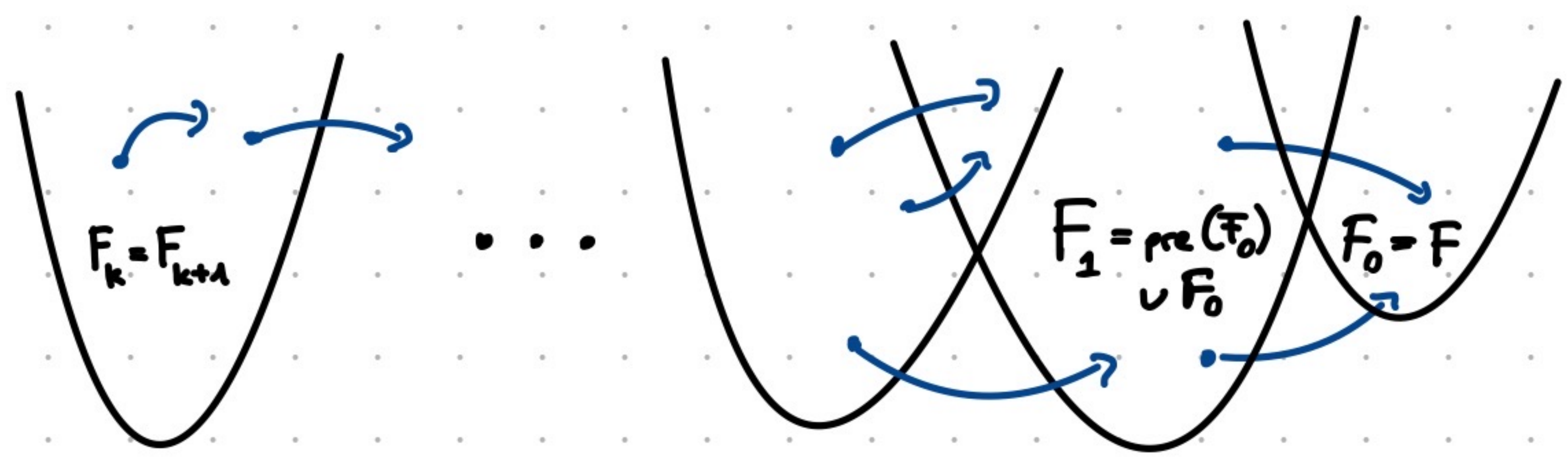
Coverability in WSTS

Goal: Decide Coverability / Upward Closed Reachability for WSTS

Given:	WSTS $W = (Q, \leq, I, \rightarrow)$ and upward closed $F \subseteq Q$
Decide:	Is there a run $q_0, q_1, \ldots, q_k$ with $q_0 \in I$ and $q_k \in F$?

Abdulla's backwards search

Idea: Perform reachability analysis of upward closed set backwards by computing the fixpoint $pre^*(F)$



with $pre(S) = \{ s \in Q \mid \exists s' \in S.\ s \rightarrow s' \}$ and $pre^*(F) = \bigcup_{i \in \mathbb{N}} pre^i(F)$

To decide upward closed reachability we then

1. Compute $pre^*(F)$
2. Check if $pre^*(F) \cap I = \emptyset$

Compute $pre^*(F)$ By Kleene iteration:

1. Compute $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ with $F_0 = \emptyset$, $F_{i+1} = F \cup pre(F_i)$
2. Check for stabilization $F_k = F_{k+1}$

Lemma If $F_k = F_{k+1}$, then $pre^*(F) = F_k$.

Proof: Note that $F_i = f^i(\emptyset)$ with $f(X) = F \cup pre(X)$:

$$F_0 = f^0(\emptyset) = \emptyset$$

$$F_{i+1} = f^{i+1}(\emptyset) = f(f^i(\emptyset)) = f(F_i) = F \cup pre(F_i)$$

Thus to show $pre^*(F) = F_k$ it remains to show that $pre^*(F)$ is the least fixed point of f.

1) $pre^*(F)$ is a fixed point: $f(pre^*(F)) = F \cup pre(pre^*(F)) = pre^*(F)$

2) $pre^*(F)$ is least fixed point: Let $f(X) = X$ fixed point.
We show $pre^i(F) \subseteq X$ by induction on i.
$i=0$: $pre^0(F) = F \subseteq f(X) = X$
$i+1$: $pre^{i+1}(F) = pre(pre^i(F)) \underset{\text{monotony + I.H.}}{\subseteq} pre(X) \subseteq f(X) = X$

Computability follows from:

(a) Increasing sequence of upward closed sets $U_0 \subseteq U_1 \subseteq \dots$ stabilizes in some $k \in \mathbb{N}$: $U_k = U_{k+1}$

(b) $F_0, F_1, \dots$ are upward closed sets

(c) Upward closed sets can be represented by finitely many minimal elements $M_i = \min(F_i)$

