# Ordinals:

## Definition:

- A well-order $(A, \leq)$ is
  - total and
  - well-founded.

- An order isomorphism is a function
$$i : (A, \leq_A) \longrightarrow (B, \leq_B)$$
between quasi orders that is
  - bijective and
  - $a \leq_A b$ iff $i(a) \leq_B i(b)$, f.a. $a, b \in A$.

Write $(A, \leq_A) \equiv (B, \leq_B)$
if such an order isomorphism exists.
Note that $\equiv$ is an equivalence.

## Definition:

An ordinal is an equivalence class wrt. $\equiv$.

Idea: It only matters how long you can count,
the names of elements are unimportant.

## Definition:

An initial segment of a well-order $(A, \leq)$
is a downward-closed subset $I \subseteq A$,
where downward-closed means: $a \leq b \in I \Rightarrow a \in I$
f.a. $a, b \in A$.

This induces an ordering on well-orders.

<u>Definition</u>:

Let $(A, \leq_A)$ and $(B, \leq_B)$ be well-orders.

We have $(A, \leq_A) \leq (B, \leq_B)$, if

$\exists$ initial segment $I \subseteq B$: $(A, \leq_A) \cong (I, \leq_B \cap (I \times I))$.

Since $\leq$ is defined up to isomorphism, it can be lifted to ordinals.

<u>Fact</u>:

The ordinals are well-ordered wrt. $\leq$.

<u>Examples and Notation</u>:

These are the names typically used for ordinals

$$0 = [(\emptyset, \emptyset)]_{\cong}$$

$$1 = [(\{e\}, e \leq e)]_{\cong} = \bullet$$

$$2 = [(\{e, f\}, \{(e,e), (f,f), (e,f)\})]_{\cong} = \bullet \longrightarrow \bullet$$

$$3 = \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$\vdots$$

$$\omega = [(\mathbb{N}, \leq_{\mathbb{N}})]_{\cong} = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\omega + 1 = [(\mathbb{N} \uplus \{e\}, \leq_{\mathbb{N}} \uplus \mathbb{N} \times \{e\})]_{\cong}$$

$$= \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots \quad \bullet$$

larger than every natural number.

<u>Note</u>:

$\omega \neq \omega + 1$, there is <u>no</u> isomorphism.

## Definition:

The order type of a well-order $(A, \leq)$
is the ordinal (which is an equivalence class)
that contains the well-order.

## Representations:

It is sometimes helpful to have
a concrete representation of ordinals.
Let $(A, \leq)$ be a well-order.
The function

$$
\begin{aligned}
i : (A, \leq) &\longrightarrow (\overbrace{\mathcal{P}_{\downarrow}}^{\text{only downward-closed subsets of } A}(A), \subseteq) \\
a &\longmapsto I(a) := \{ b \in A \mid b < a \} \\
&\qquad\quad = a{\downarrow} \setminus \{a\}.
\end{aligned}
$$

is an order isomorphism
Since the ordinals themselves form a well-order,
the isomorphism says that we can
represent every ordinal as the set of smaller ordinals:

$$
\begin{aligned}
i(3) &= \{0, 1, 2\} \qquad\qquad 3 < 4 \text{ by } i(3) \subseteq i(4). \\
i(4) &= \{0, 1, 2, 3\} \\
i(\omega) &= \mathbb{N}
\end{aligned}
$$

We will sometimes skip $i$
and just write ordinals
as sets of smaller ordinals: $3 = \{0, 1, 2\}$
$\omega = \mathbb{N}$.

This just fixes a representation
of the equivalence class, namely $(\{0, 1, 2\}, \leq)$.

There is another very common definition of ordinals.

Definition:

A set $\alpha$ is an ordinal, if

- it is transitive, meaning $\beta \in \alpha$ implies $\beta \subseteq \alpha$,
- membership $\in$ is a strict well-order on $\alpha$.

This definition leads to the following definition, often called von Neumann ordinals:

$0 := \emptyset$ $\qquad$ $2 := \{\emptyset, \{\emptyset\}\}$

$1 := \{\emptyset\}$ $\qquad$ $3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

Transfinite Induction:

We already observed in our example that there are three kinds of ordinals:

$0$ = does not have a predecessor, it is the smallest ordinal.

$1, 2, \ldots, \omega + 1$ = has a maximal predecessor

$\omega$ = does have predecessors but no maximal one.

Definition:

- A limit ordinal $\lambda$ is an ordinal $\neq 0$ that satisfies

$$\forall \alpha < \lambda . \exists \beta < \lambda . \alpha < \beta .$$

A <u>successor ordinal</u> $\alpha$ is an ordinal $\neq 0$ that is <u>not</u> a limit ordinal.

This means

$$\exists \beta < \alpha .\ \forall \gamma < \alpha .\ \neg(\beta < \gamma)$$

$$\neg(\beta < \gamma) \;\equiv\; \gamma \leq \beta .$$

We then also write $\alpha = \beta + 1$.

<u>Fact</u>:

Using the axiom of choice,

$\boxed{\text{every set can be well-ordered}}$.

