Regular Separability of VASS Reachability Languages

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1. Regular Separability

Regular Separability

Reachability languages.

$\mathbb{X} \in \{\mathbb{Z}, \mathbb{N}\}$.

$\mathbb{X}$-REGSEP:

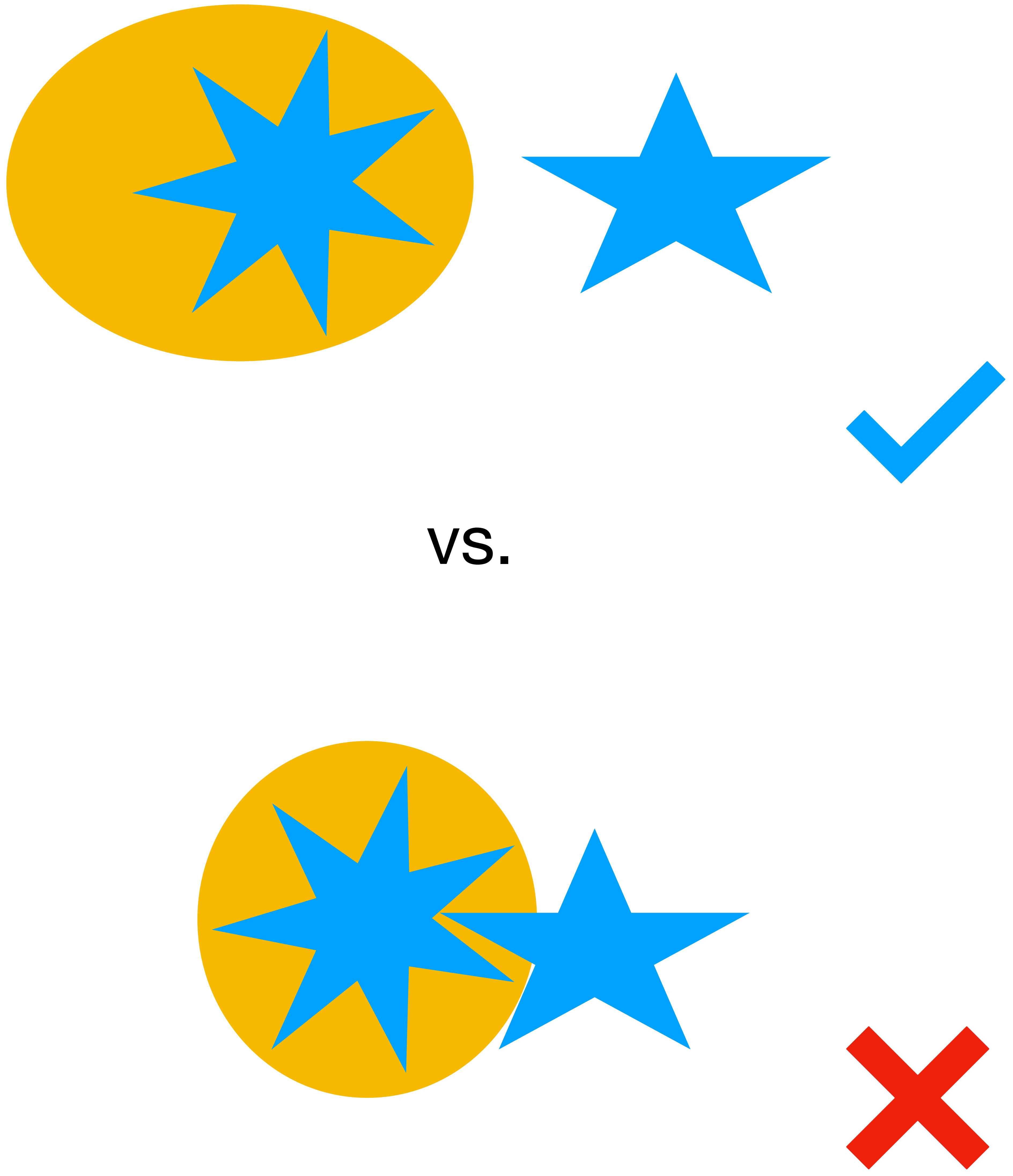
Given: Initialized VASS V_1 and V_2 over Σ .

Question: Does $L_{\mathbb{X}}(V_1) \mid L_{\mathbb{X}}(V_2)$ hold?

$L_1 \mid L_2$:

$\exists R \subseteq \Sigma^*$ regular. $L_1 \subseteq R \wedge R \cap L_2 = \emptyset$.

Write $R : L_1 \mid L_2$.



Regular Separability

Example:

1. $\{a^n . b^n \mid n \in \mathbb{N}\} \mid \{a^n . b^{n+1} \mid n \in \mathbb{N}\}$. ✓

Yes! Separator: Even.Even ∪ Odd.Odd.

2. $\{a^n . b^{\leq n} \mid n \in \mathbb{N}\} \nmid \{a^n . b^{>n} \mid n \in \mathbb{N}\}$. ✗

No! Assume $A : L_1 \mid L_2$ and A has m states.
Consider $a^{m+1} . b^{m+1} \in L_1 \subseteq L(A)$. ↯

Discussion:
Separability tries to understand the gap between languages.

Insight:
Modulo seems to play an important role!

Regular Separability

Known:

Theorem [Lorenzo, Wojtek, Slawek, Charles, ICALP'17]:
$\mathbb{Z}$-REGSEP is decidable.

Goal:

Theorem:
$\mathbb{N}$-REGSEP is decidable.

2. Transducer Trick

[Lorenzo, Wojtek, Slawek, Charles, ICALP'17]
[Wojtek and Georg, LICS'20]

Transducer Trick

Visible VAS: a_i leads to an increment of counter i.

Goal:
Take only one language as input.

Lemma:

$$
\begin{aligned}
L(V) \mid L(U) \quad &\Leftrightarrow \quad L(V) \mid T_U(D_n) \\
&\Leftrightarrow \quad T_U^{-1}(L(V)) \mid D_n \quad \text{over } \Sigma_n := \{a_i, \bar{a}_i \mid i \in dy := [1,n]\} \\
&\Leftrightarrow \quad L(V') \mid D_n \ .
\end{aligned}
$$

3. Intermezzo: Reachability

Deciding Reachability

Approximations:

Coverability graphs:
Good: Can keep counters non-negative.
Bad: Cannot guarantee precise counter values.

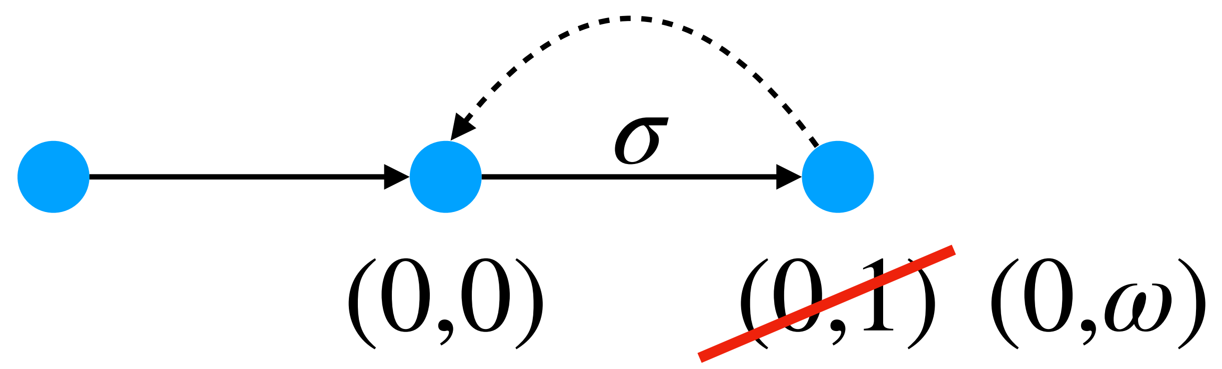
Marking Equation:
Good: Can guarantee precise counter values.
Bad: Cannot keep counters non-negative.

Solution:
Combine the two.

Deciding Reachability

Challenge:
Coverability graphs need pumping to guarantee non-negativity.
Pumping has to respect the marking equation.

Solution:
Only pump where the solution space is unbounded.



$\Rightarrow$ $x[e]$ with $e \in \sigma$, $x[j]$ with j = 2 have to be unbounded in the solution space.

Deciding Reachability

Lemma:

Consider $A \cdot x = b$ over $\mathbb{N}^k$ and variable x[i].

x[i] is unbounded in $sol(A \cdot x = b)$

$$\Leftrightarrow \quad \exists s \in sol(A \cdot x = 0) \,.\; s(x[i]) > 0.$$

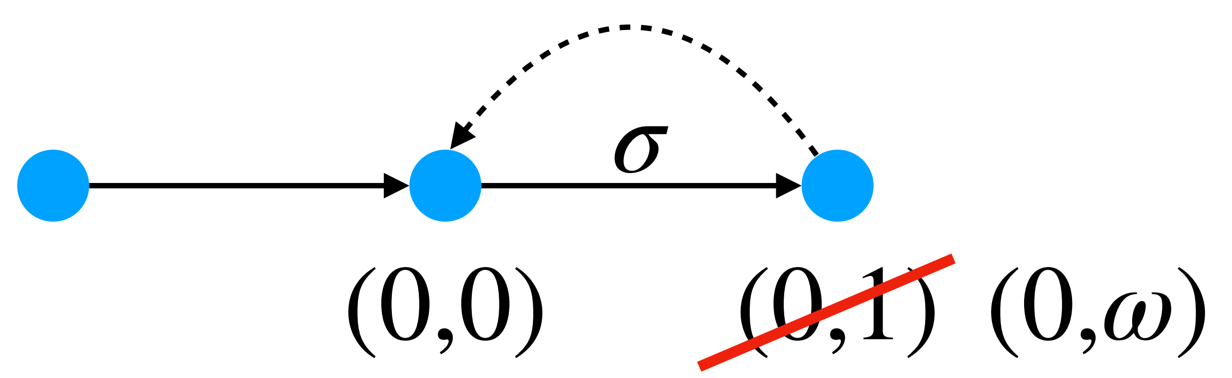
Support = the set of unbounded variables.

Support solution =

$s \in sol(A \cdot x = 0)$ giving a positive value to all variables in the support.

Note: Homogeneous solutions are stable under addition.

Deciding Reachability



$\Rightarrow$ $x[e]$ with $e \in \sigma$ / $x[j]$ with j = 2 — have to be unbounded in the solution space.

So far:
Pumping where the solution space is unbounded
= pumping should yield a support solution.

Problem:
σ may not match a support solution s.

Parikh image.

Idea:
Turn $s - \psi(\sigma)$ into a path.

Deciding Reachability

Lemma (Euler-Kirchhoff):
Let $G = (V, E)$ be a strongly connected directed graph.
Let $x : \mathbb{N}^E$ satisfy

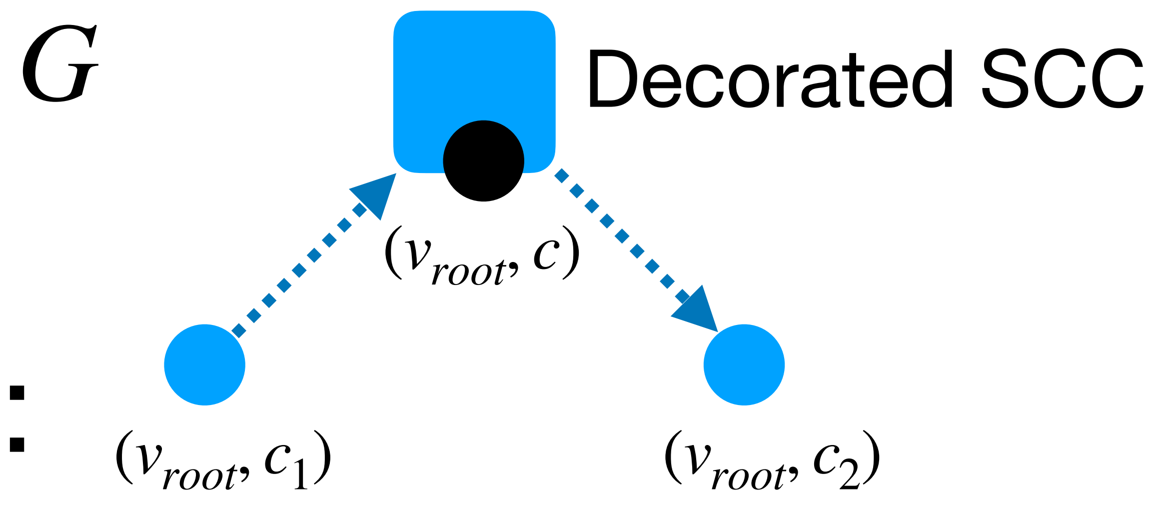
$$\sum_{e=(-,v)} x[e] = \sum_{e=(v,-)} x[e] \qquad \forall v \in V$$

$$x \geq 1$$

Then there is a cycle c in G with $\psi(c) = x$.
Also write $c = \langle x \rangle$.

