Linear Systems and Semilinear Sets

The first part of the lecture was dedicated to understanding coverability in detail.

For the rest of this lecture, we turn our focus to the (precise) reachability problem.

We will not cover it here, but whether a given channel configuration is reachable is undecidable for lossy channel system.

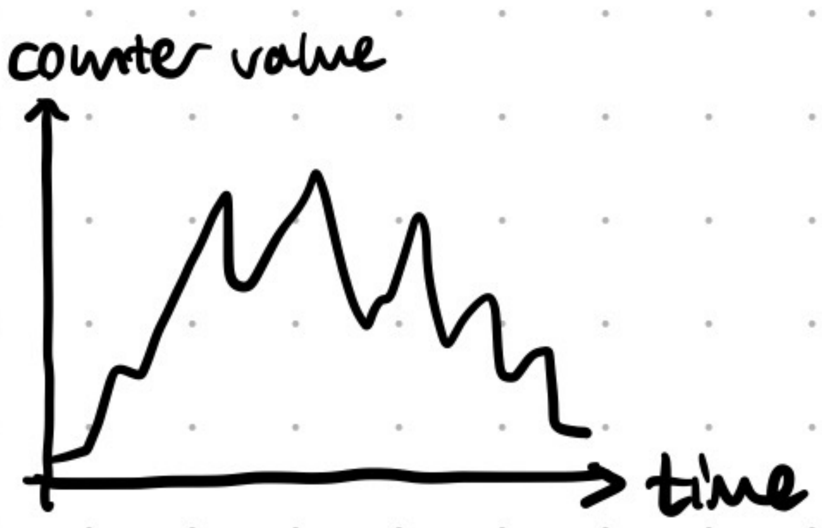
So we will focus on Petri-Nets (and related models) where the problem is (suprisingly) decidable.

Recall the problem:

> **REACH**
> Given: Petri-Net (S,T,W) and two markings $M_1, M_2 \in \mathbb{N}^S$.
> Question: Is there $\sigma \in T^*$ with $M_1[\sigma\rangle M_2$?

The $\sigma \in T^*$ must have two properties

(POS)



Keep the counter values non-negative

($\mathbb{Z}$-REACH) $M_1 + \text{effect}(\sigma) = M_2$

Provide the "exact" amount of tokens as needed.

The reachability algorithm uses approximations that deal with (POS) and ($\mathbb{Z}$-REACH). and refines the problem until the approximations can be combined to give a reaching run.

We deal with (POS) using coverability-related tools we developed shortly before.

To deal with ($\mathbb{Z}$-REACH), we use systems of linear inequalities.

↳ For the next few lectures, we dive into the world of linear integer arithmatic.

__Notation:__

For $P, R \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we write

$$P^* = \{0\} \cup \{x_1 + \ldots + x_k \mid x_1, \ldots, x_k \in P\}$$

Examples:

$P = \{(1,2)\} \longrightarrow P^* = \{(k, 2k) \mid k \in \mathbb{N}\}$

$P = \{2,3\} \longrightarrow P^* = \mathbb{N} \setminus \{1\}$

$$P + R = \{x_1 + x_2 \mid x_1 \in P,\ x_2 \in R\}$$

$$x + P = \{x + x_1 \mid x_1 \in P\}$$

Also:
We refer to $(0,\ldots,0) \in \mathbb{R}^k$ by 0, and $(1,\ldots,1)$ by 1.

Also:
We apply functions $f: X \to Y$ to sets $S \subseteq X$ by
$f(S) = \{f(s) \mid s \in S\}$

Today, we look into systems of linear inequalities.

Linear Integer Systems

We call systems S of the form subject to

$$x \in \mathbb{N}^V \text{ s.t.}$$

Product order

$$A \cdot x \geq b$$

linear integer systems for a finite set of variables V, a set of constraints C, $A \in \mathbb{Z}^{C \times V}$ and $b \in \mathbb{Z}^C$.

We denote such a system also by $[A \cdot x \geq b]$ shortly.

We write

$$sol(S) = \{ y \in \mathbb{N}^V \mid A \cdot y \geq b \}$$

to denote the set of values that satisfy the system $S = [A \cdot x \geq b]$.

