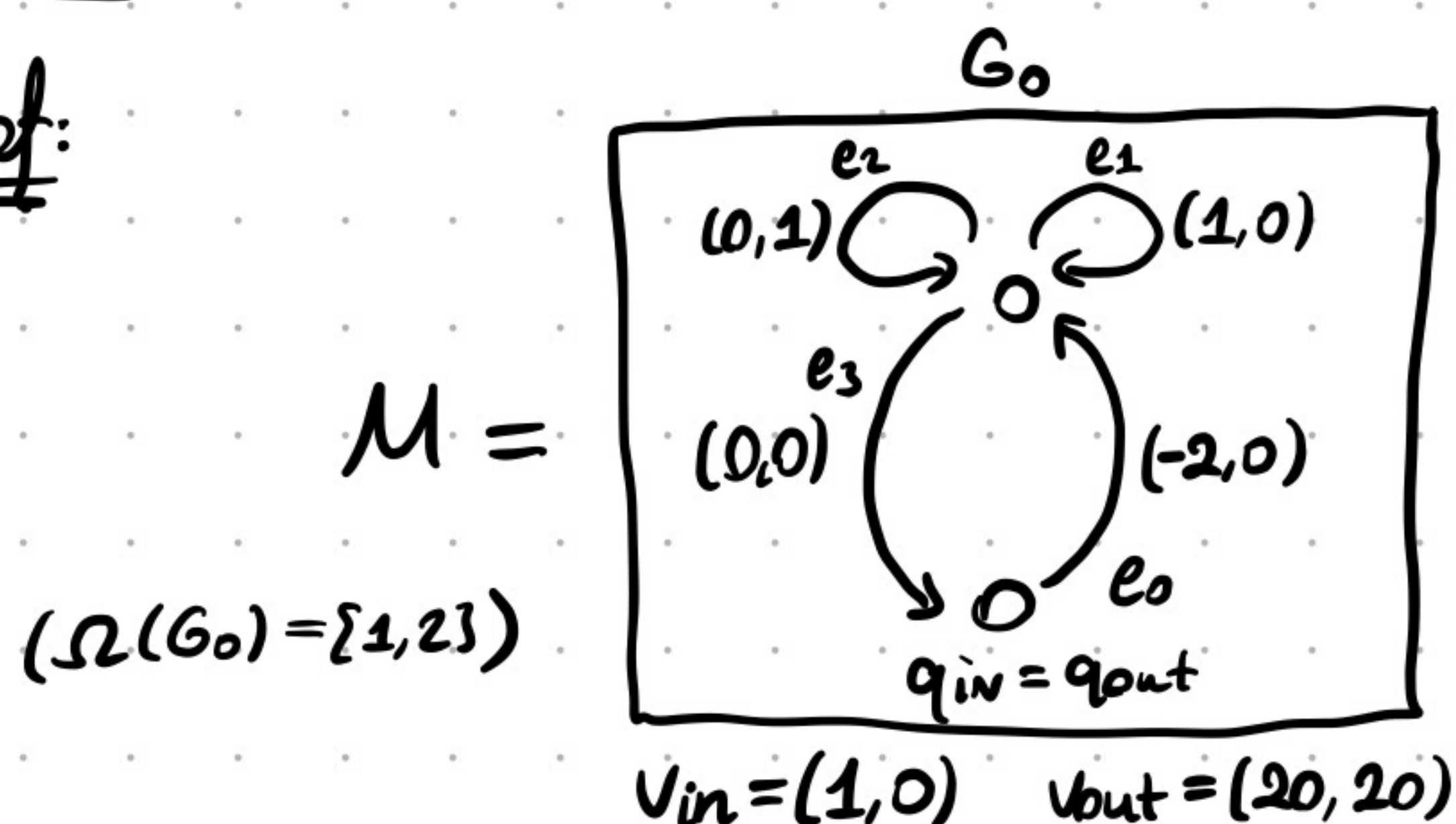
- $x[e_1], x[e_2] \leq 3$: This is because $3 - x[e_1] - x[e_2] \geq 0$ must hold. (MARKING) + targets
- $x[\text{out}, 0, 1]$ is at most 10, achieved by taking e_1 3 times.
- $x[\text{in}, 1, 1] = x[\text{out}, 1, 1]$ also holds.
- $x[\text{out}, 1, 1] = x[\text{in}, 1, 1]$, since the second preceq graph edges all have 0 effect on the first counter.