Realization.

Deciding Reachability



Definition:
A precovering graph (PG) is a strongly connected VASS:

- The nodes are decorated by gen. markings, like in coverability graphs.
- These markings agree on where to put ω.
- The PG has a root (v_{root}, c) with decoration c.
- There are gen. entry/exit markings $(v_{root}, c_1), (v_{root}, c_2)$ with $c_1, c_2 \sqsubseteq_\omega c$.

Specialization:
Preserve concrete values,
may concretize ω.

Deciding Reachability

Definition:
A PG is perfect, if

- all edge variables are in the support,
- all variables decorated ω in the entry and exit markings are in the support,
- $Up(G) \neq \emptyset \neq Down(G)$:

  $u \in Up(G)$ = cycle in G exec. from c_1 increasing the counters in $\Omega(c) \backslash \Omega(c_1)$.
  $v \in Down(G)$ = cycle in G bw exec. from c_2 decreasing $\Omega(c) \backslash \Omega(c_2)$.

Deciding Reachability

Pumping should yield a support solution:

Let s be a support solution with

$$d := s - \psi(u) - \psi(v) \geq 1 \ .$$

This is why we have connectivity
and all edges
should be in the support!

By the Euler-Kirchhoff Lemma, the difference can be realized by a cycle

$$w = \langle d \rangle \ .$$

Now $\psi(u) + \psi(w) + \psi(v) = s$ and we say they match.

Deciding Reachability

Insight:

v has a strictly negative effect on the ω counters

$\Rightarrow \quad u\,.\,w$ must have a strictly positive effect.

Pumping:

u, w, v and s match $\quad \Rightarrow \quad u^c\,.\,w^c\,.\,v^c$ and $c \cdot s$ match.

With

$k :=$ least number of $u\,.\,w$ needed to execute w .

$c :=$ k + least number of further u needed to execute $u^k\,.\,w^k$

the sequence becomes an ℕ-run/executable.

Deciding Reachabilit

Trick 1:
Add this slack to Lambert's Iteration Lemma.

Lambert's Iteration Lemma [TCS'92]:

For c large enough, one can even fit in a $\mathbb{Z}$-cycle that reaches the exit from the entry marking:

$$u^c \,.\, \rho \,.\, w^c \,.\, v^c \,.$$

Since pumping happens in a support solution, this still solves reachability. Notably, it stays non-negative.

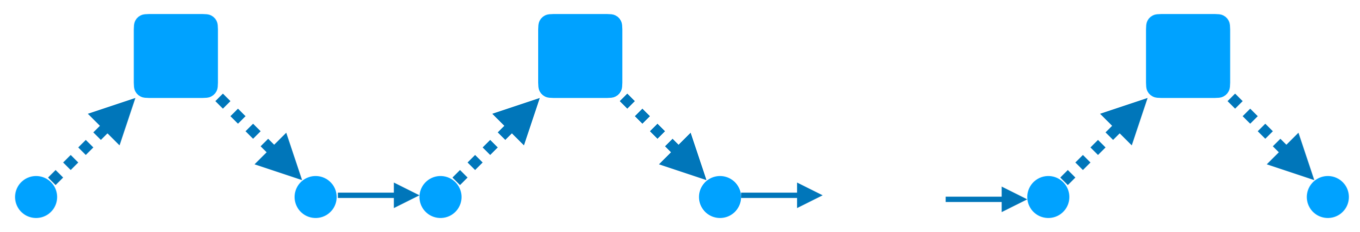
Note:

This works for all $\mathbb{Z}$-runs, and all choices of (u, w, v) that match a support solution.

Deciding Reachability

Problem: Precovering graphs may not be perfect.

Solution: Decompose them into sequences of precovering graphs, MGTS:

MGTS 

Deciding Reachability

Deciding Reachability:

As long as perfectness fails, decomposition is guaranteed to succeed.

It yields finite sets of MGTS that are smaller in a well-founded order.

Hence, perfectness will eventually hold.

For perfect MGTS,

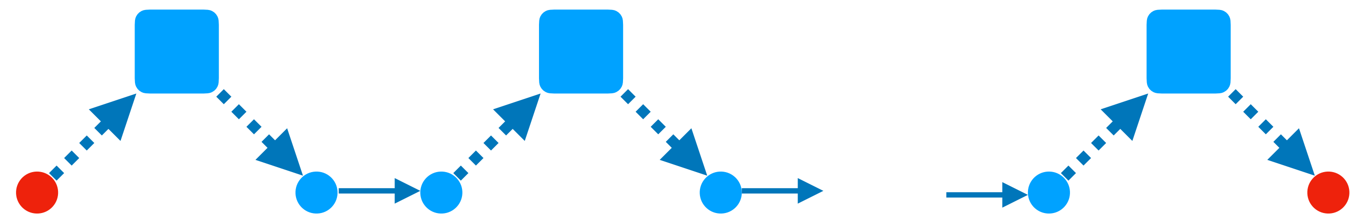
$$\mathbb{N}\text{-reachability holds} \quad \Leftrightarrow \quad \mathbb{Z}\text{-reachability holds.}$$

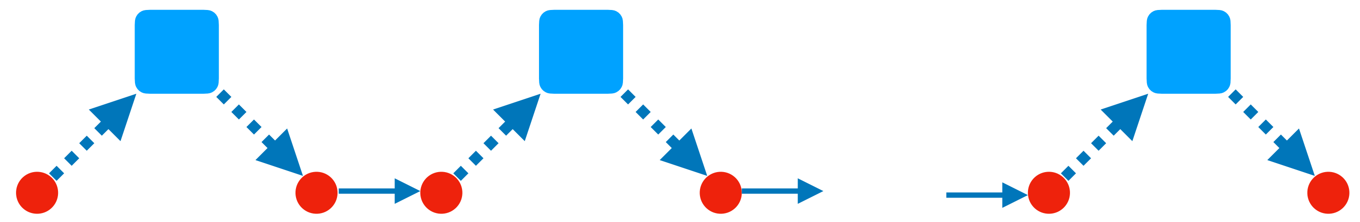
Deciding Reachability

Acceptance on MGTS:

C := Counters that have to stay non-negative.

$\leq$:= Preorder to compare markings at red nodes for acceptance.

$Acc_{C,\leq}(U) :=$ 

$IAcc_{C,\leq}(U) :=$ 

The $\mathbb{Z}$-runs for reachability satisfy $IAcc_{\mathbb{Z},\sqsubseteq_\omega}$.

4. DMGTS

DMGTS

Doubly-Marked MGTS $W = (U, \mu)$:

$U =$ MGTS over Σ_n with counters $sj \uplus dy$ with dy visible.

$\mu \geq 1$.

Strategy:
Define language $L_{sj}(W)$ and $L_{dy}(W)$.
Use perfectness to achieve

$$L_{sj}(W) \mid D_n \qquad \Leftrightarrow \qquad L_{\mathbb{Z},sj}(W) \mid L_{\mathbb{Z},dy}(W) \ .$$

Trick 2:
Defining two languages on a single DMGTS.

DMGTS

Specialization only makes requirements on dy .

Keep dy counters non-negative.

Trick 3: Intersection.

Trick 4: Modulo-μ Specialization.

Acceptance:

$$(I)Acc_{dy}(W) := (I)Acc_{dy,\sqsubseteq_\omega[dy]}(W)$$

$$(I)Acc_{\mathbb{Z},dy}(W) := (I)Acc_{\mathbb{Z},\sqsubseteq_\omega[dy]}(W)$$

$$IAcc_{sj}(W) := IAcc_{sj,\sqsubseteq_\omega[sj]}(W) \cap IAcc_{dy,\sqsubseteq^\mu_\omega[dy]}(W)$$

$$IAcc_{\mathbb{Z},sj}(W) := IAcc_{\mathbb{Z},\sqsubseteq_\omega[sj]}(W) \cap IAcc_{\mathbb{Z},\sqsubseteq^\mu_\omega[dy]}(W)$$

DMGTS

Modulo-μ Specialization:

$x \sqsubseteq^{\mu}_{\omega} k$, if $k = \omega$ or $x \equiv k \mod \mu$.

Lemma (Monotonicity of Modulo-μ Intermediate Acceptance):

$$\rho \in IAcc_{\mathbb{Z}, \sqsubseteq^{\mu}_{\omega}[dy]}(W) \quad \Rightarrow \quad \rho + \mu \in IAcc_{\mathbb{Z}, \sqsubseteq^{\mu}_{\omega}[dy]}(W) \ .$$

Increase the Dyck counters in all configurations by μ.

Trick 5: Monotonicity.

Thanks to this, we could have replaced dy by $\mathbb{Z}$ in $IAcc_{sj}(W)$.

DMGTS

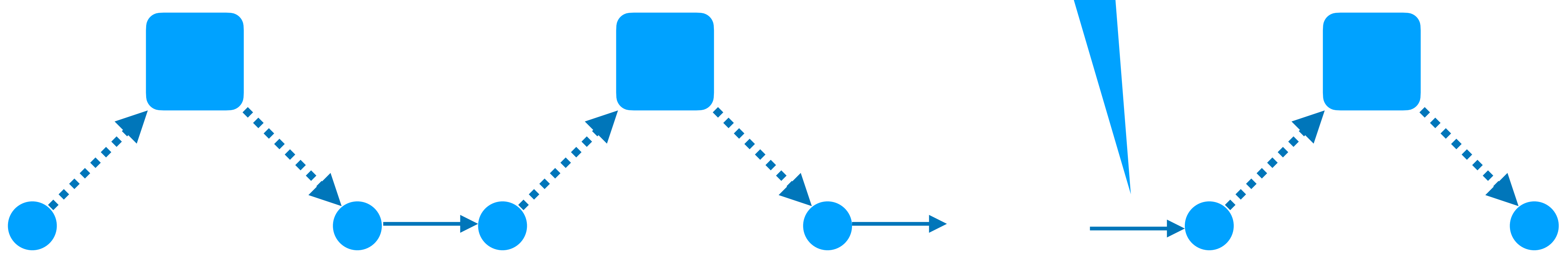
Trick 6:
Use of $\lambda_\sharp$.

Languages:

$$L_{sd}(W) := \{\lambda(\rho) \mid \rho \in IAcc_{sd}(W)\}$$

$$L_{\mathbb{Z},sd}(W) := \{\lambda_\sharp(\rho) \mid \rho \in IAcc_{\mathbb{Z},sd}(W)\}$$

$\lambda_\sharp(\rho)$ puts here
$(a, \sharp)$ instead of a.



DMGTS

Trick 7: Faithfulness.

Zero-Reaching:

$$W\,.\,c_{in}[dy] = 0 = W\,.\,c_{out}[dy]\ .$$

Faithful: Intermediate acceptance modulo-μ ⇔ ordinary intermediate acceptance, provided we fix initial and final values.

Faithfulness: Zero-reaching +

$$Acc_{\mathbb{Z},dy}(W) \cap IAcc_{\mathbb{Z},\sqsubseteq^{\mu}_{\omega}[dy]}(W) \quad \subseteq \quad IAcc_{\mathbb{Z},dy}(W)\ .$$

DMGTS

Trick 8:
Share $Up(G)$ and $Down(G)$ between sj and dy .
Make sure the edges are in the support of both sides.