We call S feasible, if $sol(S) \neq \emptyset$.

We also define a homogeneous system S_{hom} of the system $S = [A \cdot x \geq b]$ as $S_{hom} = [A \cdot x \geq 0]$. We call a system homogeneous, if it is of the form $[A \cdot x \geq 0]$.

We use equalities in our systems as well, i.e.

$$A \cdot x \geq b \quad \text{and}$$
$$B \cdot x = c$$

This is shorthand for $[A \cdot x \geq b \wedge B \cdot x \geq c \wedge -B \cdot x \geq -c]$ which can be packed to one matrix $\left[\begin{bmatrix} A \\ B \\ -B \end{bmatrix} \cdot x \geq \begin{bmatrix} b \\ c \\ -c \end{bmatrix} \right]$.

Motivation for studying these systems

In the reachability algorithm, we will use these system to approximate the PN by the perspective of ($\mathbb{Z}$-REACH) using these systems.

To get an idea, consider the marking equation

$$x \in \mathbb{N}^T \text{ s.t.}$$
$$A_W \cdot x = M_2 - M_1$$

for the PN (S, T, W) with source marking M_1, target M_2 where

$$A_W = \begin{array}{cc} \text{col. } t \in T & \\ \begin{bmatrix} \ddots & \vdots & \iddots \\ \cdots & W(t,s) - W(s,t) & \cdots \\ \iddots & \vdots & \ddots \end{bmatrix} & \text{row } s \in S \end{array}$$

For any $y \in \mathbb{N}^T$, $A_W \cdot y$ is the effect of applying transition $t_0 \in T$, $x[t_0]$ times; $t_1 \in T$, $x[t_1]$ times; $\dots$; $t_k \in T$, $x[t_k]$ times where $T = \{t_0, \dots, t_k\}$.

If $sol(A_W \cdot x = M_2 - M_1) = \emptyset$, then M_2 is already unreachable from M_1.

If not, we cannot say whether M_2 is reachable, but $sol(A_W \cdot x = 0)$ tells us about which transitions can be repeated together without changing the token balance at the end.

Goal for today: Show

Theorem: Let $S = [A \cdot x \geq b]$ be a LIS. Then if S is feasible,

$$sol(S) = B + P^*$$

for finite $B, P \subseteq \mathbb{N}^V$.

To this end, we define $\leq_A \subseteq \mathbb{Z}^V \times \mathbb{Z}^V$ wrt. $A \in \mathbb{Z}^{C \times V}$ such that

$$x \leq_A y \quad \text{iff} \quad x \leq y \quad \text{and} \quad A \cdot x \leq A \cdot y.$$

The solutions to $S = [A \cdot x \geq b]$ form a wqo under $\leq_A$.

Lemma: $(sol(S), \leq_A)$ is a wqo for any $S = [A \cdot x \geq b]$

Proof: Let $(y_i)_{i \in \mathbb{N}}$ be an infinite sequence.

We show that there are $i < j$ with $y_i \leq_A y_j$.

Consider the infinite sequence $((y_i, A \cdot y_i))_{i \in \mathbb{N}}$ in $\mathbb{Z}^{V \cup C}$.

Since $y_i \in sol(S)$ for all $i \in \mathbb{N}$, $A \cdot y_i \in \mathbb{N}^C$ and $y_i \in \mathbb{N}^V$ as well.

So $((y_i, A \cdot y_i))_{i \in \mathbb{N}}$ is an infinite sequence in $\mathbb{N}^{V \cup C}$.

Because $(\mathbb{N}^{V \cup C}, \leq)$ is a wqo, there are $i < j$ with $(y_i, A \cdot y_i) \leq (y_j, A \cdot y_j)$.

Then $y_i \leq y_j$ and $A \cdot y_i \leq A \cdot y_j$, and thus $y_i \leq_A y_j$. $\square$

Now we show that solutions to homogeneous systems are closed under addition.

Lemma: Let $S = [A \cdot x \geq 0]$. Then, $sol(S) + sol(S) \subseteq sol(S)$.

Proof: Let $y, z \in sol(S)$. We show $y + z \in sol(S)$. We get $A \cdot y \geq 0$ and $A \cdot z \geq 0$.