(d) computability of $\min(pre(\uparrow M_i))$ for finite M_i

will be an assumption

Kleene iteration terminates

Kleene iteration is effective

(a) and (c) follow from the fact that $\leq$ is a wqo and

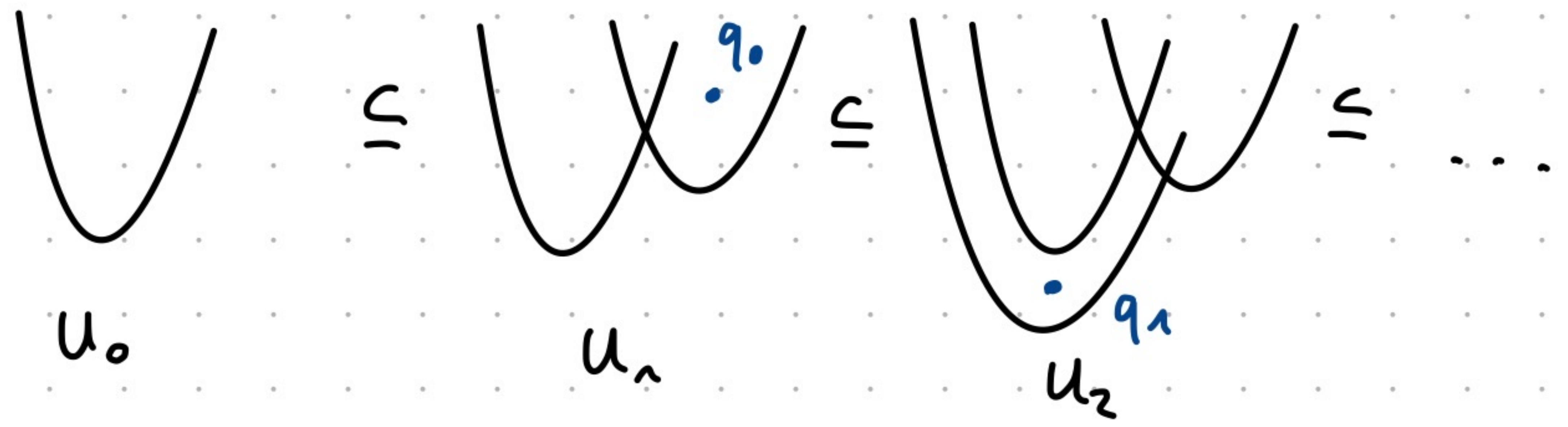
(b) follows from the fact that $\leq$ is a simulation

(d) will be an assumption on the given WSTS

The following lemmas show (a), (b).

__Lemma__ Let $(Q, \leq)$ be a quasi order. Then $(Q, \leq)$ is a wqo iff any increasing sequence of upward closed sets $U_0 \subseteq U_1 \subseteq \dots$ stabilizes, ie. there is $k \in \mathbb{N}$. $U_k = U_{k+1}$

"$\Rightarrow$": Towards a contradiction assume $U_0 \subsetneq U_1 \subsetneq \dots$
Then there are $q_i \in U_{i+1} \setminus U_i$:



Since U_i's are upward closed $q_i \not\leq q_j$ for $i<j$ and $q_0, q_1, \dots$ is an infinite bad sequence. ↯

"$\Leftarrow$" Consider a infinite sequence $q_0, q_1, \dots$
By assumption $\uparrow\{q_0\} \subseteq \uparrow\{q_0, q_1\} \subseteq \dots$ stabilizes $U_k = U_{k+1}$.
Then there is $i \leq k$ $q_i \leq q_{k+1}$.

Lemma Let $(Q, I, \to)$ be a transition system and $\leq \subseteq Q \times Q$.
Then $\leq$ is a simulation iff $pre(U)$ is upward closed for every upward closed set $U \subseteq Q$

Proof: Homework

Let $(Q, \leq)$ be a qo.
We define $min_{qo} : \mathbb{P}(Q) \to \mathbb{P}(Q)$ with

$$min_{qo}(U) = \{q \in U \mid \forall u \in U.\ q \leq u\}$$

Recall the Lemma below from the last VL

Lemma Let $(Q, \leq)$ be a wqo and $U \subseteq Q$ upcl. set.
Then there is a finite $B \subseteq U$ with $\uparrow B = U$.

Then in a wqo, $min_{qo}(U)$ is made up of finitely many equivalence classes.

$$min_{qo}(U) \subseteq \bigcup_{b \in B} [b]_{\equiv}$$

We define $min(U)$ as a choice of one representative from each equivalence class.

- $min(U)$ is an antichain
- $min_{qo}(U) = \{x \in U \mid \exists q \in min(U).\ x \equiv q\}$

We show that the Kleene iteration is effective if (d) holds.

Let $M_0 = min(F)$
$M_{i+1} = min(M_i \cup min(pre(\uparrow M_i)))$

and $min(B)$ is finite.

Lemma $\uparrow M_i = F_i$ for all $i \in \mathbb{N}$

Proof: by induction

$i = 0$: $\uparrow M_0 = \uparrow min(F) = F = F_0$

$i = i+1$: $\uparrow M_{i+1} = \uparrow min(M_i \cup min(pre(\uparrow M_i)))$

$= \uparrow M_i \cup \uparrow pre(\uparrow M_i)$

$\overset{\text{Lemma above}}{=} \uparrow M_i \cup pre(\uparrow M_i)$

$\overset{IH}{=} F_i \cup pre(F_i)$

$= F_{i+1}$

<u>Theorem</u> [Abdulla 96'] Let $(Q, \leq, I, \to)$ be a WSTS with

- $\min(\mathrm{pre}(\uparrow M))$ for finite $M \subseteq Q$ computable
- $\leq$ decidable
- $I \cap \uparrow\{x\}$ is decidable (follows from above if I finite)

and an upcl. set $F \subseteq Q$ given by $\min(F)$.
Then it is decidable whether F is reachable from I.