Perfectness:
W is perfect, if it is faithful and for all $G \in W$:

$$Up(G) \neq \emptyset \neq Down(G) .$$

$$\forall e \in G.E.e \in supp(Char_{sj}(W)) \wedge e \in supp(Char_{dy}(W)) .$$

$$\forall j \in sd.G.c_{io}[j] = \omega \quad \Rightarrow \quad x[G, io, j] \in supp(Char_{sd}(W)) .$$

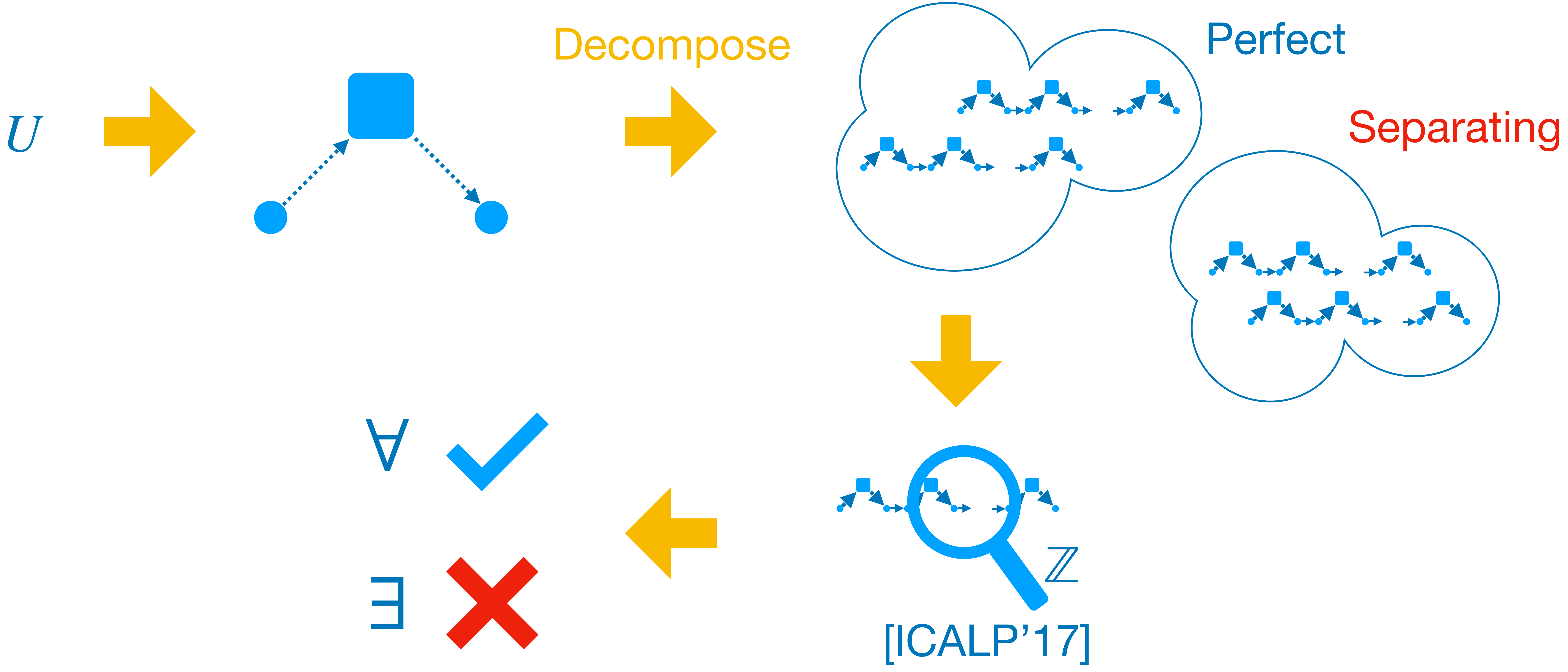
5. Deciding Regular Separability

Theorem: Let U be an initialized VASS over Σ_n.

Then $L(U) \mid D_n$ is decidable.

Deciding Regular Separability

Algorithm:



Deciding Regular Separability

Reduce $\mathbb{N}$-REGSEP to $\mathbb{Z}$-REGSEP using perfectness!

Algorithm:

1. Turn the given VASS U into an initial DMGTS W.

2. Decompose W into finite sets $Perf$ and Fin.

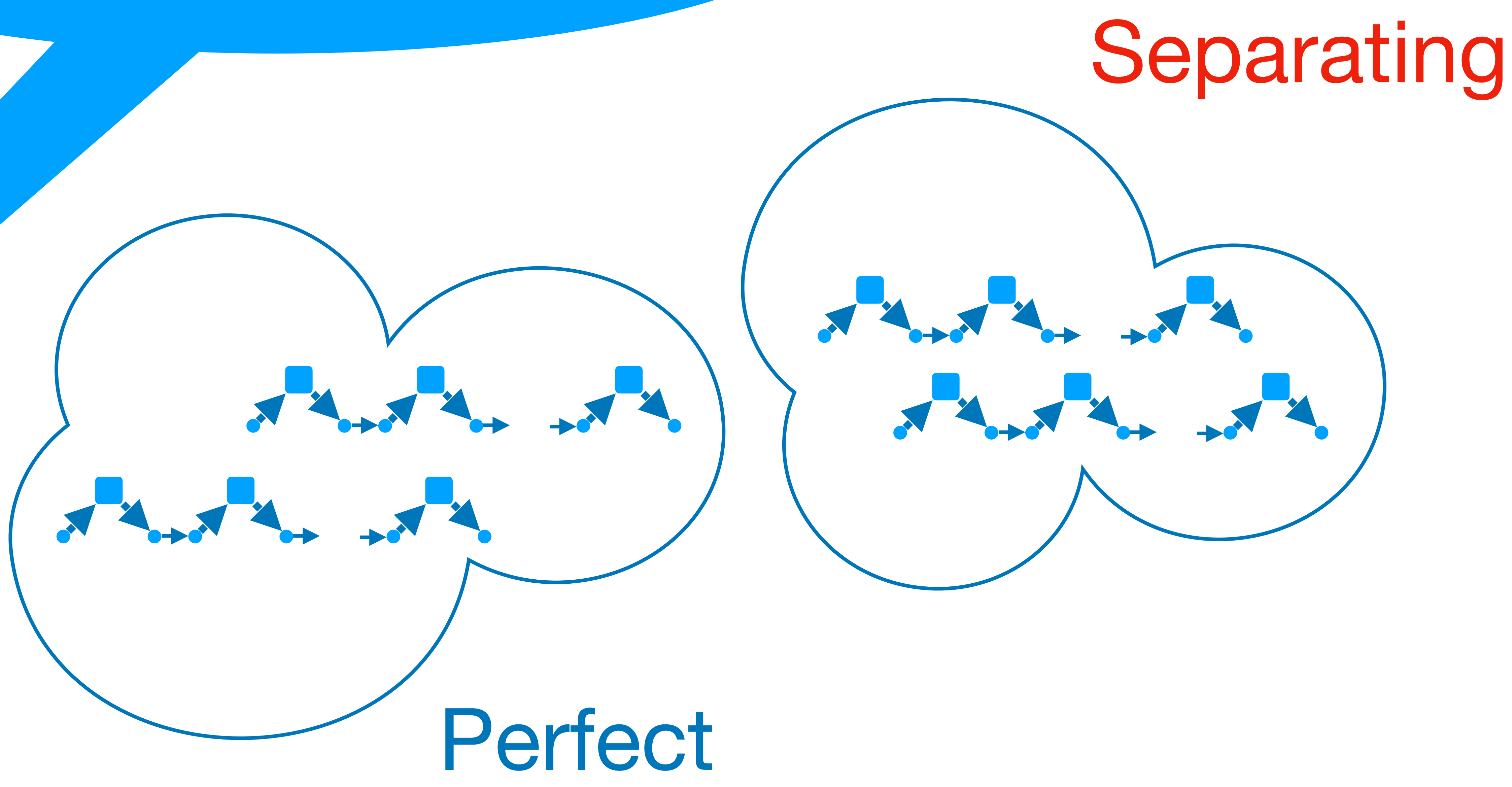
For the DMGTS $T \in Fin$,

$$L_{sj}(T) \mid D_n \, .$$

For the DMGTS $S \in Perf$,

$$L_{sj}(S) \mid D_n \quad \Leftrightarrow \quad L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S) \, .$$

3. Check $L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S)$ using [ICALP'17].
If all checks pass return true, else return false.



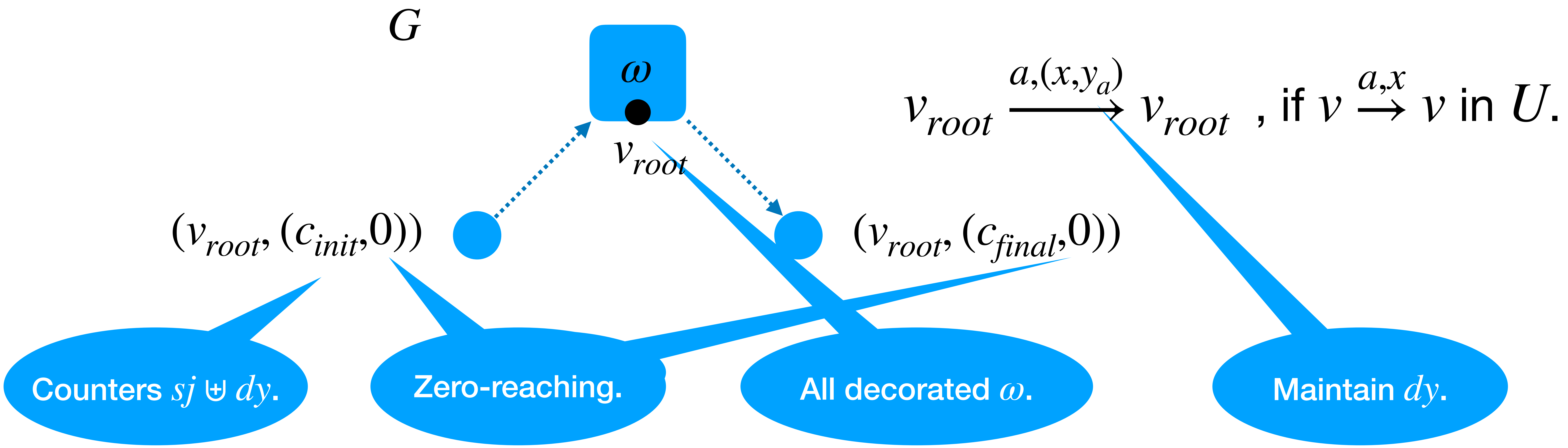
Needed: Initial DMTS, decomposition, separability transfer.

Deciding Regular Separability: Initial DMGTS

Definition:

Let (U, c_{init}, c_{final}) be a VAS with counters sj.

The associated initial DMGTS is $W = (G, \mu)$ with $\mu = 1$ and



Deciding Regular Separability: Initial DMGTS

Lemma (Initial DMGTS):

1. $L_{sj}(W) = L(U)$.
2. W is faithful.

We can now show $L_{sj}(W) \mid D_n$ and rely on faithfulness.

Proof:

1. $L_{sj}(W)$ additionally requires acceptance modulo μ on dy.
As $\mu = 1$ and the extremal markings are 0 on dy, this is no restriction.

2. W is zero-reaching by definition.
Moreover, there are no intermediate markings.
Hence, acceptance and intermediate acceptance on dy coincide:

$$Acc_{\mathbb{Z},dy}(W) \cap IAcc_{\mathbb{Z},\sqsubseteq^{\mu}_{\omega}[dy]}(W) \quad \subseteq \quad Acc_{\mathbb{Z},dy}(W) \quad = \quad IAcc_{\mathbb{Z},dy}(W)\,.$$

$\square$

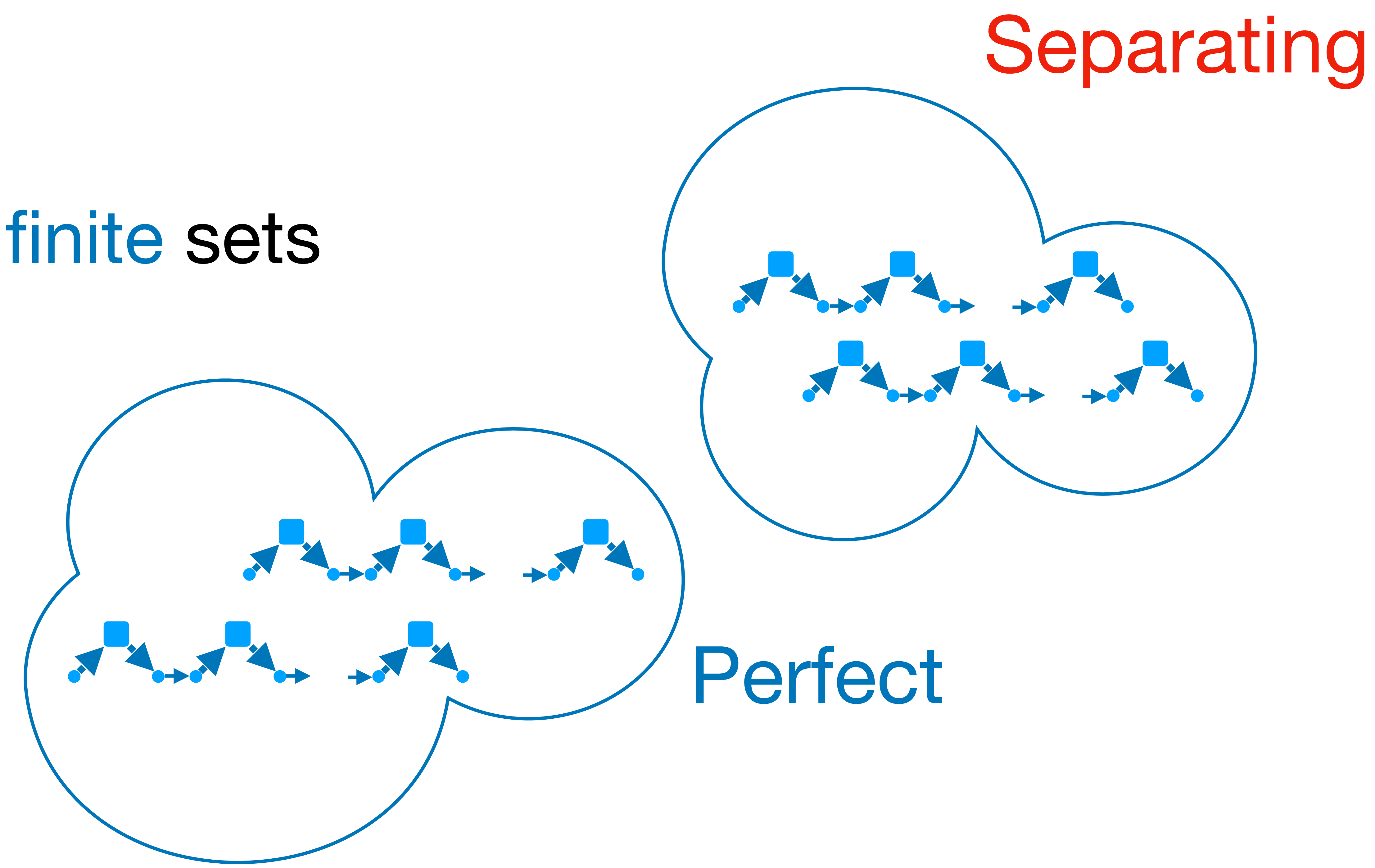
Deciding Regular Separability: Decomposition

