	Theoretical Computer Science	
Prof. Dr. Roland Meyer	**Exercise Sheet 2**	TU Braunschweig
René Maseli		Winter Semester 2025/26

Release: 2025-11-10 Due: 2025-11-20 23:59

Hand in your solutions to the Vips directory of the StudIP course until Thursday, November 20th 2025 23:59. You should provide your solutions either directly as .pdf file or as a readable scan/photo of your handwritten notes. State **all** members of your group with **student id, name and course**.

Homework Exercise 2.1: Join-Meet-Continuity [4 points]

Prove or disprove, if the following statements hold.

a) [1 point] Let M be a finite set, $R \subseteq M \times M$ a binary relation over M. Consider the complete lattice $\langle \mathcal{P}(M), \subseteq \rangle$. Show that the function post : $\mathcal{P}(M) \to \mathcal{P}(M)$ is $\sqcup$-continuous.

$$\text{post}(X) := \{\, y \in M \mid \forall x \in M : \langle x, y \rangle \in R \Rightarrow x \in X \,\}$$

b) [2 points] Let $\langle B, E, F \rangle$ be a control flow graph. Consider the lattice $\langle \mathcal{P}(\text{Vars})^B, \subseteq^B \rangle$, where all elements $f, g : B \to \mathcal{P}(\text{Vars})$ are ordered by $F \subseteq^B G$ iff $\forall b \in B : f(b) \subseteq g(b)$. A block is said to *use* a variable, if the variable appears inside any expression contained in the block (except the assigned variable of an assignment). Show that the function $G : \mathcal{P}(\text{Vars})^B \to \mathcal{P}(\text{Vars})^B$ is $\sqcap$-continuous.

$$G(X)(b) = \left\{ x \in \text{Vars} \;\middle|\; \exists b' \in B : b \; F \; b' \text{ and } (b' \text{ uses } x \text{ or } (b' \neq [x := e]^\ell \text{ and } x \in X_{b'})) \right\}$$

c) [1 point] Betrachten Sie die totale Ordnung $\omega + 2 := \langle D, \leq_{\omega+2} \rangle$ mit $D := \mathbb{N} \cup \{\omega, \omega{+}1\}$ and

$$\leq_{\omega+2} := \leq \cup \{\, \langle n, \omega \rangle \mid n \in \mathbb{N} \cup \{\omega\} \,\} \cup \{\, \langle n, \omega{+}1 \rangle \mid n \in D \,\} .$$

Zeigen Sie, dass die Funktion succ : $D \to D$ with $\text{succ}(n) := \begin{cases} \omega{+}1 & \text{falls } n = \omega{+}1 \\ n+1 & \text{sonst} \end{cases}$ $\sqcap$-stetig ist.

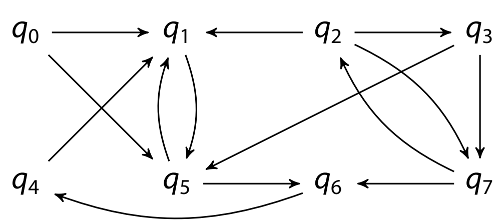
Homework Exercise 2.2: Graph Unreachability [2 points]

Let $\langle V, E \rangle$ be a graph, i.e. V finite and $E \subseteq V \times V$. Consider the function $f : \mathcal{P}(V) \to \mathcal{P}(V)$ with

$$f(X) = \{\, v \in V \mid v \neq q_0 \text{ and } (\forall x \in V \setminus X : \langle x, v \rangle \notin E) \,\} .$$

a) [1 point] Show that f is monotone and $\sqcap$-continuous in $\langle \mathcal{P}(V), \subseteq \rangle$.

b) [1 point] Apply the construction from a), to find all vertices in the following graph $G = \langle V, E \rangle$, that are **not** reachable from the start node q_0. Give all elements, up to the point, where the sequence stabilizes. Compute gfp(f) using the sequence in Kleene's fixed point theorem.

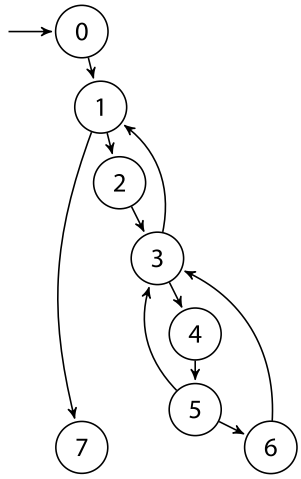


Homework Exercise 2.3: Reaching Definitions [5 points]

Given the following program with variables Vars $:= \{x, y, z\}$ and data flow graph $G = \langle B, E, F\rangle$ as given beneath. The goal is to compute for each block b, the set of blocks, that may be the latest writer of some variable, before the control flow reaches b.

```
if [¬(x = y)]^0 then
    [z := 3 · x · x + 2]^1
    if [x < y]^2 then
        [y := y − z]^3
    else
        [x := x − z]^4
    end if
else
    while [y · x > 64]^5 do
        [y := y − 1]^6
    end while
end if
[x := x + y]^7
```

```
[x := 0]^0
while [x < 2^4]^1 do
    [y := 3x + 2]^2
    while [y < 5x]^3 do
        [y := y + 2]^4
        if [3x < y]^5 then
            [x := x + 1]^6
        end if
    end while
end while
[x := x − 14]^7
```



a) [1 point] Consider the program on the left side. Draw the control flow graph $G = \langle B, E, F\rangle$. Mark its extremal blocks.

