SS 2025 1.05.2025

Exercises to the lecture
Semantics
Sheet 3

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Delivery until 7.05.2025 at 23:59

Exercise 3.1 (Compactness of Colorings)
A *proper coloring* of a graph (V, E) with colors C is a labelling $l : V \mapsto C$ such no two vertices sharing the same edge have the same color, i.e. for all edges $\{v, w\} \in E. l(v) \neq l(w)$. A coloring using at most k colors is called a (proper) k-coloring. Show the following using König's Lemma:
A (countably) infinite graph G is k-colorable if and only if every finite subgraph of G is k-colorable.

Exercise 3.2 (Variation of Ramsey's Theorem)
Let (V, E) be an infinite graph such that for every infinite set of vertices $X \subseteq V$ there are $v, v' \in X$ with $(v, v') \in E$. Prove that (V, E) contains an infinite complete subgraph.

Exercise 3.3 (Abstraction Refinement)
Consider the following program. It computes the product $z = x \cdot y$ for $x, y \in \mathbb{N}$. Use the CEGAR loop to show that block 8 is not reachable. Begin with the empty set of predicates.

$[z := 0]^1$
if $[x > 0]^2$ **then**
  if $[y > 0]^3$ **then**
    while $[x > 0]^4$ **do**
      $[z := z + y]^5$
      $[x := x - 1]^6$
    if $[z = 0]^7$ **then**
      $[\text{skip}]^8$
    else
      $[\text{skip}]^9$
  else
    $[\text{skip}]^{10}$
else
  $[\text{skip}]^{11}$

Exercise 3.4 (Spurious Counterexamples)
Let $r = r_1, \ldots, r_n$ be a spurious counterexample and

$$s_i = \mathsf{sp}(\mathit{true}, r_1, \ldots, r_i) \qquad\qquad w_i = \mathsf{wp}(r_{i+1}, \ldots, r_n, \mathit{false}).$$

Show that $s_i \models w_i$ holds for all $i = 1, \ldots, n - 1$.