Proposition (Decomposition):
Given a faithful DMTS W, we can compute finite sets

$$Perf \text{ and } Fin \text{ of DMGTS,}$$

where

- $\forall S \in Perf . \quad S$ is perfect,
- $\forall T \in Fin . \quad L_{sj}(T) \mid D_n$,
- $L_{sj}(W) = L_{sj}(Perf) \cup L_{sj}(Fin)$.



We only have to show $L_{sj}(Perf) \mid D_n$ and can rely on perfectness.

Deciding Regular Separability: Decomposition

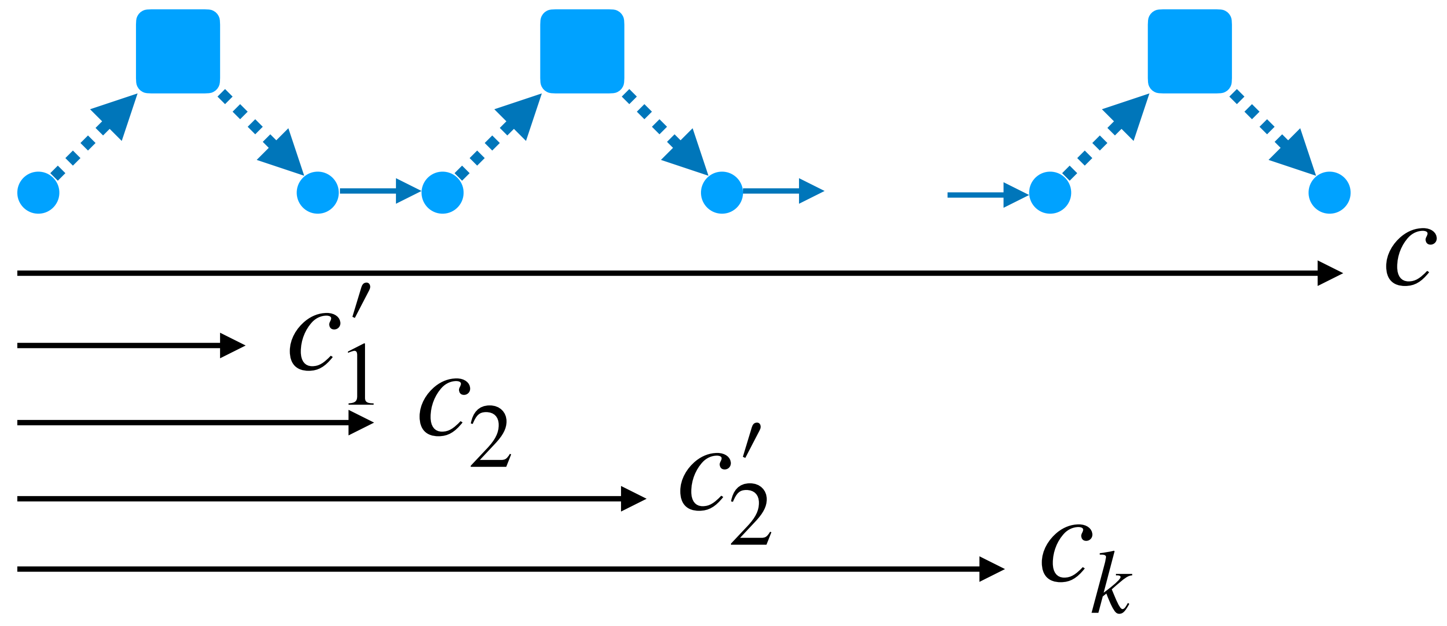
Proposition (Separability Transfer):
If S is perfect,

$$L_{sj}(S) \mid D_n \quad \Leftrightarrow \quad L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S)\,.$$

We can rely on the decision procedure for $\mathbb{Z}$-REGSEP from [ICALP'17].

Lemma:
Given a DMGTS W, we can compute ($\mathbb{Z}$-)VASS U_{sj} and U_{dy} with $L_{\mathbb{Z},sd}(S) = L_{\mathbb{Z}}(U_{sd})\,.$

Proof:
Auxiliary counters for each intermediate marking.
Maintain them until that marking is reached.
Check their values at the end.



□

Deciding Regular Separability

Algorithm:

1. Turn the given VASS U into an initial DMGTS W.

2. Decompose W into finite sets $Perf$ and Fin.

For the DMGTS $S \in Perf$,

$$L_{sj}(S) \mid D_n \quad \Leftrightarrow \quad L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S)\ .$$

It remains to prove
decomposition and separability transfer!

3. For each $S \in Perf$, compute VASS U_{sj} and U_{dy} with $L_{\mathbb{Z}}(U_{sd}) = L_{\mathbb{Z},sd}(S)$.

4. Check $L_{\mathbb{Z}}(U_{sj}) \mid L_{\mathbb{Z}}(U_{dy})$ using [ICALP'17].

5. If all $S \in Perf$ pass the check, then return true, else return false.

6. Separability Transfer

Proposition: Let S be perfect. Then

$$L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S) \quad \Leftrightarrow \quad L_{sj}(S) \mid D_n \, .$$

6.1 Separability

Lemma: Let S be faithful. Then

$$L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S) \quad \Rightarrow \quad L_{sj}(S) \mid D_n \,.$$

Separability

Language intersection.

Every transition as an a and $(a, \sharp)$ variant.

Approach:

Reuse a separator for the $\mathbb{Z}$-languages:

$$B^\sharp : L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S) \quad \overset{!}{\Rightarrow} \quad B^\sharp \times A^\sharp : L_{\mathbb{Z},sj}(S) \mid D_n^\sharp \quad \Rightarrow \quad B \times A : L_{sj}(S) \mid D_n \; .$$

Note:

Every $\mathbb{Z}$-separator can be turned into an $\mathbb{N}$-separator.

$A^\sharp$ only depends on S, but is independent of $B^\sharp$.

Separability

Lemma:

Let S be faithful.

We can construct an NFA $A^\sharp$ so that for all $B^\sharp$.

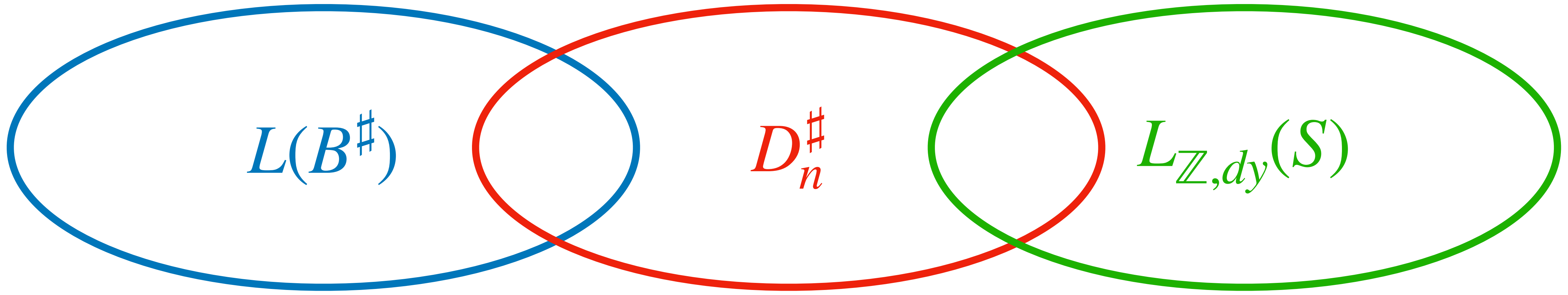
$$B^\sharp : L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S) \quad \Rightarrow \quad B^\sharp \times A^\sharp : L_{\mathbb{Z},sj}(S) \mid D_n^\sharp \ .$$

Task:

Restrict $B^\sharp$ to make it disjoint from $D_n^\sharp$.

Separability: Disjointness

Situation:



Observation: $B^\sharp$ is disjoint from $L_{\mathbb{Z},dy}(S)$.

Lemma: Let $L(B^\sharp) \cap L_{\mathbb{Z},dy}(S) = \varnothing$. Then

Easier to prove!

$$L(B^\sharp \times A^\sharp) \cap D_n^\sharp = \varnothing \quad \Leftrightarrow \quad L(B^\sharp \times A^\sharp) \cap D_n^\sharp \subseteq L_{\mathbb{Z},dy}(S) \; .$$

Proof: $\Rightarrow$ ✓

$$\Leftarrow \quad L(B^\sharp \times A^\sharp) \cap D_n^\sharp \overset{\text{assumption}}{\subseteq} L(B^\sharp) \cap L_{\mathbb{Z},dy}(S) \overset{\text{premise}}{=} \varnothing \; .$$

$\square$

Separability: Disjointness

1. Failure of $L(B^\sharp) \cap D_n^\sharp \subseteq L_{\mathbb{Z},dy}(S)$:

$B^\sharp$ may not follow the control flow of S .

1. Definition:

$$A_S^\sharp := NFA(S).$$

1. Check of $L(B^\sharp \times A_S^\sharp) \cap D_n^\sharp \subseteq L_{\mathbb{Z},dy}(S)$:

Consider $w \in L(B^\sharp \times A_S^\sharp) \cap D_n^\sharp$.

Then w labels a run ρ through S.

As $w \in D_n^\sharp$ and S is visible, ρ takes the Dyck counters in S from 0 to 0 .

Hence,

$$\rho \in Acc_{\mathbb{Z},dy}(S) \ .$$

Separability: Disjointness

2. Failure of $L(B^\sharp \times A^\sharp_S) \cap D^\sharp_n \subseteq L_{\mathbb{Z},dy}(S)$:

$L_{\mathbb{Z},dy}(S)$ is not defined via $Acc_{\mathbb{Z},dy}(S)$ but via $IAcc_{\mathbb{Z},dy}(S)$.
The run may not reach intermediate values.

2. Solution: Faithfulness

$$Acc_{\mathbb{Z},dy}(S) \cap IAcc_{\mathbb{Z},\sqsubseteq^{\mu}_{\omega}[dy]}(S) \subseteq IAcc_{\mathbb{Z},dy}(S) .$$

Track the control flow as before.
Track the dy counters modulo μ.
Check the dy counters when entering and exiting precovering graphs.

Definition of $A^\sharp$.

Separability: Disjointness

Proof of $L(B^\sharp \times A^\sharp) \cap D_n^\sharp \subseteq L_{\mathbb{Z},dy}(S)$:

Consider $w \in L(B^\sharp \times A^\sharp) \cap D_n^\sharp$.

Then w labels a run ρ through S.

As before, we have $\rho \in Acc_{\mathbb{Z},dy}(S)$.

But additionally, we now get $\rho \in IAcc_{\mathbb{Z},\sqsubseteq^{\mu}_{\omega}[dy]}(S)$.

Faithfulness yields

$$\rho \in IAcc_{\mathbb{Z},dy}(S) \ .$$

$\square$

Trick 7 in Action:
Faithfulness gives us disjointness from $D_n^\sharp$.

