SS 2025 03.07.2025

Exercises to the lecture
Semantics
Sheet 9

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Jan Grünke Delivery until 15.07.2025 at 11:30

Exercise 9.1 (Transfinite Iteration)
In this exercise, we prove a fixed point theorem that does not require the given function to be continuous (in contrast to Kleene's theorem).
Let $(D, \leq)$ be a complete lattice, and let $f : D \to D$ be a monotone function. The transfinite iterates of f from $\perp$ are defined as follows:

$$\begin{aligned} f^0 &= \perp \\ f^{\alpha+1} &= f(f^\alpha) \\ f^\lambda &= \bigsqcup_{\beta<\lambda} f^\beta \end{aligned}$$

where α is a successor ordinal and λ is a limit ordinal.

1. Show that the transfinite iterates of f form an increasing chain.
2. Show that the chain becomes stationary, i.e., there exists some ordinal ϵ such that for all $\delta \geq \epsilon$, we have $f^\delta = f^\epsilon$.
3. Let ϵ be the ordinal where the chain becomes stationary. Show that f^ϵ is the **least** fixed point of f.

Bonus Exercise 9.2 (Transition Invariants & Linear Ranking Functions)
Consider a program with variables $x_1, x_2, \ldots, x_n$ ranging over $\mathbb{Z}$, consisting of a single while loop with a body f:

```
int x_1, x_2, ..., x_n
while x_1 > 0 ∧ x_2 > 0 ∧ ... ∧ x_n > 0 do
    (x_1, x_2, ..., x_n) ← f(x_1, x_2, ..., x_n)
end while
```

Let f be a linear function in $x_1, x_2, \ldots, x_n$. Assume we obtain a k-ary disjunctive termination argument of the form $R^* \subseteq T_1 \cup \cdots \cup T_k$ with each T_i well-founded. Suppose we have synthesized linear ranking functions $r_j : \mathbb{Z}^n \to \mathbb{N}$ for each T_j. This means that the ranking function r_j maps a program state (an assignment to the variables) to a natural number such that

$$r_j(s) > r_j(s') \quad \text{whenever } (s, s') \in T_j.$$

Determine the maximal number of loop iterations by classifying it within the Grzegorczyk hierarchy.