Thus $A \cdot (y + z) = A \cdot y + A \cdot z \geq 0 + A \cdot z \geq 0 + 0 = 0$ and $y + z \in sol(S)$.

$\square$

Now we show a version of the theorem for homogeneous systems.

<u>Theorem</u>: Let $S = [A \cdot x \geq 0]$. Then $sol(S) = P^*$ for some finite $P \subseteq \mathbb{N}^V$.

<u>Proof</u>: Let $P = \min_A(sol(S) \setminus \{0\})$. Since $(\mathbb{N}^V, \leq_A)$ is a wqo, $\uparrow_A P = \uparrow_A(sol(S)\setminus\{0\})$

wrt. $\leq_A$ wrt. $\leq_A$

and P is finite.

We show that $P^* = sol(S)$.

$\subseteq$: Clearly, $P \subseteq sol(S)$ and $\{0\} \subseteq sol(S)$ and $sol(S)$ is closed under addition, so $P^* \subseteq sol(S)$.

$\supseteq$: Let $y \in sol(S)$. We show by induction on $\|y\|_1 = \sum_{i \in V} |y[i]|$ that $y \in P^*$.

<u>Base Case, $\|y\|_1 = 0$</u>: $y = 0$, so clearly $y \in P^*$.

<u>Inductive Case, $\|y\|_1 = k+1$</u>: Since $\|y\|_1 = k+1$, $y \neq 0$.

Then $y \in sol(S)\setminus\{0\} \subseteq \uparrow_A(sol(S)\setminus\{0\})$, so $y \in \uparrow_A P$ as well.

Thus, there is a $p \in P$ with $p \leq_A y$. Then $p \leq y$ and $A \cdot p \leq A \cdot y$.

This means that $y - p \geq 0$ and $A \cdot (y-p) \geq 0$. Thus $(y-p) \in sol(S)$.

Because $y-p, y, p \in \mathbb{N}^V$, $\|y-p\|_1 < \|y\|_1$ and the I.H. applies.

We get $y - p = p_0 + p_1 + \ldots + p_k$ for some $p_0, p_1, \ldots, p_k \in P$ or $y - p = 0$.

Then, $y = p_0 + p_1 + \ldots + p_k + p$ or $y = p$ and thus $y \in P$.

$\square$

Now we show a final helper lemma (which will also be useful later).

<u>Lemma</u>: Let $S = [A \cdot x \geq b]$, $s \in sol(S)$ and $h \in sol(S_{hom})$. Then $s + h \in sol(S)$.

<u>Proof</u>: We have $A \cdot s \geq b$ and $A \cdot h \geq 0$. Thus

$$A \cdot (s+h) = A \cdot s + A \cdot h \geq b + A \cdot h \geq b + 0 = b$$

and $s + h \in sol(S)$.

$\square$

Finally we show the main theorem:

Proof: Let $S = [A\cdot x \geq b]$. Let $B = \min_A sol(S)$. Similarly to before, B is finite and $\uparrow_A B = \uparrow_A sol(S)$. We claim $B + sol(S_{hom}) = sol(S)$.

Because $sol(S_{hom}) = P^*$ for some finite P by the prev. theorem, we get the result we seek.

$\subseteq$: Let $b \in B \subseteq sol(S)$ and $h \in sol(S_{hom})$. Then, $b+h \in sol(S)$ by the prev. lemma.

$\supseteq$: Let $s \in sol(S)$.

Since $s \in \uparrow_A B$, there is $b \in B$ with $b \leq_A s$.

Thus $b \leq s$ and $A\cdot b \leq A\cdot s$.

This means $s-b \geq 0$ and $A\cdot(s-b) \geq 0$.

We get $s-b \in sol(S_{hom})$.

Then $b+(s-b) \in B + sol(S_{hom})$.

$\square$

In fact, we can even say something about the sizes of the contents of B and P: (we wont show this here)

Theorem: For $S = [A\cdot x \geq b]$, $sol(S) = B + P^*$ for finite $B, P \subseteq S$, where for all $s \in B \cup P$, $\|s\|_1 \leq O(\|S\|^{c+\ell})$ for $A \in \mathbb{Z}^{\ell \times d}$ and c independent of S.