Separability: Inclusion

Problem: $L_{\mathbb{Z},sj}(S) \subseteq L(B^\sharp \times A^\sharp)$?

Yes!

$L_{\mathbb{Z},sj}(S) \subseteq L(B^\sharp)$ by assumption.

For $L_{\mathbb{Z},sj}(S) \subseteq L(A^\sharp)$, note that

$$IAcc_{\mathbb{Z},sj}(S) \quad = \quad IAcc_{\mathbb{Z},\sqsubseteq_\omega[sj]}(S) \ \boxed{\cap \quad IAcc_{\mathbb{Z},\sqsubseteq^\mu_\omega[dy]}(S)\ .}$$

The latter intersection guarantees the inclusion!

Trick 3 in Action:
The intersection in the definition of $IAcc_{\mathbb{Z},sj}(S)$
is what allows us to restrict the $\mathbb{Z}$-separator!

The $\sharp$ is not needed for this direction
of separability transfer!

6.2 Intermezzo: Büchi Boxes

Intermezzo: Büchi Boxes

Goal: Understand what a separator can distinguish [Büchi'62].

Definition:
An NFA A over Σ induces an equivalence on Σ^* by

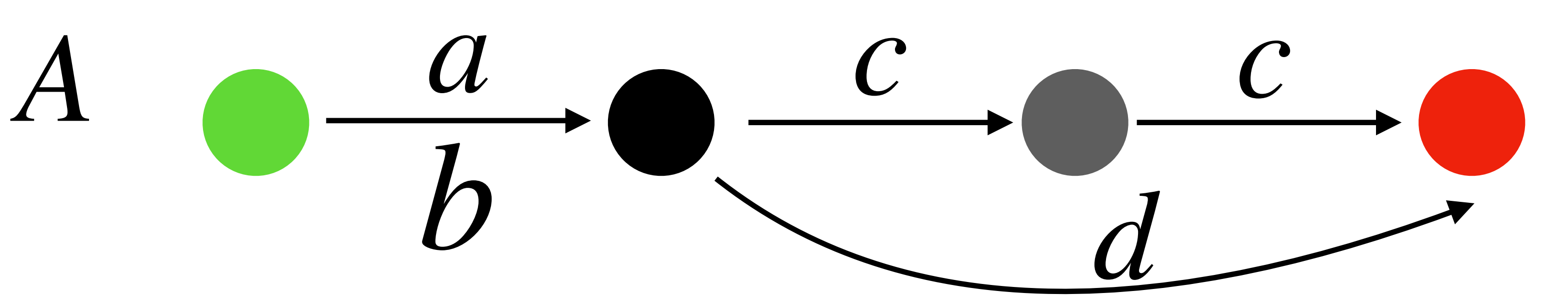
$$u \sim_A v, \text{ if } \quad \forall p, q \in A.Q. \quad p \xrightarrow{u} q \quad \Leftrightarrow \quad p \xrightarrow{v} q.$$

Intuition:
Words are equivalent, if they induce the same state changes.
Equivalence classes therefore correspond to relations on states.

□

Intermezzo: Büchi Boxes

Example:

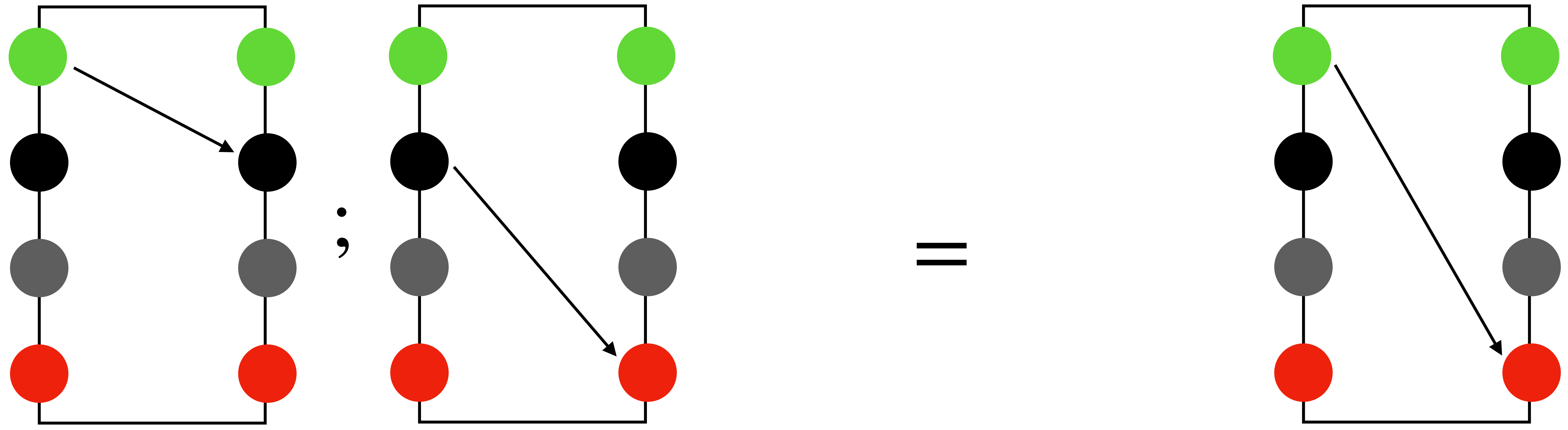


$a \sim_A b$ $\quad$ $a \nsim_A v \quad v \neq b$

$c \,.\, c \sim_A d$ $\quad$ $c \,.\, c \,.\, c \sim_A a \,.\, a$

Classes = relations on states:

$$[a]_{\sim_A} \,.\, [c \,.\, c]_{\sim_A} = \{a, b\} \,.\, \{c \,.\, c, d\} \quad = \quad \{a \,.\, c \,.\, c, a \,.\, d, b \,.\, c \,.\, c, b \,.\, d\} = [a \,.\, c \,.\, c]_{\sim_A}$$



$Box(a)$ $\quad$ $Box(c \,.\, c)$ $\quad$ $Box(a \,.\, c \,.\, c)$

Intermezzo: Büchi Boxes

Lemma (Büchi):

1. $\sim_A$ is a congruence wrt. concatenation:

$$\forall u_1, u_2, v_1, v_2 \,.\quad u_1 \sim_A u_2 \;\wedge\; v_1 \sim_A v_2 \quad \Rightarrow \quad u_1 . v_1 \sim_A u_2 . v_2 \,.$$

2. $\sim_A$ has finite index.

3. $\forall c \in \Sigma^*/_{\sim_A} . \quad c \subseteq L(A) \quad \vee \quad c \cap L(A) = \varnothing \,.$

4. $\forall c \in \Sigma^*/_{\sim_A} . \quad c$ is a regular language.

Proof:

1. routine, 2. count the boxes, 3. by definition, 4.

$$[u]_{\sim_A} \quad = \quad \bigcap_{\substack{p,q \in A.Q \\ p \xrightarrow{u} q}} L(A_{p,q}) \cap \bigcap_{\substack{p,q \in A.Q \\ p \not\xrightarrow{u} q}} \overline{L(A_{p,q})} \,.$$

□

6.3 Inseparability

Lemma: Let S be perfect. Then

$$L_{\mathbb{Z},sj}(S) \nmid L_{\mathbb{Z},dy}(S) \quad \Rightarrow \quad L_{sj}(S) \nmid D_n \ .$$

Inseparability

Strategy:

Towards a contradiction, assume $A : L_{sj}(S) \mid D_n$.

We construct words

$$o_{sj} \in L_{sj}(S) \quad \text{and} \quad o_{dy} \in L_{dy}(S) \subseteq D_n \quad \text{with} \quad o_{sj} \sim_A o_{dy} .$$

Contradiction:

$$o_{sj} \in L(A) \quad \overset{\text{Büchi 3.}}{\Rightarrow} \quad o_{dy} \in L(A) \quad \Rightarrow \quad L(A) \cap D_n \neq \emptyset . \quad ↯$$

$$o_{sj} \notin L(A) \quad \Rightarrow \quad L_{sj}(S) \nsubseteq L(A) .$$

□

Inseparability

Trick 8 in Action:
The pumping sequences u_i and v_i
are shared between $L_{sj}(S)$ and $L_{dy}(S)$.

Construction:
Use Lambert's iteration lemma twice:

$$o_{sj} = \lambda(u_0^c . g_0^c . w_{sj,0}^c . v_0^c . t_1 \ldots t_k . u_k^c . g_k^c . w_{sj,k}^c . v_k^c) \in L_{sj}(S)$$
$$o_{dy} = \lambda(u_0^c . h_0^c . w_{dy,0}^c . v_0^c . t_1 \ldots t_k . u_k^c . h_k^c . w_{dy,k}^c . v_k^c) \in L_{dy}(S) \; .$$

Note: We can assume a common pumping constant c .

Strategy (cont.):
For $o_{sj} \sim_A o_{dy}$, using Büchi 1. we need

$$\forall 0 \leq i \leq k \, . \quad \lambda(g_i) \sim_A \lambda(h_i) \quad \wedge \quad \lambda(w_{sj,i}) \sim_A \lambda(w_{dy,i}) \; .$$

Inseparability: $\lambda(g_i) \sim_A \lambda(h_i)$

Construction:

$$o_{sj} \quad = \quad \lambda(u_0^c \,.\, {g_0}^c \,.\, w_{sj,0}^c \,.\, v_0^c \,.\, t_1 \ldots t_k \,.\, u_k^c \,.\, {g_k}^c \,.\, w_{sj,k}^c \,.\, v_k^c)$$

$$\qquad\qquad\qquad \wr_A \qquad\qquad\qquad\qquad\qquad \wr_A$$

$$o_{dy} \quad = \quad \lambda(u_0^c \,.\, {h_0}^c \,.\, w_{dy,0}^c \,.\, v_0^c \,.\, t_1 \ldots t_k \,.\, u_k^c \,.\, {h_k}^c \,.\, w_{dy,k}^c \,.\, v_k^c)$$

When solving reachability, $g_0 \ldots g_k$ resp. $h_0 \ldots h_k$ can be arbitrary $\mathbb{Z}$-runs.

We need $\lambda(g_i) \sim_A \lambda(h_i)$.

The premise $L_{\mathbb{Z},sj}(S) \nmid L_{\mathbb{Z},dy}(S)$ provides equivalent $\mathbb{Z}$-runs.

Inseparability: $\lambda(g_i) \sim_A \lambda(h_i)$

Goal: Use the premise $L_{\mathbb{Z},sj}(S) \nmid L_{\mathbb{Z},dy}(S)$ to obtain equivalent $\mathbb{Z}$-runs.

Idea: Understand how $\sim_A$ yields separability, then use contraposition.

Lemma:
Let A be an NFA so that

for all pairs of words

$$w_0 \,.\, (a_1, \sharp)\ldots(a_k, \sharp) \,.\, w_k \in L_{\mathbb{Z},sj}(S)$$
$$v_0 \,.\, (a_1, \sharp)\ldots(a_k, \sharp) \,.\, v_k \in L_{\mathbb{Z},dy}(S)$$

there is $0 \leq i \leq k$ with $w_i \nsim_A v_i$.

Then $L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S)$.

Inseparability: $\lambda(g_i) \sim_A \lambda(h_i)$

Lemma: Let A be an NFA so that

for all pairs of words … there is $w_i \nsim_A v_i$.

Then $L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S)$.

Construction of g_i and h_i:
Apply the lemma in contraposition to the premise $L_{\mathbb{Z},sj}(S) \nmid L_{\mathbb{Z},dy}(S)$.

This yields a pair of words as in the lemma with $w_i \sim_A v_i$ for all i .

Then the g_i and h_i are loops in the PGs of S with

$$\lambda(g_i) = w_i \qquad \lambda(h_i) = v_i \quad \text{for all } i .$$

Inseparability: $\lambda(g_i) \sim_A \lambda(h_i)$

Lemma: Let A be an NFA so that

for all pairs of words … there is $w_i \nsim_A v_i$.

Then $L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S)$.

Proof: Define

$$L \quad := \bigcup_{w_0.(a_1,\sharp)\ldots(a_k,\sharp).w_k \in L_{\mathbb{Z},sj}(S)} [w_0]_{\sim_A} . (a_1,\sharp)\ldots(a_k,\sharp) . [w_k]_{\sim_A} .$$

L is regular:

The union is finite as $\sim_A$ has finite index by Büchi 2.
The classes are regular by Büchi 4.

L is a separator:

$L_{\mathbb{Z},sj}(S) \subseteq L$ by definition.

