Order Theory II: Some WQO Theory

Goal: Learn more about wqo's.

Some definitions: Notation:

an index set ($\mathbb{N}, \{0,\dots,k\}$, etc.)

Equivalences in $(Q,\le)$: We write $q \equiv p$ if $q \le p$ and $p \le q$

Sequences $(q_i)_{i\in I}$ or $q_0, q_1, \dots$

Subsequences $(q_{\varphi(i)})_{i\in\mathbb{N}}$ is a subsequence of $(q_i)_{i\in\mathbb{N}}$

When we use this notation, we implicitly use a function $\varphi : \mathbb{N} \to \mathbb{N}$ that is monotonous and injective.

Increasing sequence: $(q_i)_{i\in I}$ with $q_i \le q_{i+1}$ for all $i \in I$

Decreasing sequence: $(q_i)_{i\in I}$ with $q_i \ge q_{i+1}$ for all $i \in I$

We say strict increasing/decreasing if $q_i \not\equiv q_{i+1}$ for all $i \in I$ as well.

Good sequence: $(q_i)_{i\in I}$ with $i<j$ and $q_i \le q_j$.

Bad sequence: Not good sequence

Anti-Chain: $(q_i)_{i\in I}$ where for all $i \ne j$, $q_i \not\le q_j$ and $q_j \not\le q_i$ (we also write $q_i \not\sim q_j$)

The definition of wqo's is then "all infinite sequences are good".

Lemma: Let $(q_i)_{i\in\mathbb{N}}$ be a sequence in the qo $(Q,\le)$. Then for a subsequence $(q_i)_{i\in\mathbb{N}}$

if $(q_i)_{i\in\mathbb{N}}$ is (strictly) increasing then so is $(q_{\varphi(i)})_{i\in\mathbb{N}}$.
(strictly) decreasing
anti-chain
bad sequence

Proof: Directly follows from the definitions.

There are many properties that equivalently define wqo's.

Theorem: The following statements are equivalent for the quasi order $(Q,\le)$.

(1) $(Q,\le)$ is a wqo

(2) Every infinite sequence $(q_i)_{i\in\mathbb{N}}$ in Q contains an infinite subsequence $(q_{\varphi(i)})_{i\in\mathbb{N}}$ that is increasing, i.e. for all $i \in \mathbb{N}$, $q_{\varphi(i)} \le q_{\varphi(i+1)}$.

(3) There is no infinite sequence $(q_i)_{i\in\mathbb{N}}$ that is strictly decreasing or an antichain.

(4) For any $S \subseteq Q$, there is a finite $S_0 \subseteq S$ such that $\uparrow S_0 = \uparrow S$.

Proof: (1)$\Rightarrow$(4): Let $S \subseteq Q$. Suppose there is no finite $B \subseteq S$ with $\uparrow B \ne \uparrow S$.

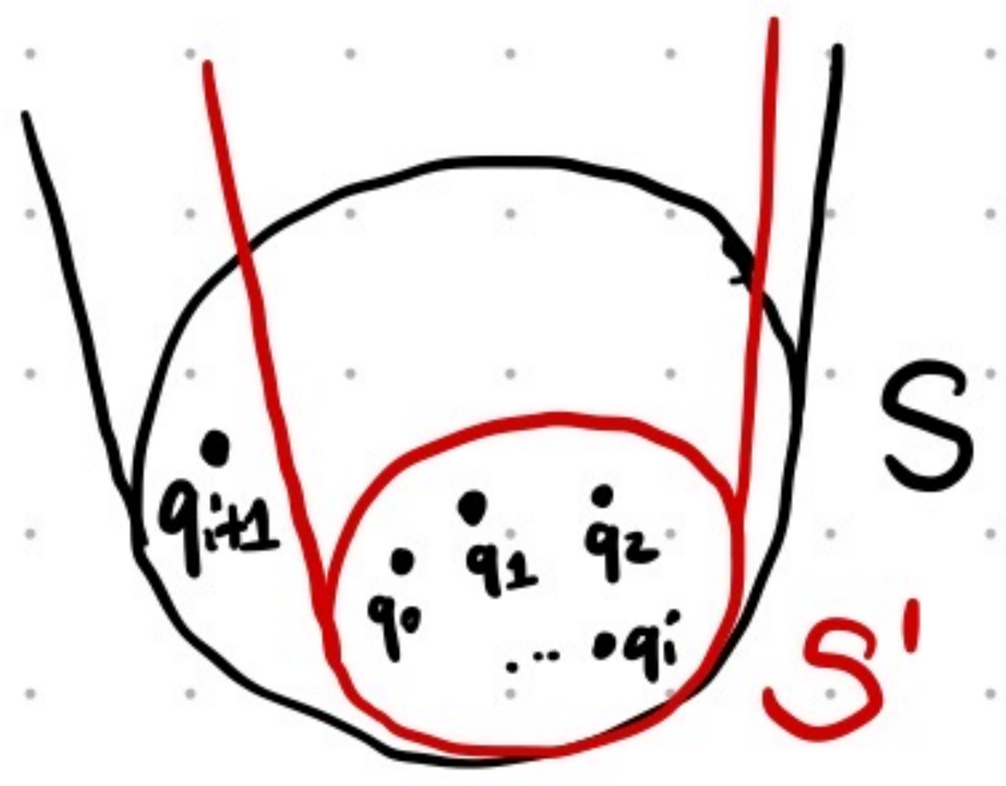
We construct a bad and infinite $(q_i)_{i\in\mathbb{N}}$ in S inductively.