The sum of the absolute value + 1 of each value used in A or b

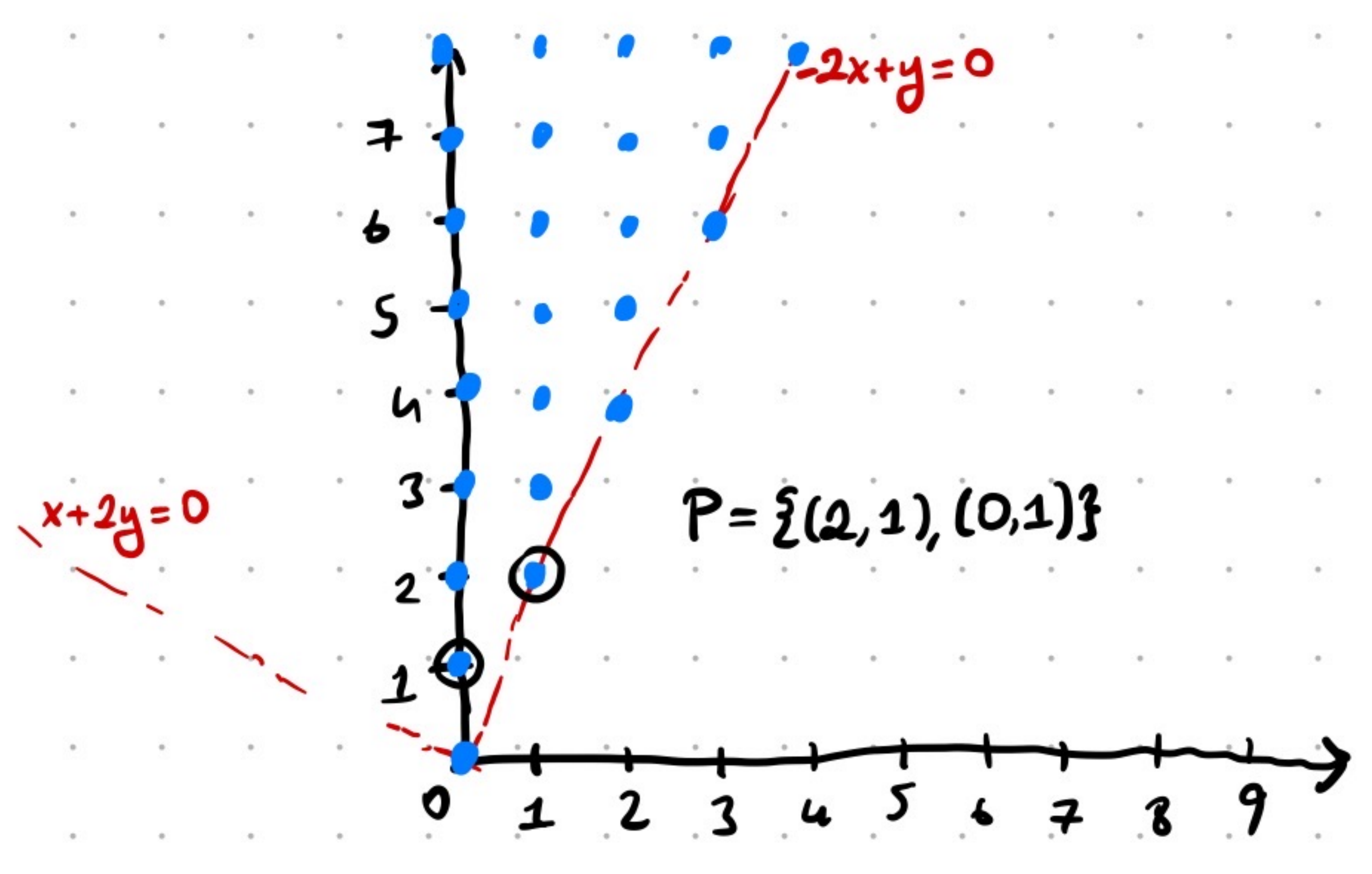
Example

$$-2x + y \geq 2$$
$$x + 2y \geq 7$$

$\rightarrow A\cdot x \geq b$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$sol(A\cdot x \geq 0)$:



$sol(A\cdot x \geq b)$: