Exercise 9.1

1. We do transfinite induction.
Assuming $(f^\beta)_{\beta<\alpha}$ is an increasing chain we show that $(f^\beta)_{\beta\leq\alpha}$ is an increasing chain for all α.

Base Case $\alpha = 0$

Chain has only one element.

Inductive Step successor ordinal $\alpha = \alpha' + 1$

By IH $(f^\beta)_{\beta\leq\alpha'}$ is an increasing chain.
Thus, we have $f^{\beta'} \subseteq f^{\alpha'}$ for all $\beta' \leq \alpha'$.
By monotony of f we get

$$f^{\beta'+1} = f(f^{\beta'}) \leq f(f^{\alpha'}) = f^\alpha$$

So for all β with $1 \leq \beta \leq \alpha'+1 = \alpha$
we have $f^\beta \leq f^\alpha$
With $f^0 = \perp \leq f^\alpha$ we the get

$$f^\beta \leq f^\alpha$$

for all $\beta \leq \alpha$

Inductive Step Limit ordinal $\alpha = \lambda$

By IH $(f^\beta)_{\beta<\alpha}$ is an increasing chain with
Limit $f^\alpha = \bigsqcup_{\beta<\alpha} f^\beta$. (which exists because $(D,\leq)$ is a complete lattice)

By definition of "$\sqcup$" we have $f^\beta \subseteq f^\alpha$ for all $\beta \leq \alpha$.

2. Idea: Iterate f until we must have seen an element twice.

Let α be the least ordinal which cardinality is equal to that of D:

$|D| = |\alpha|$ (i.e. there is a bijection from D to α)

Then $f^{\alpha+1}$ is an element in $\{f^\beta \mid \beta \leq \alpha\}$. (pigeon hole principle)

Pick such an element $\beta < \alpha+1$ with $f^\beta = f^{\alpha+1}$.
We have (by 1.)

$$f^\beta \subseteq f^{\beta+1} \subseteq \dots \subseteq f^{\alpha+1}$$

Since $f^\beta = f^{\alpha+1}$ it follows that

$$f^\beta = f^{\beta+1} = \dots = f^{\alpha+1}$$

So, f^β is a fixed point of f:

$$f(f^\beta) = f^{\beta+1} = f^\beta$$

3. Let x be a fixed point of f.

We show $f^\alpha \leq x$ for all α by transfinite induction.

<u>Base Case</u> $\alpha = 0$

$$f^0 = \bot \leq x \quad .$$

<u>Inductive Step</u> successor ordinal $\alpha = \alpha' + 1$

$$f^\alpha = f(f^{\alpha'}) \leq f(x) = x$$

by monotony of + IH

<u>Inductive Step</u> Limit ordinal $\alpha = \lambda$

$$f^\alpha = \bigsqcup_{\beta < \alpha} f^\beta \leq \bigsqcup_{\beta < \alpha} x = x$$

monotony of $\sqcup$ + IH

In particular $f^\varepsilon \leq x$ holds.

Exercise 9.2

1. Let $s_0, s_1, s_2, \dots$ be a sequence of program states after each loop iteration.
   This induces a sequence

$$\underbrace{\langle r_1(s_0), r_2(s_0), \dots r_k(s_0)\rangle}_{r(s_0)}, \underbrace{\langle r_1(s_1), \dots, r_k(s_1)\rangle}_{=: r(s_1)}, \dots$$

Note that this sequence is finite:
Otherwise there is a comparable pair $r(s_{i_1}) \leq r(s_{i_2})$ with $i_1 < i_2$ by Dickson's Lemma. Since $(s_{i_1}, s_{i_2}) \in R^+$ we have $(s_{i_1}, s_{i_2}) \in T_j$ for some j. This implies $r_j(s_{i_1}) > r_j(s_{i_2})$ contradicting $r(s_{i_1}) \leq r(s_{i_2})$.

Since f and the ranking functions are linear, the above sequence is also linearly controlled (in $\mathcal{F}_1$).
Applying the LFT the number of loop iterations is bounded by $\mathcal{F}_{k+1}$
(with k = # transition invariants $T_1, \dots T_k$)