For the base case, we choose any $q_0 \in S$.

For the inductive case, let $q_0, \dots, q_i$ be a bad sequence in S. We extend this sequence. Let $S' = \{q_0, \dots, q_i\}$. We have $\uparrow S' \ne \uparrow S$. Since $S' \subseteq S$, $S \setminus \uparrow S' \ne \emptyset$. Choose $q_{i+1} \in S \setminus \uparrow S'$.

Clearly there is no $j \in \{0,\dots,i\}$ with $q_j \le q_{i+1}$ since $q_{i+1} \notin \uparrow S'$. ✓

Picture:



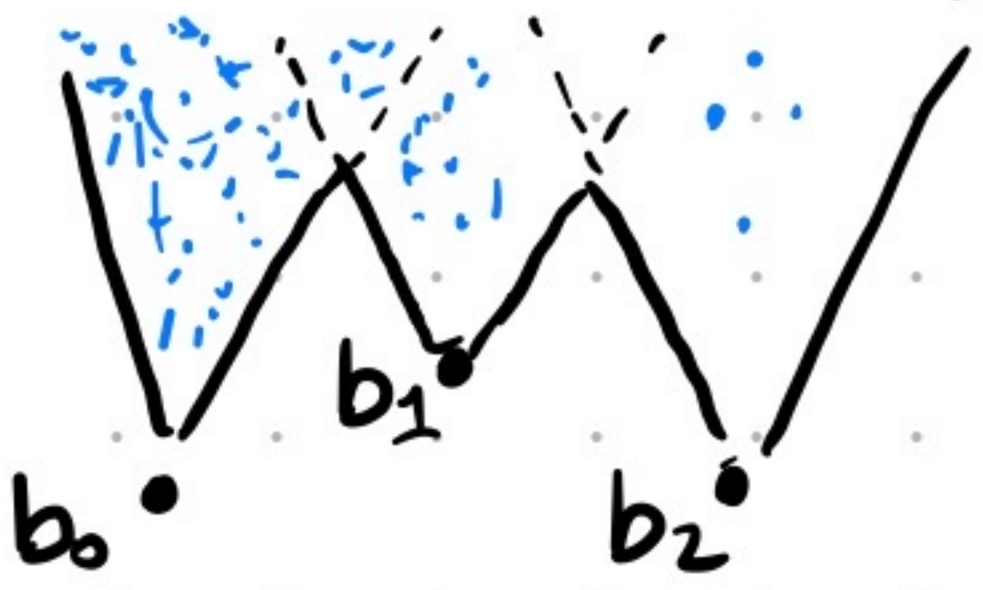
$(4) \Rightarrow (1)$: Let $(q_i)_{i \in \mathbb{N}}$ be an infinite sequence. We define the set $S = \{q_i \mid i \in \mathbb{N}\}$.

If S is finite, we are done: there must be at least two indices $i < j$ with $q_i = q_j$.

Let S be infinite. The set has a finite $B \subseteq S$ with $\uparrow B = \uparrow S$ per (4).

Then, there must be a $b \in B$ with infinitely many $q \in S$ in $\uparrow\{b\}$.

Picture:



Infinitely many S elements ($\cdot$'s).