The <u>proof technique of transfinite induction</u> can then be used to establish properties for all elements in the set in a way that is very similar to induction.

<u>Theorem (Soundness of Transfinite Induction)</u>:

Let $(A, \leq)$ be a well-order. Let $P \subseteq A$.

If $\forall x \in A .\ (I(x) \subseteq P \Rightarrow x \in P)$, then $P = A$.

<u>Proof</u>:

Towards a contradiction, assume $P \subsetneq A$.

Then $P \setminus A \neq \emptyset$ and thus has a $\leq$-minimal element $a$.

Since $a$ is $\leq$-minimal, $I(a) \subseteq P$.

By the assumption, $a \in P$. $\lightning$. $\square$

How does this look in practice?

still well-founded induction

normal induction

**Base case 0**: Show $0 \in P$.

**Inductive step $\alpha+1$**: Consider a successor ordinal $\alpha+1$, show $\alpha+1 \in P$ assuming $\alpha \in P$.

**Inductive step $\lambda$**: Consider a limit ordinal $\lambda$, show $\lambda \in P$ assuming $\alpha \in P$ f.a. $\alpha < \lambda$.

## Ordinal Arithmetic:

Our goal is to move from the semantic understanding of ordinals to a syntactic manipulation.

**Definition**:
Let $(A, \leq_A)$ and $(B, \leq_B)$ be well-orders.

Define

$$A + B := (A \uplus B, \leq_A \uplus \leq_B \uplus A \times B)$$

$$A \cdot B := (A \times B, (a \times b) \leq (a' \times b'),$$
$$\text{if } b < b' \vee (b = b' \wedge a \leq a'))$$

**Illustration**:

$$A + B = \overbrace{\cdot \longrightarrow \cdot \longrightarrow \cdots}^{A} \ \overbrace{\cdot \longrightarrow \cdot \longrightarrow \cdots}^{B}$$

$$A \cdot B = \underbrace{\cdot \longrightarrow \cdots}_{A \times \{0\}} \ \underbrace{\cdot \longrightarrow \cdots}_{A \times \{1\}} \ \underbrace{\cdot \longrightarrow \cdots}_{A \times \{2\}} \ \underbrace{\cdot \longrightarrow \cdots}_{A \times \{3\}} \ \cdots$$

one copy of $A$ for each element in $B$.

We can reflect these operations on the syntactic representation of ordinals.

Definition:

$+1$ is the successor function from above

$$\alpha + 0 := \alpha$$

$$\alpha + (\beta + 1) := (\alpha + \beta) + 1$$

$$\alpha + \lambda := \sup\{\alpha + \beta \mid \beta < \lambda\}.$$

$$\alpha \cdot 0 := 0$$

$$\alpha \cdot (\beta + 1) := \alpha \cdot \beta + \alpha$$

$$\alpha \cdot \lambda := \sup\{\alpha \cdot \beta \mid \beta < \lambda\}.$$

This is transfinite recursion in the 2nd operand.

Lemma:

$+, \cdot$ are associative.

$+, \cdot$ are not commutative: $1 + \omega = \omega \neq \omega + 1$

$$2 \cdot \omega = \omega \neq \omega \cdot 2.$$

- $\cdot$ distributes over $+$ from the left: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
- does not distribute over $+$ from the right:

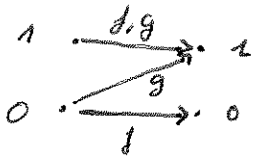
$$(1 + 1) \cdot \omega = \omega \neq 1 \cdot \omega + 1 \cdot \omega = \omega \cdot 2.$$

To define exponentiation $\alpha^\beta$ semantically, we consider $\alpha$ and $\beta$ as sets of smaller ordinals and consider functions $f : \beta \to \alpha$ with finite support, meaning $f(x) \neq 0$ for only finitely many elements.

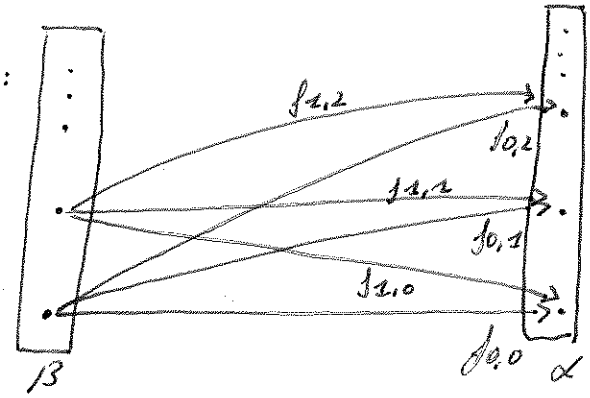
These functions are well-ordered by

$$f < g \quad \text{if} \quad \exists x \in \beta.\ f(x) < g(x)$$
$$\wedge\ \forall x < y \in \beta.\ f(x) = g(x).$$

<u>Example:</u>




<u>Idea:</u>




$f_{0,0} < f_{0,1} < \dots$
$< f_{1,0} < f_{1,1} < \dots$
$< f_{2,0} < f_{2,1} < \dots$

There are $\alpha$-many functions $f_{0,*}$.