Assume $L \cap L_{\mathbb{Z},dy}(S) \neq \emptyset$.
Then there is $v_0 . (a_1,\sharp)\ldots(a_k,\sharp) . v_k \in L_{\mathbb{Z},dy}(S)$
for which there is $w_0 . (a_1,\sharp)\ldots(a_k,\sharp) . w_k \in L_{\mathbb{Z},sj}(S)$

with $w_i \sim_A v_i$ for all i . ↯

Trick 6 in Action:
The $\sharp$ is essential here.
To conclude $w_i \sim_A v_i$ for all i , we use that
$\sim_A$ only relates words without $\sharp$.

□

Inseparability: $\lambda(g_i) \sim_A \lambda(h_i)$

Construction:

$$o_{sj} = \lambda(u_0^c \,.\, g_0{}^c \,.\, w_{sj,0}^c \,.\, v_0^c \,.\, t_1 \ldots t_k \,.\, u_k^c \,.\, g_k{}^c \,.\, w_{sj,k}^c \,.\, v_k^c)$$

$$\wr_A \qquad\qquad \wr_A$$

$$o_{dy} = \lambda(u_0^c \,.\, h_0{}^c \,.\, w_{dy,0}^c \,.\, v_0^c \,.\, t_1 \ldots t_k \,.\, u_k^c \,.\, h_k{}^c \,.\, w_{dy,k}^c \,.\, v_k^c)$$

Inseparability: $\lambda(w_{sj,i}) \sim_A \lambda(w_{dy,i})$

Construction:

$$o_{sj} \quad = \quad \lambda(u_0^c \,.\, g_0{}^c \,.\, w_{sj,0}{}^c \,.\, v_0^c \,.\, t_1 \ldots t_k \,.\, u_k^c \,.\, g_k{}^c \,.\, w_{sj,k}{}^c \,.\, v_k^c)$$

$$\sim_A \qquad\qquad \sim_A$$

$$o_{dy} \quad = \quad \lambda(u_0^c \,.\, g_0{}^c \,.\, w_{dy,0}{}^c \,.\, v_0^c \,.\, t_1 \ldots t_k \,.\, u_k^c \,.\, g_k{}^c \,.\, w_{dy,k}{}^c \,.\, v_k^c)$$

Actually: We will also modify the support solutions and covering sequences.

Inseparability: $\lambda(w_{sj,i}) \sim_A \lambda(w_{dy,i})$

Goal: Construct support solutions s_{sj} and s_{dy} and for all $0 \leq i \leq k$

$$u_i \in Up(G_i) \qquad v_i \in Down(G_i) \qquad w_{sj,i} \qquad w_{dy,i}$$

with $\lambda(w_{sj,i}) \sim_A \lambda(w_{dy,i})$ so that

$$\psi(u_i) + \psi(w_{sj,i}) + \psi(v_i) = s_{sj}[G_i . E]$$
$$\psi(u_i) + \psi(w_{dy,i}) + \psi(v_i) = s_{dy}[G_i . E] \ .$$

(Matching)

Need matching to invoke Lambert's iteration lemma.

Inseparability: $\lambda(w_{sj,i}) \sim_A \lambda(w_{dy,i})$

Notation:

Fix an index $0 \leq i \leq k$ and call the

$$u_i \in Up(G_i) \qquad v_i \in Down(G_i) \qquad w_{sj,i} \qquad w_{dy,i}$$

we want to construct u, v, w_{sj}, and w_{dy} .

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Idea:

For the construction of w_{sj} and w_{dy}, use pumping.

Construction:

Assume A has n states.
We define

$$\begin{aligned} w_{sj} &:= \mathit{diff}^{n} \,.\, \mathit{rem} \\ w_{dy} &:= \mathit{diff}^{n+c\cdot n!} \,.\, \mathit{rem} \;. \end{aligned}$$

The runs diff and rem and the constant c will be fixed when we analyze (Matching).

No matter how, $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$ will hold.

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Lemma:

Let A be a DFA over Σ with n states and let $c \in \mathbb{N}$.
Then for all $u, v \in \Sigma^*$, we have

$$u^n . v \sim_A u^{n+c \cdot n!} . v \ .$$

Proof:

Consider states p and q in A.
To show

$$p \xrightarrow{u^n . v} q \quad \Leftrightarrow \quad p \xrightarrow{u^{n+c \cdot n!} . v} q$$

it suffices to show that A reaches the same state when reading u^n and $u^{n+c \cdot n!}$ from p .

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Lemma:
Let A be a DFA over Σ with n states and let $c \in \mathbb{N}$.
Then for all $u, v \in \Sigma^*$, we have

$$u^n . v \sim_A u^{n+c\cdot n!} . v \ .$$

Proof:
We show that A reaches the same state
when reading u^n and $u^{n+c\cdot n!}$ from p .

Let q_i be the state in A reached after reading u^i from p, where $u^0 := \varepsilon$.
By the pigeonhole principle, there are

$$0 \leq i < j \leq n \quad \text{with} \quad q_i = q_j \ .$$

As A is a DFA, u^n and $u^j . u^{j-i} . u^{n-j} = u^{n+(j-i)}$ both end up in q_n .
We not only repeat u^{j-i} once, but

$$\frac{c \cdot n!}{j-i} \text{ many times.}$$

Thanks to the factorial and $c \in \mathbb{N}$, this is a positive integer.
This means also $u^{n+c\cdot n!}$ ends up in q_n .

$\square$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Want: $u \in Up(G)$, $v \in Down(G)$, $diff$, and rem, and support solutions s_{sj} and s_{dy} that match.

Have: By perfectness, support solutions s'_{sj} and s'_{dy} and for all $0 \leq i \leq k$.

$$u'_i \in Up(G_i) \qquad v'_i \in Down(G_i)$$

so that

$$s'_{sd}[G_i\,.\,E] - \psi(u'_i) - \psi(v'_i) \geq 1 \ .$$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Needed:

$$\psi(u) + \psi(w_{sj}) + \psi(v) = s_{sj}[E]$$
$$\psi(u) + \psi(w_{dy}) + \psi(v) = s_{dy}[E]\ .$$

(Matching)

Recall: $w_{sj} = \mathit{diff}^n\,.\,\mathit{rem}$ and $w_{dy} = \mathit{diff}^{n+c\cdot n!}\,.\,\mathit{rem}$.

Consequence: Need

$$\psi(u) + n \cdot \psi(\mathit{diff}) + \psi(\mathit{rem}) + \psi(v) = s_{sj}[E]$$
$$\psi(u) + (n + c \cdot n!) \cdot \psi(\mathit{diff}) + \psi(\mathit{rem}) + \psi(v) = s_{dy}[E]\ .$$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Consequence: Need

$$\cancel{\psi(u)} + \cancel{n} \cdot \cancel{\psi(diff)} + \cancel{\psi(rem)} + \cancel{\psi(v)} = s_{sj}[E]$$
$$\cancel{\psi(u)} + (\cancel{n} + c \cdot n!) \cdot \psi(diff) + \cancel{\psi(rem)} + \cancel{\psi(v)} = s_{dy}[E] \ .$$

Consequence: We subtract the equations to isolate $\psi(diff)$:

$$c \cdot n! \cdot \psi(diff) = s_{dy}[E] - s_{sj}[E] = (s_{dy} - s_{sj})[E] \ .$$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Consequence: We subtract the equations to isolate $\psi(diff)$ and get

$$c \cdot n! \cdot \psi(diff) = (s_{dy} - s_{sj})[E] \ .$$

Define:

$$\boxed{s_{sd} \ := \ c \cdot n! \cdot s'_{sd} \, .}$$

Consequence: We can factor out $c \cdot n!$ and get rid of it,

$$\cancel{c \cdot n!} \cdot \psi(diff) = \cancel{c \cdot n!} \cdot (s'_{dy} - s'_{sj})[E] \ .$$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Definition: To obtain $\psi(\mathit{diff}) = (s'_{dy} - s'_{sj})[E]$, we set

$$\boxed{\mathit{diff} \ := \ \langle (s'_{dy} - s'_{sj})[E] \rangle \ .}$$

Remark:

To invoke Euler-Kirchhoff, we need $(s'_{dy} - s'_{sj})[E] \geq 1$.

We can assume s'_{dy} has been scaled to guarantee this.

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Recall: We need matching

$$\psi(u) + \psi(w_{sj}) + \psi(v) = s_{sj}[E] \ .$$

Consequence: Inserting the choice of s_{sj} yields

$$\psi(u) + n \cdot \psi(diff) + \psi(rem) + \psi(v) = c \cdot n! \cdot s'_{sj}[E] \ .$$

Consequence:

$$\psi(rem) = c \cdot n! \cdot s'_{sj}[E] - \psi(u) - \psi(v) - n \cdot \psi(diff) \ .$$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Consequence:

$$\psi(rem) = c \cdot n! \cdot s'_{sj}[E] - \psi(u) - \psi(v) - n \cdot \psi(diff)\ .$$

Idea: To apply Euler-Kirchhoff, the right-hand side has to be ≥ 1 .

Define:

$$u := (u')^{c \cdot n!} \qquad v := (v')^{c \cdot n!}\ .$$

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Consequence:

$$
\begin{aligned}
\psi(rem) &= c \cdot n! \cdot s'_{sj}[E] - \psi(u) - \psi(v) - n \cdot \psi(diff) \\
&= c \cdot n! \cdot s'_{sj}[E] - c \cdot n! \cdot \psi(u') - c \cdot n! \cdot \psi(v') - n \cdot \psi(diff) \\
&= c \cdot n! \cdot \underbrace{(s'_{sj}[E] - \psi(u') - \psi(v'))}_{\geq 1} - n \cdot \psi(diff) \ .
\end{aligned}
$$

Definition:

$c :=$ least value so that $\psi(rem) \geq 1$.

Defininition:

$rem := \langle c \cdot n! \cdot (s'_{sj}[E] - \psi(u') - \psi(v')) - n \cdot \psi(diff) \rangle$.

Inseparability: $\lambda(w_{sj}) \sim_A \lambda(w_{dy})$

Remark:

The choice of c is not local to G

but global in that it has to hold for all PGs in S .

7. Decomposition

Proposition: Given a faithful DMGTS W, we can compute finite sets $Perf$ and Fin of DMGTS so that

(i) $\forall S \in Perf$. S is perfect .

(ii) $\forall T \in Fin$. $L_{sj}(T) \mid D_n$.

(iii) $L_{sj}(W) = L_{sj}(Perf) \cup L_{sj}(Fin)$.

Decomposition

Faithfulness is an invariant!

Approach:
Capture a single decomposition step.
Rely on well-foundedness.

Lemma (Step):
There is a computable function $dec(-)$ that takes a DMGTS W as follows

faithful, imperfect, $sol(Char_{sj}(W)) \neq \emptyset \neq sol(Char_{dy}(W))$.

It returns finite sets $(X, Y) = dec(W)$ of DMGTS with

(a) $\forall S \in X .\ S$ is faithful and $S < W$.
(b) $\forall T \in Y .\ L_{sj}(T) \mid D_n$.
(c) $L_{sj}(W) = L_{sj}(X) \cup L_{sj}(Y)$.

If not perfect, you can decompose.

(b) and (c) as required by decomposition.

Decomposition

algo(input: a faithful DMGTS W, output: $Perf$ and Fin)
    if W is perfect then
        return $Perf = \{W\}$, $Fin = \varnothing$;
    else if $sol(Char_{sj}(W)) = \varnothing$ then
        return $Perf = \varnothing$, $Fin = \varnothing$;
    else if $sol(Char_{dy}(W)) = \varnothing$ then
        return $Perf = \varnothing$, $Fin = \{W\}$;
    else
        $(X, Y) = dec(W)$;
        $Perf = \varnothing$; $Fin = Y$;
        for all $S \in X$ begin
            $(Perf_S, Fin_S)$ = algo(S);
            $Perf = Perf \cup Perf_S$;
            $Fin = Fin \cup Fin_S$;
        end for all
    end else
end

(i), (ii), (iii) trivial

$$sol(Char_{sj}(W)) = \varnothing \quad \Rightarrow \quad L_{\mathbb{Z},sj}(W) = \varnothing$$
$$\Rightarrow \quad L_{sj}(W) = \varnothing \, .$$

Goal:

(i) $\forall S \in Perf \, . \, S$ is perfect .

(ii) $\forall T \in Fin \, . \, L_{sj}(T) \mid D_n \, .$

(iii) $L_{sj}(W) = L_{sj}(Perf) \cup L_{sj}(Fin) \, .$

$$sol(Char_{dy}(W)) = \varnothing \quad \Rightarrow \quad L_{\mathbb{Z},dy}(W) = \varnothing$$
$$\Rightarrow \quad L_{\mathbb{Z},sj}(W) \mid L_{\mathbb{Z},dy}(W)$$
{Separability Transfer} $\Rightarrow$ $L_{sj}(W) \mid D_n$.