Thus, there is a $i \in \mathbb{N}$ with infinitely many $j_0, j_1, \ldots \in \mathbb{N}$ where $q_i \leq q_{j_k}$ for all $k \in \mathbb{N}$.

Then, there must be some $k \in \mathbb{N}$ where $i < j_k$ and $q_i \leq q_{j_k}$. ✓

$(1) \Rightarrow (2)$: Let $(q_i)_{i \in \mathbb{N}}$ be an infinite sequence. We construct an increasing subsequence.

Let $u_0 < u_1 < u_2 < \ldots$ be the indices of elements that are not covered by a later element.

This means the u_k's where for all $j \geq u_k$, $q_{u_k} \not\leq q_j$.

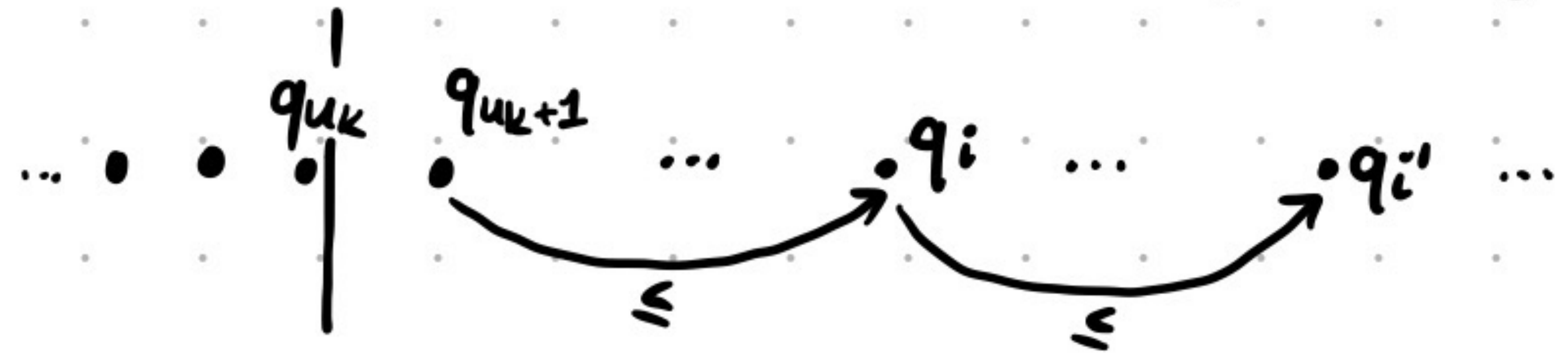
The definition guarantees that no element in $q_{u_0}, q_{u_1}, \ldots$ is covered by an element later in the sequence, so it is bad.

By the wqo definition, this sequence must be finite with the max. index u_k.

Then for all $j > u_k$, there is $j' > j$ with $q_j \leq q_{j'}$.

Since this process gives us larger indices, we can repeat this step infinitely often to construct an infinite increasing subsequence. ✓

Small picture:



$(2) \Rightarrow (3)$: Suppose there is an infinite $(q_i)_{i \in \mathbb{N}}$ that is strictly decreasing or an antichain.

We know that there must be an increasing subsequence $(q_{\varphi(i)})_{i \in \mathbb{N}}$.

Then, $q_{\varphi(0)} \leq q_{\varphi(1)}$, so $(q_i)_{i \in \mathbb{N}}$ cannot be an anti-chain or strictly decreasing. ✓

$(3) \Rightarrow (1)$: Let $(q_i)_{i \in \mathbb{N}}$ be an infinite sequence. Suppose that it is a bad sequence.

Idea:
Do the reversed version of the u_i construction in $(1) \Rightarrow (2)$ to eliminate decreasing pairs

Let $u_0 < u_1 < u_2 < \ldots$ be the indices of elements that do NOT cover a later element.

(Reverse of $(1) \Rightarrow (2)$)

First, let this sequence be finite with the maximal index u_k.

Then similar to $(1) \Rightarrow (2)$ we could find a decreasing subsequence $(q_{\varphi(i)})_{i \in \mathbb{N}}$ following u_k.

This sequence cannot be strictly decreasing by (3), so we find $\varphi(i) < \varphi(j)$ with $q_{\varphi(i)} \leq q_{\varphi(j)}$.

Now let this sequence be infinite, and consider $(q_{u_i})_{i \in \mathbb{N}}$.