There are $\alpha$-many ways to define $f_{1,i}$.

Hence, there are $\alpha^2$-many functions $f_{1,*}$.

There are $\alpha^3$-many functions $f_{2,*}$.

<u>Definition:</u>

$$\alpha^0 := 1$$
$$\alpha^{\beta+1} := (\alpha^\beta)\cdot\alpha$$
$$\alpha^\lambda := \sup\{\alpha^\beta \mid \beta < \lambda\}$$

<u>Lemma:</u>

- $1^\alpha = 1$
- $\alpha^1 = \alpha$
- $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$
- $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$
- $(\alpha\cdot\beta)^\gamma \neq \alpha^\gamma\cdot\beta^\gamma$

<u>Counterexample:</u>

$$(\omega\cdot 2)^2$$
$$= \omega\cdot 2\cdot(\omega+\omega)$$
$$= \omega\cdot(2\cdot\omega) + \omega\cdot(2\cdot\omega)$$
$$= \omega\cdot\omega + \omega\cdot\omega$$
$$= \omega^2\cdot 2$$
$$\neq \omega^2\cdot 4 = \omega^2\cdot 2^2.$$

## Definition:

$$\varepsilon_0 = \sup\{0, 1, \omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

This ordinal is the least solution to $\omega^x = x$.

## Theorem (Cantor Normal Form (CNF)):

Every ordinal $\alpha < \varepsilon_0$ can be written as

$$\alpha = \omega^{\beta_1} + \dots + \omega^{\beta_m} = \sum_{i=1}^{m} \omega^{\beta_i}$$

where $\alpha > \beta_1 \geq \dots \geq \beta_m \geq 0$

and each $\beta_i$ is again in CNF.

We write $CNF(\alpha)$ for the set of ordinals $\beta < \alpha$ in CNF.

## Definition:

- The natural sum of ordinals in CNF is

$$\sum_{i=1}^{m} \omega^{\beta_i} \oplus \sum_{j=1}^{n} \omega^{\beta_j'} := \sum_{k=1}^{m+n} \omega^{\gamma_k},$$

where $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{m+n}$ reorders $\beta_1, \dots, \beta_m, \beta_1', \dots, \beta_n'$.

- The natural product is

$$\sum_{i=1}^{m} \omega^{\beta_i} \otimes \sum_{j=1}^{n} \omega^{\beta_j'} := \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \omega^{\beta_i \oplus \beta_j'}.$$

## Example:

$5 \oplus 3 = 8$ $\qquad$ $5 \otimes 3 = 15$

$\omega^5 + \omega^3 \oplus \omega^4 = \omega^5 + \omega^4 + \omega^3$ $\qquad$ $\omega^5 + \omega^3 \otimes \omega^4 = \omega^9 + \omega^7.$

# Fundamental Sequences and Predecessors

Our goal is to approximate limit ordinals by sequences of smaller ordinals.

**Definition:**

Let $\lambda$ be a limit ordinal.

A *fundamental sequence* for $\lambda$ is a sequence of ordinals $(\lambda_x)_{x \in \mathbb{N}}$

with $\cdot\ \lambda_x < \lambda$ f.a. $x \in \mathbb{N}$

$\cdot\ \lambda = \sup\{\lambda_x \mid x \in \mathbb{N}\}$.

The latter condition in particular means

$$\forall \alpha < \lambda.\ \exists x.\ \alpha < \lambda_x.$$

**Construction:**

One typically fixes a fundamental sequence for $\omega$, like

$$\omega_x := x + 1.$$

One then lifts this to all ordinals in CNF by

$$(\gamma + \omega^{\beta+1})_x := \gamma + \omega^\beta \cdot \omega_x$$

$$(\gamma + \omega^\lambda)_x := \gamma + \omega^{\lambda_x}.$$

The definition $\omega_x := x+1$ guarantees $\lambda_x > 0$ f.a. $x \in \mathbb{N}$.

From now on, we assume to have an assignment of fundamental sequences.

## Definition:

The $x \in \mathbb{N}$ - indexed predecessor $P_x(\alpha) < \alpha$ of $\alpha \neq 0$ is

$$P_x(\alpha + 1) := \alpha$$
$$P_x(\lambda) := P_x(\lambda_x).$$

Since $\lambda_x < \lambda$, we have $P_x(\alpha) < \alpha$.

Example:

- $P_n(\omega^0) = P_n(1) = 0$
- $P_n(\omega^d) = P_n(\omega^{d-1} \cdot \omega_n)$

$$= P_n(\omega^{d-1} \cdot (n+1))$$
$$= P_n(\omega^{d-1} \cdot n + \omega^{d-1})$$
$$\{\text{needs a proof}\} = \omega^{d-1} \cdot n + P_n(\omega^{d-1}).$$

The property

$$\boxed{P_x(\gamma + \alpha) = \gamma + P_x(\alpha)}$$

reminds us of

$$\partial_n \alpha = \{ \gamma \oplus \partial_n \omega^d \mid \alpha = \gamma \oplus \omega^d \},$$

we just choose to pick the summand with the smallest exponent.