$\Rightarrow$ "Simple" proof for unreachability.

Unfortunately, the overapproximation is not complete.

Lemma: There are MGTS M where $\text{sol}(\text{Char}_M(x)) \neq \emptyset$ but $\text{Runs}_N(M) = \emptyset$

Proof:



All ≥ 0 effects are possible in G_0 , so $\text{sol}(\text{Char}_M(x)) \neq \emptyset$.
But in an N -run, $(-2, 0)$ cannot ever be taken starting from $(1, 0)$, so only $(1, 0)$ can be reached.
 $\Rightarrow \text{Runs}_N(M) = \emptyset$.

Tomorrow, we will see that for special MGTs, named perfect, feasible $\text{Char}_M(x)$ does imply a reaching run.

Lemma: For perfect MGTs M , if $\text{sol}(\text{Char}_M(x)) \neq \emptyset$ then $\text{Runs}_M(M) \neq \emptyset$.
(Coming soon)

In order to prepare for this result, we need to talk about the homogeneous version of the characteristic equations, $\text{HChar}_M(x)$.

As we discussed in Linear Integer Systems, the homogeneous system is obtained by replacing all constants by 0.

For the sake of completeness:

$$\text{HChar}_M(x) = \bigwedge_{0 \leq i \leq k} \text{HChar}_{G_i}(x[\text{in}, i], x[\text{out}, i], x[G_i.T]) \wedge \bigwedge_{1 \leq i \leq k} x[\text{out}, i] - x[\text{in}, i] = 0$$

where

$$\text{HChar}_{G_i}(y_{\text{in}}, y_{\text{out}}, y_T) = y_{\text{out}} - y_{\text{in}} - \Delta_{G_i} \cdot y_T = 0$$

$$\bigwedge_{q \in G_i.Q} \sum_{(p,u,q) \in G_i.T} y_T(p,u,q) - \sum_{(q,u,p) \in G_i} y_T(q,u,p) = 0$$

$$\bigwedge_{\substack{j \leq d \\ G_i.v_{\text{in}}[j] \neq w}} y_{\text{in}}[j] = 0 \wedge \bigwedge_{\substack{j \leq d \\ G_i.v_{\text{out}}[j] \neq w}} y_{\text{out}}[j] = 0$$

The homogeneous system tells us the cycles we can repeat without breaking a solution.

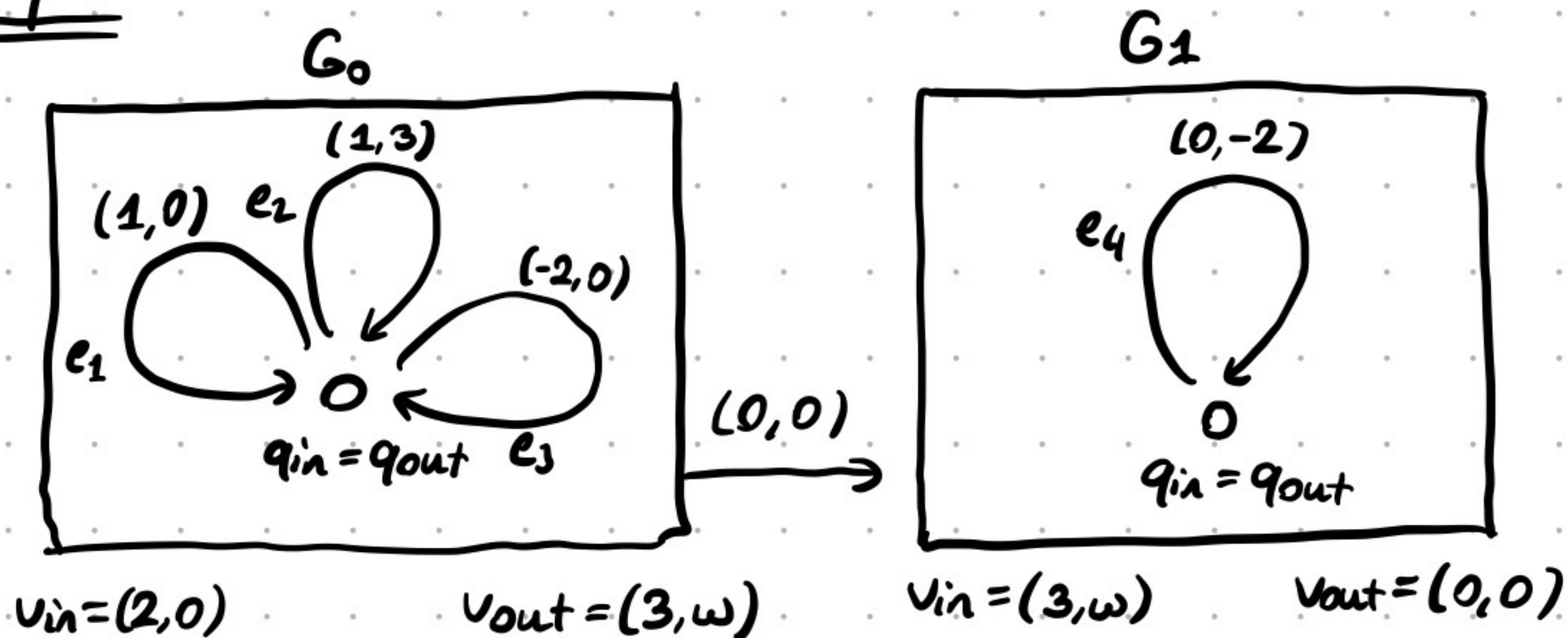
The effects of the bridge transitions are ignored.