We claim that $(q_{u_i})_{i \in \mathbb{N}}$ and thus $(q_i)_{i \in \mathbb{N}}$ are good.

Suppose $(q_{u_i})_{i \in \mathbb{N}}$ is bad.

By definition of u_i's, if $i < j$, then $q_{u_i} \not\geq q_{u_j}$ (because $u_i < u_j$).

Since $(q_{u_i})_{i \in \mathbb{N}}$ is bad, $q_{u_i} \not\leq q_{u_j}$ for all $i < j$ as well.

Then $(q_{u_i})_{i \in \mathbb{N}}$ is an infinite antichain, which contradicts (3). ✓

$\square$

We can combine orders using operations to construct new orders.

Let $(Q, \leq)$ and $(P, \leq)$ be two qo's. The wqo's are closed under the following operations (as we will see).

Disjoint Union Order $(P \dot\cup Q, \leq)$: For all $t, t' \in P$, $t \leq t'$.
For all $t, t' \in Q$, $t \leq t'$.
If neither hold for $t, t' \in P \dot\cup Q$, they are incomparable.

Product Order $(P \times Q, \leq)$: For all $(t_0, t_1), (s_0, s_1) \in P \times Q$, we have $(t_0, t_1) \leq (s_0, s_1)$ iff $t_0 \leq s_0$ and $t_1 \leq s_1$.

Subsequence Order $(P^*, \leq)$: The order $\leq \subseteq P^*$ is the smallest order that has the following properties:

(1) $\varepsilon \leq w$ for all $w \in P^*$

(2) $s \leq s'$ for all $s, s' \in P$ with $s \leq s'$

(3) $w.v \leq w'.v'$ for all $w, w', v, v' \in P^*$ with $w \leq w'$ and $v \leq v'$.

Example: $P = \{a, b, c\}$, $\leq$ is equality.

$$ab \leq accb \leq acccb \leq acbacbacb$$

Closure under disjoint unions is left as an exercise to the reader (in particular, sheet 1).

Theorem (Dickson's Lemma): If $(Q_0, \leq)$ and $(Q_1, \leq)$ are wqo's then so is the product order $(Q_0 \times Q_1, \leq)$.

Proof: Let $(q_0, p_0), (q_1, p_1), \ldots$ be an infinite sequence in $Q_0 \times Q_1$.

We show that there are $k < l$ with $(q_k, p_k) \leq (q_l, p_l)$.

Note that $(q_i)_{i \in \mathbb{N}}$ is an infinite sequence in Q_0,

Since Q_0 is a wqo, there is an infinite subsequence $(q_{\varphi(i)})_{i \in \mathbb{N}}$ that is increasing.

Consider the subsequence $p_{\varphi(0)}, p_{\varphi(1)}, \ldots$.

Using the same trick, we can find an increasing subsequence $(p_{\psi(\varphi(i))})_{i \in \mathbb{N}}$ in $(p_{\varphi(i)})_{i \in \mathbb{N}}$.

Since the subsequence of an increasing subsequence is also increasing, $(q_{\psi(\varphi(i))})_{i \in \mathbb{N}}$ is increasing as well.

In particular, $(q_{\psi(\varphi(0))}, p_{\psi(\varphi(0))}) \leq (q_{\psi(\varphi(1))}, p_{\psi(\varphi(1))})$.

This concludes the proof. ☐

Idea (Demonstrated on $\mathbb{N}^2$):

$(q_i, p_i)_{i \in \mathbb{N}}$:	$(0,5)$,	$(1,2)$,	$(6,1)$,	$(7,0)$,	$(8,1)$,	$(1,9)$,	$(8,10)$,	$(9,11), \ldots$
$(q_{\varphi(i)}, p_{\varphi(i)})_{i \in \mathbb{N}}$: 🟦 is increasing		$(1,2)$,	$(6,1)$,	$(7,0)$,	$(8,1)$,		$(8,10)$,	$(9,11), \ldots$
$(q_{\psi(\varphi(i))}, p_{\psi(\varphi(i))})_{i \in \mathbb{N}}$: 🟦 is increasing 🟥 is increasing			$(6,1)$,		$(8,1)$,		$(8,10)$,	$(9,11), \ldots$

Lemma: $(\mathbb{N}, \leq)$ is a wqo.

Proof: Any infinite sequence is bounded and repeats an element, or has an infinite strictly increasing subsequence.

Corollary: $(\mathbb{N}^d, \leq)$ is a wqo.