Decomposition: Step Lemma

Fact: Let W be faithful.

W is not perfect $\Leftrightarrow$ $\exists G \in W\,.\,(1) \vee (2) \vee (3)$ with

(1) $G\,.\,c_{io}[j] = \omega \;\wedge\; G\,.\,c_{io}[j] \notin supp(Char_{sd}(G))$.

(2) $e \in G\,.\,E \;\wedge\; e \notin supp(Char_{sd}(G))$.

(3) $Up(W) = \emptyset \;\vee\; Down(G) = \emptyset$.

Approach: Case distinction.

7.1 Case $j \notin supp(Char_{sd}(W))$

Step Lemma: Case $j \notin supp(Char_{sd}(W))$

Fact: If $j \notin supp(Char_{sd}(W))$,

$$A_{sd} \;:=\; \{s[j] \mid s \in sol(Char_{sd}(W))\}$$

is finite, non-empty, and $\subseteq \mathbb{N}$.

Lemma in the beginning.

$sol(Char_{sd}(W)) \neq \varnothing$.

Shape of $Char_{sd}(W)$.

7.1.1 Case $sd = sj$

Step Lemma: Case $j \notin supp(Char_{sj}(W))$

Let $W = (U, \mu)$.

Define:

$$X := \{(U_a, \mu) \mid a \in A_{sj}\} \qquad Y := \varnothing \ .$$

U with the value of j
at the moment of interest
modified from ω to a .

Step Lemma: Case $j \notin supp(Char_{sj}(W))$

Proof:

(c) $L_{sj}(W) = L_{sj}(X)$.

$\subseteq$ Consider $\rho \in IAcc_{sj}(W)$.
Then ρ solves the characteristic equations.
Hence, counter j assumes a value $a \in A_{sj}$ at the moment of interest.
Hence, $\rho \in IAcc_{sj}(U_a, \mu)$, and $(U_a, \mu) \in X$.

$\supseteq$ Concrete values make intermediate acceptance stronger.

(b) $\forall T \in Y\ldots$ There is nothing to show.

Step Lemma: Case $j \notin supp(Char_{sj}(W))$

Proof (cont.):

(a) Faithfulness.

We neither modified the edges nor the dy markings.
Hence, faithfulness holds by the faithfulness of W.

(a) Descent

$\Omega(G)$, $G.E$, and $G.c_{\overline{io}}$ stay unchanged.
We reduce $|\Omega(G.c_{io})|$.

7.1.2 Case $sd = dy$

This is the complicated case!

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Setting:

We change an extremal marking for a Dyck counter

from ω to a concrete value.

As a consequence, we have to check faithfulness.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Setting: We have to check faithfulness.

Lemma (Modulo Trick):
Consider $0 \leq a, b < \nu$.

$$a \equiv b \mod \nu \quad \Rightarrow \quad a = b \; .$$

Trick 9:
The Modulo Trick is essential for faithfulness.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Discussion:

(i) We will have $b \in A_{dy}$.

Hence, to apply the Modulo Trick, we need to

modify μ to ν with $\nu > \max A_{dy}$.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Discussion:

(ii) We canot simply increase ν to exceed μ .
We need

$$\text{acceptance modulo } \nu \quad \Rightarrow \quad \text{acceptance modulo } \mu \ .$$

This works, if μ divides ν . We thus set

$$\nu \ := \ \mu \cdot l$$

for an l defined later.

Trick 10:
Maintaining divisibility among the μ values.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Discussion:

(iii) If we modify μ to ν, we need to

modify the extremal markings of all PGs.

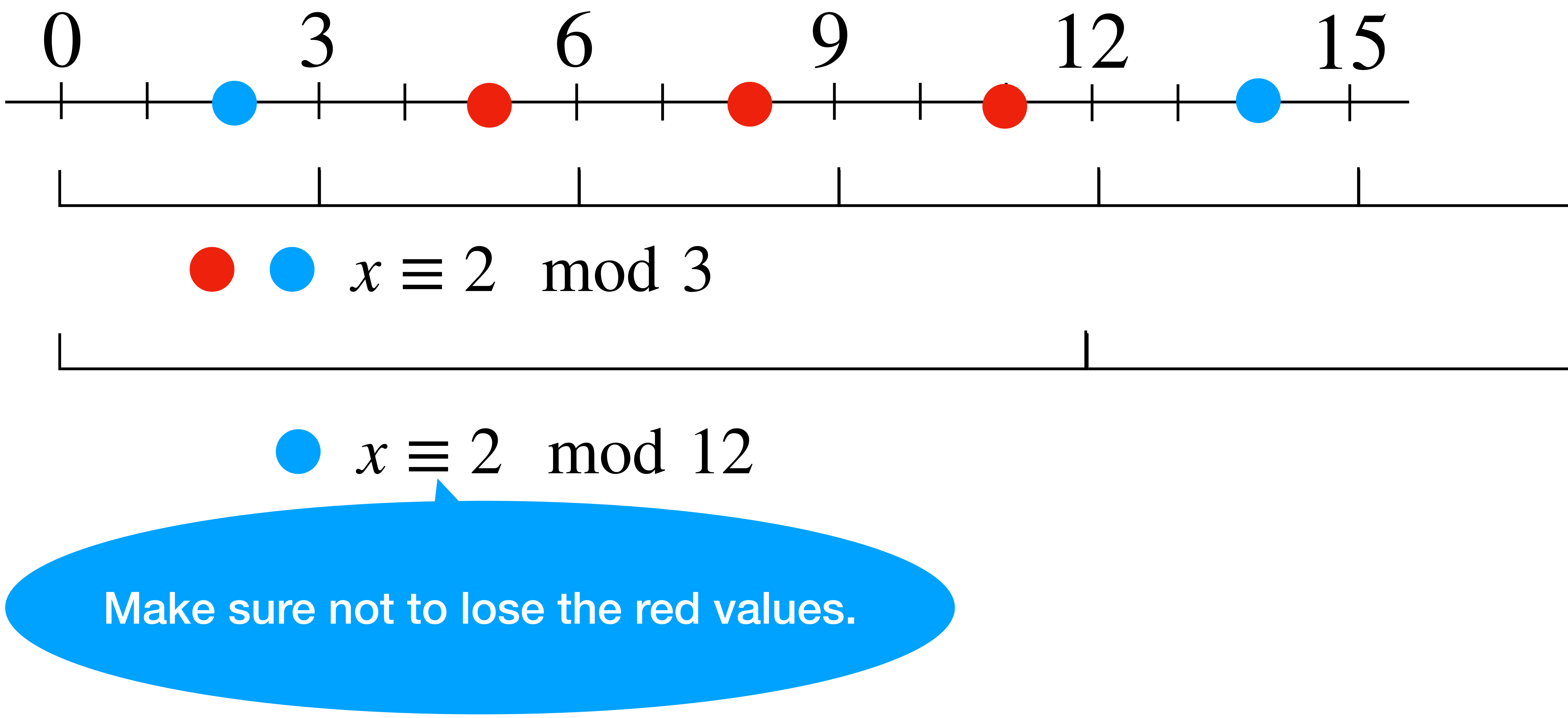
Example:

$$x \equiv 2 \mod 3.$$

Let $l = 4$ and thus $\nu = 3 \cdot 4 = 12$.
Then

$$x \equiv 2 \mod 12$$

does not yield all solutions.



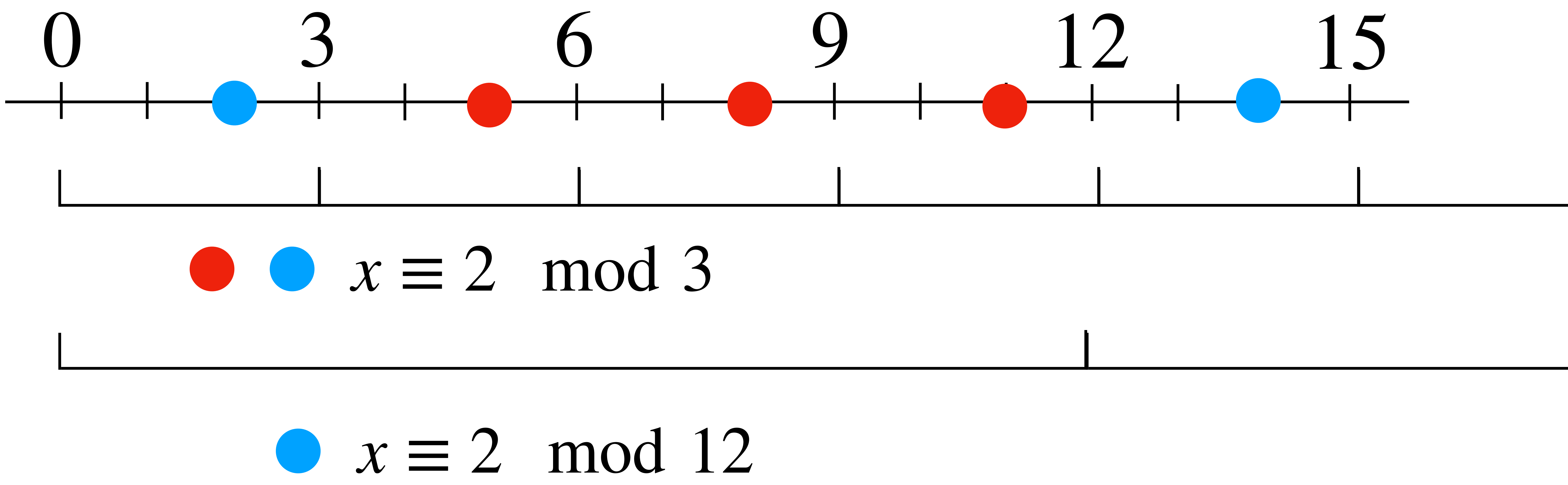
Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Example:

$$x \equiv 2 \mod 3.$$

Then

$$x \equiv 2 \mod 12$$
$$x \equiv 5 \mod 12$$
$$x \equiv 8 \mod 12$$
$$x \equiv 11 \mod 12$$



together yield all solutions.

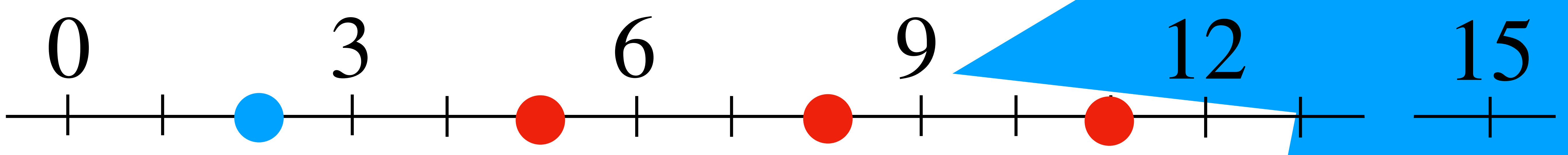
Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Lemma ($\mu - \nu-$Modification): Let μ divide ν and consider $x, k \in \mathbb{Z}$.

$$x \equiv k \mod \mu \quad \Leftrightarrow \quad \exists 0 \leq i < \nu \,. \begin{array}{l} x \equiv i \mod \nu \\ i \equiv k \mod \mu \end{array}$$

Example:

$$x \equiv 2 \mod 3 \quad \Leftrightarrow \quad \exists i \in \{2,$$



Trick 11:
Adapt intermediate markings to $i \equiv k \mod \mu$.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Goal: Transfer the adaptation lemma to DMGTS.

Approach: Equate MGTS up to modulo equivalence

$$k \equiv i \mod \mu$$

on the Dyck counters.

Definition (μ−Modification Equivalence):

$$G_1 \equiv_\mu G_2, \quad \text{if} \quad \begin{array}{ll} G_1 . V = G_2 . V & G_1 . c_{io}[sj] = G_2 . c_{io}[sj] \\ G_1 . E = G_2 . E & G_1 . c_{io}[dy] \equiv G_2 . c_{io}[dy] \mod \mu \end{array}$$

$$S_1 . up . S_2 \equiv_\mu S_1' . up . S_2', \quad \text{if} \quad S_1 \equiv_\mu S_1' \wedge S_2 \equiv_\mu S_2' .$$

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Lemma ($\mu - \nu-$Modification for Intermediate Acceptance):
Assume μ divides ν.

$$IAcc_{sj}(U,\mu) \;=\; \bigcup_{\substack{V \equiv_\mu U \\ 0 \le V < \nu}} IAcc_{sj}(V,\nu)\,.$$

All extremal markings take values from $[0,\nu-1] \cup \{\omega\}$.

Note:
This is a direct lift of the $\mu - \nu-$Modification Lemma.