The Euler-Kirchoff equation no longer fixes a start/end point: it only asks for cycles.

Instead of reaching the concrete targets, our solution must have 0 effect where a target is concrete.

Intuition: If you add this to a solution, the value for that point remains stable.

Example:



for $x[\text{in}, 0] = x[\text{out}, 1] = 0$:

→ $x[e_1] = 2, x[e_3] = 1$ is a homogeneous solution (0 effect, we can repeat)

→ $x[e_1] = 1$ is not, the out target in G_1 not stable.

→ $x[e_2] = 2, x[e_3] = 1, x[e_4] = 3$

is a homogeneous solution. Effect at the exit of G_0 is $(2,6) + (-2,0) = (0,6) \neq (0,0)$, but $(0,6) \leq_{\text{spec}} (w,0)$ holds, and this effect is cancelled out by 3 e_4 's, exiting with $(0,0)$. Without $x[e_4] = 3$, it is not a hom. sol.

As we saw with Linear Integer Systems, we can add homogeneous solutions to get another hom. sol. and we can add them to solutions, to get other solutions.

Lemma: Let $h, g \in \text{sol}(\text{HChar}_n(x))$ and $s \in \text{sol}(\text{Char}_n(x))$.

Then $h+g \in \text{sol}(\text{HChar}_n(x))$ and $s+h \in \text{sol}(\text{Char}_n(x))$.

The Support

An important object we will use is the support of the equation system.

We write $\sigma \in \text{sup}(\text{HChar}_n(x))$, if there is a solution $s \in \text{sol}(\text{HChar}_n(x))$ with $s[\sigma] \neq 0$.

(Here σ is a component name e.g. $(\text{in}, 0), (\text{out}, 0), \dots, (\text{in}, k), (\text{out}, k)$, or an edge.)

We shorten $\text{sup}(\text{HChar}_n(x))$ to $\text{sup}(M)$.

We call $s \in \text{HChar}_n(x)$ a full-support solution, if $s[\sigma] \neq 0$ for all $\sigma \in \text{sup}(M)$.

A very important property (which we will use tomorrow) is the following:

Lemma: There is always a full-support solution $h \in \text{sol}(\text{HChar}_n(x))$.

Proof: For each $\sigma \in \text{sup}(M)$, let $h_\sigma \in \text{HChar}_n(x)$ be the hom. solution with $h_\sigma[\sigma] \neq 0$.

Since $h_\sigma \geq 0$ must hold, $h_\sigma[\sigma] \geq 1$.

Let $h = \sum_{\sigma} h_\sigma$. Since hom. solutions are closed under addition, $h \in \text{sol}(\text{HChar}_n(x))$.

Let $\sigma \in \text{sup}(M)$. Then $h[\sigma] \geq h_\sigma[\sigma] \geq 1$.

Example: In the prev. example, $(\text{out}, 0, 1), (\text{in}, 1, 1), e_1, e_2, e_3, e_4 \in \text{sup}(M)$.

Setting $x[\text{out}, 0, 1] = x[\text{in}, 1, 1] = 6$,

$x[e_1] = 2, x[e_2] = 2, x[e_3] = 2, x[e_4] = 3$ and all else 0 gives a full-support solution

(Summed all valid examples)