b) [1 point] Now consider the program on the right side, as well as the lattice $\langle D, \subseteq\rangle$ over the domain $D = \mathcal{P}(\text{Vars}\times(B+\{?\}))$. Assign for each block $b \in B$ elements $\text{kill}(b), \text{gen}(b) \in D$ such, that the transfer functions $f_b : D \to D$ become suitable for the analysis: $f_b(X) = (X \setminus \text{kill}(b)) \cup \text{gen}(b)$.

c) [3 points] Solve the induced equation system from b) by determining its least solution using Kleene's fixed point theorem.

Homework Exercise 2.4: Live Variables [5 points]

Consider the following program. Map each block in the program to the set of variables that may be read by some other block later in the program order.

a) [1 point] Draw the control flow graph G. Note that this is a backwards analysis.

b) [1 point] Consider the lattice $D = \langle \mathcal{P}(\{x, y, z\}), \subseteq\rangle$. Assign for each block $b \in B$ a suitable, monotone transfer function f_b over this lattice.

c) [3 points] Consider the data flow system $S := \langle G, D, \{x, y, z\}, (f_b)_{b\in B}\rangle$. Write down the induced equation system and determine its least solution using Kleene's fixed point theorem.

```
[x := 24 · y · z · z + 12 · y · y · z + 2]^0
if [y < z]^1 then
    [x := 16 · y · y]^2
else
    while [y < z]^3 do
        [x := 2 · y · y − y + 1]^4
        [y := y + x]^5
    end while
end if
[skip]^6
```

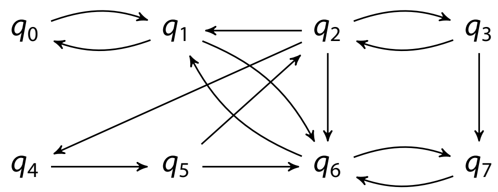
Tutorial Exercise 2.5:

Let $n_1, n_2, n_3 \in \mathbb{N} \setminus \{0\}$ be positive integers and $D := \{ k \in \mathbb{N} \mid k \mid n_1 \cdot n_2 \cdot n_3 \}$ the set of divisors of the product $n_1 \cdot n_2 \cdot n_3$. Consider the lattice $\langle D, | \rangle$. Show that the function $f_a : D \to D$ is $\sqcap$-continuous.

$$f_a(x) := \gcd(n_1, \operatorname{lcm}(n_2, n_3 \cdot x))$$

Tutorial Exercise 2.6:

Consider the graph $G = \langle V, E \rangle$ on the right, as well, as the powerset lattice $\langle \mathcal{P}(V), \subseteq \rangle$. Für each initial vertex $v \in V$, consider the function $f_v : \mathcal{P}(V) \to \mathcal{P}(V)$ with



$$f_v(X) := \{ y \mid y = v \text{ or } \exists x \in X : \langle x, y \rangle \in E \} .$$

Give the sequence from Kleene's fixed point theorem for $\operatorname{lfp}(f_{q_0})$ and $\operatorname{lfp}(f_{q_2})$ each until they stabilize.

Tutorial Exercise 2.7:

Consider a lattice $\langle D, \sqsubseteq \rangle$ and arbitrary elements $\text{keep}, \text{get} \in D$. Prove, that the function $f(x) = (x \sqcap \text{keep}) \sqcup \text{get}$ is monotonic.

Tutorial Exercise 2.8:

Let $S = \langle \langle B, E, F \rangle, \langle D, \sqsubseteq \rangle, i, (f_b)_{b \in B} \rangle$ be a Data Flow System of a may-analysis, with Control Flow Graph $\langle B, E, F \rangle$, a complete lattice $\langle D, \sqsubseteq \rangle$ with ACC, initial data flow value $i \in D$ for extremal blocks and monotonic transfer functions $f_b : D \to D$ for each block $b \in B$.

Consider the lattice $\langle D^B, \sqsubseteq^B \rangle$ of functions $f \in D^B$, which assigns for each block b an element $f(b) \in D$. A pair $f_1, f_2 \in D^B$ is ordered (i.e. $f_1 \subseteq^B f_2$), if and only if all blocks $b \in B$, $f_1(b) \subseteq f_2(b)$ holds.

Show, that the function $g_S : D^B \to D^B$ is monotonic and $\sqcup$-continuous:

$$g_S(f)(b) = \bigsqcup_{\langle b', b \rangle \in F} f(b') \sqcup \begin{cases} i & \text{if } b \in E \\ \bot & \text{otherwise} \end{cases}$$