- $V \equiv_\mu U$ corresponds to $i \equiv k \mod \mu$.
- $0 \le V < \nu$ corresponds to $0 \le i < \nu$.
- The union is the existential quantifier.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Discussion:
(iv) If we modify the extremal markings of all PGs,
we have to check faithfulness also there.

To apply the Modulo Trick,

ν has to be larger than all values in extremal markings.

Recall $\nu := \mu \cdot l$. We thus set

$l := \max A_{dy} \cup$ values in extremal markings.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Remark:
We do not maintain the invariant that
μ is larger than the values in the extremal markings.

This would force us to repeat the argument for Case (1)
in Cases (2) + (3).

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

$\mu - \nu-$Modification .

Not just A_{dy}!

Definition:

$$Z := \{(V,\nu) \mid V \equiv_\mu U_a,\ 0 \leq V < \nu,\ 0 \leq a < \mu,\ V.c_{in}[dy] = 0\}$$

$$X := \{(V,\nu) \in Z \mid V.c_{out}[dy] = 0\}$$

$$Y := Z \backslash X\ .$$

Zero-reaching.

Dyck counters $\not\equiv 0 \mod \nu$.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

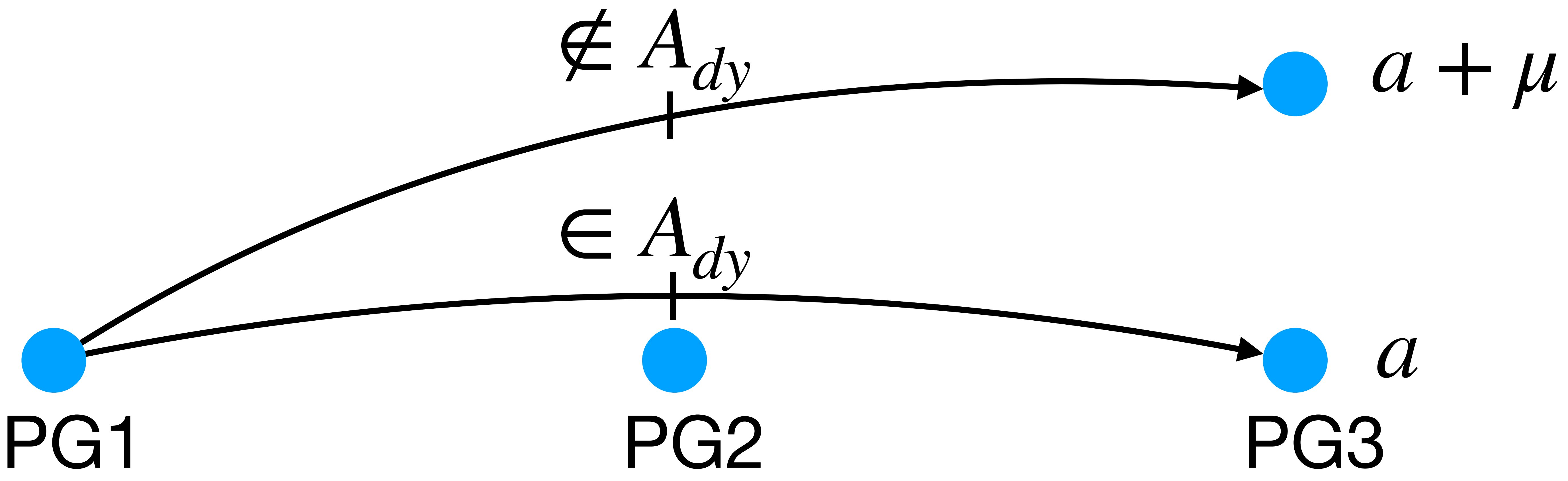
Note:

We cannot just take the values from A_{dy} .

They stem from $Char_{dy}(W)$ which reaches intermediate values precisely.

In $IAcc_{sj}(W)$, we only need to reach intermediate Dyck values modulo μ.

Hence, A_{dy} may not contain enough values.



Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (of the Step Lemma):
Let $W = (U, \mu)$.

(c) $L_{sj}(W) = L_{sj}(X) \cup L_{sj}(Y)$.

Similar to the Case $sd = sj$, we have

$$IAcc_{sj}(U, \mu) = \bigcup_{0 \leq a < \mu} IAcc_{sj}(U_a, \mu) .$$

With the $\mu - \nu -$Modification Lemma for Intermediate Acceptance,

$$IAcc_{sj}(U_a, \mu) = \bigcup_{\substack{V \equiv_\mu U_a \\ 0 \leq V < \nu}} IAcc_{sj}(V, \nu) .$$

We argue that we do not lose words by assuming in V

the initial values for dy zero modulo ν instead of zero modulo μ .

Consider $\rho \in IAcc_{sj}(U_a, \mu)$.
As U_a is zero-reaching, ρ starts from a multiple of μ on dy , say μ for simplicity.
By the monotonicity of modulo acceptance, $\rho + (\nu - \mu) = \rho + (l-1) \cdot \mu \in IAcc_{sj}(U_a, \mu)$.
This run is labeled by the same word and starts from ν on dy .
Hence, it will be accepted by $V \equiv_\mu U_a$ where the Dyck counters are initially 0 .

Trick 5 in Action:
Monotonicity of modulo-μ intermediate acceptance.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (cont.):

(b) $\forall T \in Y \,.\, L_{sj}(T) \mid D_n$.

Consider $T \in Y$.

By construction, $T.c_{in}[dy] = 0$ and $T.c_{out}[dy] \neq 0$.

This means $\rho \in IAcc_{sj}(T)$ has an effect $c \not\equiv 0 \mod \nu$ on dy.

By visibility of T and the VAS accepting D_n, we have $\lambda(\rho) \notin D_n$.

Hence, an NFA that tracks the Dyck counters modulo ν

and accepts upon values $\neq 0$ shows separability.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (cont.):

(a) Descent

As in the case $sd = sj$.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (cont.): Recall that $S = (V, \nu)$ and $W = (U, \mu)$.
Faithfulness

$$Acc_{\mathbb{Z},dy}(S) \;\cap\; IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \quad\subseteq\quad IAcc_{\mathbb{Z},dy}(S)$$

is a consequence of

$$Acc_{\mathbb{Z},dy}(S) \;\cap\; IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \quad\subseteq\quad IAcc_{\mathbb{Z},dy}(W) \qquad (1)$$

$$IAcc_{\mathbb{Z},dy}(W) \;\cap\; IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \quad\subseteq\quad IAcc_{\mathbb{Z},dy}(S)\,. \qquad (2)$$

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (cont.): For

$$Acc_{\mathbb{Z},dy}(S) \;\cap\; IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \quad\subseteq\quad IAcc_{\mathbb{Z},dy}(W) \qquad (1)$$

we use

$$Acc_{\mathbb{Z},dy}(S) \;\subseteq\; Acc_{\mathbb{Z},dy}(W)$$

$$IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \;\subseteq\; IAcc_{\mathbb{Z},\sqsubseteq^{\mu}_{\omega}[dy]}(W)$$

and the faithfulness of W.

S and W are zero-reaching.
We only change an intermediate value,
which acceptance does not see.

μ divides ν.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (cont.): For

$$IAcc_{\mathbb{Z},dy}(W) \;\cap\; IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \quad\subseteq\quad IAcc_{\mathbb{Z},dy}(S)\,.$$

Consider ρ in the intersection.
Consider counter j that we changed from ω to a concrete value.

As $\rho \in IAcc_{\mathbb{Z},dy}(W)$, ρ solves $Char_{dy}(W)$.
Hence, it reaches a value $b \in A_{dy}$ at the moment of interest.

As $\rho \in IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S)$, it also reaches the value a that replaces ω in S, but only modulo ν.

We have $0 \leq a, b$ by the definition of intermediate acceptance.

We have $b < \nu$ by the choice of ν.
We have $a < \nu$ by the construction of S.

Modulo intermediate acceptance means $a \equiv b \mod \nu$.
The Modulo Trick shows a=b.

Step Lemma: Case $j \notin supp(Char_{dy}(W))$

Proof (cont.): For

$$IAcc_{\mathbb{Z},dy}(W) \ \cap \ IAcc_{\mathbb{Z},\sqsubseteq^{\nu}_{\omega}[dy]}(S) \quad \subseteq \quad IAcc_{\mathbb{Z},dy}(S)\ .$$

Consider a counter different from j or j but another moment.

As $\rho \in IAcc_{\mathbb{Z},dy}(W)$, ρ reaches an intermediate value b given in W.

We again have $b < \nu$ by the choice of ν.

Now the same argument applies.

□

7.2 Reasoning Locally about Faithfulness

Reasoning Locally about Faithfulness

Case (1): Modified entire DMGTS.

Cases (2) + (3): Modify a single PG.

Goal: Develop techniques that allow us to

reason about a single PG and
lift the result to the entire DMGTS.

Focus on faithfulness.

Reasoning Locally about Faithfulness

Definition:
MGTS context

$$C[\,\bullet\,] \quad ::= \quad \bullet \mid C[\,\bullet\,]\,.\,up\,.\,W \mid W\,.\,up\,.\,C[\,\bullet\,]\,.$$

DMGTS insertion: For $W = (S, \mu)$ let

Replace • by S.

$$C[W] \;:=\; (C[S], \mu)\,.$$

Lemma: Well-founded order stable under insertion

$$W_1 \leq W_2 \qquad \Rightarrow \qquad C[W_1] \leq C[W_2]\,.$$

Reasoning Locally about Faithfulness

Approach: For Cases (2) + (3), consider $C[(G, \mu)]$,

decompose (G, μ) into sets of DMGTS U and V,
define

$$X := C[U] := \{C[(S, \mu)] \mid (S, \mu) \in U\} \qquad\qquad Y := C[V] .$$

Reasoning Locally about Faithfulness

Goal: Lift faithfulness of $C[(G, \mu)]$ to $C[U]$.

Approach: Establish a relation between (G, μ) and the DMGTS in U.

Same μ.

Definition:

- (S, μ) is a specialization of (G, μ), if

Smaller language.

Preserve faithfulness.

1. $S.c_{io} \sqsubseteq_{\omega} G.c_{io}$.
2. $\forall \rho \in Runs_{\mathbb{Z}}(S).\ \exists \sigma \in Runs_{\mathbb{Z}}(G).\quad \sigma \approx \rho$.
3. $\forall \rho \in IAcc_{\mathbb{Z}, \sqsubseteq^{\mu}_{\omega}[dy]}(S)$ with $\rho[first/last][dy] \sqsubseteq_{\omega} G.c_{io}.\quad \rho \in IAcc_{\mathbb{Z},dy}(S)$.

- If W_1 is a specialization of W_2, then $C[W_1]$ is a specialization of $C[W_2]$.

Reasoning Locally about Faithfulness

Lemma: Let W_1 be a specialization of W_2.

Only need to worry about $L_{sj}(W) \subseteq L_{sj}(X \cup Y)$.

$L_{sj}(W_1) \subseteq L_{sj}(W_2)$.

W_2 faithful $\Rightarrow$ W_1 faithful.

Intuition: Why does decomposition for Cases (2) + (3) guarantee

$\forall \rho \in IAcc_{\mathbb{Z},\sqsubseteq^{\mu}_{\omega}[dy]}(S)$ with $\rho[first/last][dy] \sqsubseteq_{\omega} G\,.\,c_{io}\,.\quad \rho \in IAcc_{\mathbb{Z},dy}(S)$?

Decompositions for (2) + (3) unroll G into DMGTS.

New intermediate counter values = consistent assignments in G or values in coverability graph for G.

Hence, runs in the new DMGTS respects these values.

7.3 Case $e \notin supp(Char_{sd}(W))$

Case (2): $e \notin supp(Char_{sd}(W))$

Observation: If e is not in the support, there is

an upper bound $l \in \mathbb{N}$

on the number of times e can be taken.

Idea: Decompose G so that every occurrence of e leads to a new PG.

Definition:
$U =$ DMGTS that admit at most l occurrences of e.
$V_{sj} = \emptyset$.
V_{dy} = DMGTS that expect $l + 1$ occurrences of e,
afterwards return to the root of G.

Case (2): $e \notin supp(Char_{sd}(W))$

Faithfulness already done!

Lemma: Let (G, μ) contain edge e with $e \notin supp(Char_{sd}(C[(G, \mu)]))$.
With elementary resources, we can compute sets U and V
containing specializations of (G, μ) that satisfy:

Descent also done!

$$\forall S \in U\,.\, S < (G, \mu)\,.$$
$$\forall \rho \in IAcc_{sj}(G, \mu)\,.\, \exists \sigma \in IAcc_{sj}(U \cup V)\,. \quad \sigma \approx \rho\,.$$
$\forall T \in V\,. \quad Char(C[T])$ is infeasible.

Separability also done!