Proof of the Length Function Theorem

Step 1: · Study properties of controlled bad sequences.

· Derive a bounding function $M_{g,A}(n)$

for the length of (g,n)-controlled bad sequences.

Step 2: · Study properties of the functions $M_{g,A}(n)$.

· Classify them in the Grzegorczyk hierarchy.

Step 1

Residual NWQOs and a Descent Equation

- In the case of $\mathbb{N}$, we have $L_{g,\mathbb{N}}(n) = n+1$,

  the longest (g,n)-controlled bad sequence is

  $$n, n-1, \ldots, 1, 0 \quad \text{with } n+1 \text{ elements.}$$

- Actually, the sequence only shows $\boxed{L_{g,\mathbb{N}}(n) \geq n+1}$.

  We have not argued why it is the longest.

  Here is a sketch.

  In every (g,n)-controlled bad sequence $k, l, m, \ldots$

  once $k \leq n$ (holds by control) has been fixed,

  the remaining elements $l, m, \ldots$ have to stem

  $$\text{from } \{0, \ldots, k-1\} \text{ of size } k.$$

  So the suffix, which is again bad,

  is of length at most

  $$L_{g,\Gamma_k}(n) = k, \quad \text{by the pigeonhole principle.}$$

Choosing $k = n$ maximizes the length of the suffix, so

$$\boxed{L_{g,\mathbb{N}}(n) \le n+1}.$$

Strategy:

Consider a (g,n)-controlled bad sequence $a_0 a_1 \dots a_n$ over an nwqo A.

Distinguish between a_0 with $|a_0| \le n$ and the suffix $a_1 \dots a_n$.

The suffix satisfies

(i) $a_0 \not\le a_i$ for all i (otherwise the full sequence was good)

(ii) it is bad — " —

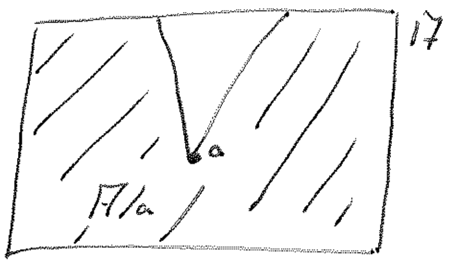
(iii) it is $(g, g(n))$-controlled (otherwise the full sequence was not controlled).

Definition:

Let A be an nwqo and $a \in A$.

The <u>residual nwqo</u> A/a is the nwqo induced by the subset

$$A/a := \{ b \in A \mid a \not\le b \}.$$



Examples:

- $\mathbb{N}/\ell = \underset{\{0,\dots,\ell-1\}}{[\ell]} = [k]/\ell$ with $\ell \le k-1$.
- $\underset{\{a_0,\dots,a_{k-1}\}}{T_k/a_i} \equiv T_{k-1}$ with $a_i \in T_k$.

- The conditions (i) to (iii) guarantee
  that the suffix $a_1 a_2 \ldots$
  of a (g,n)-controlled bad sequence $a_0 a_1 \ldots$ in A
  is a $(g, g(n))$-controlled bad sequence in A/a_0.
- If we now pick $a_0' \in A_{\leq n}$
  that maximizes $L_{g,A/a_0'}(g(n))$, say through $a_1' a_2' \ldots$
  then we have constructed the (g,n)-controlled bad sequence $a_0' a_1' a_2' \ldots$

  Hence, $$\boxed{L_{g,A}(n) \geq \max_{a \in A_{\leq n}} \{1 + L_{g,A/a}(n)\}}.$$
- The inequality also holds in the reverse direction,
  which is easy to see.
  Let $a_0^n . a_1^n . a_2^n \ldots$ be a maximal (g,n)-controlled bad sequence in A,
  meaning the length is $L_{g,A}(n)$.
  If this sequence is not empty, then
  - $a_0^n \in A_{\leq n}$
  - $a_1^n . a_2^n \ldots$ has length $L_{g,A/a_0^n}(g(n))$.

  If the latter condition was not true,
  we could substitute the suffix for a longer bad sequence.

<u>Proposition (Descent Equation)</u>:

$$L_{g,A}(n) = \max_{a \in A_{\leq n}} \{1 + L_{g,A/a}(g(n))\}.$$

Residuation is <u>well-founded</u>:

$$A \supsetneq A/a_0 \supsetneq A/a_0/a_1 \supsetneq \dots \quad \text{has to stabilize.}$$

Why? The sequence $a_0 . a_1 \dots$ one forms is a bad sequence in a wqo, hence finite.

Well-foundedness allows us to use induction in combination with the descent equation to understand $L_{g,A}(n)$.

<u>Example</u>:

(1) Claim: $L_{g,[k]}(n) = k$, provided $k-1 \leq n$.

<u>Proof</u>: We proceed by induction on k.

<u>$k=0$</u>: The only sequence over $[0] = \emptyset$ is ε. It has length $0 = k$.

<u>$k>0$</u>: Assume the equality holds for k, provided $k-1 \leq n$. Consider $k+1$, and moreover n so that $(k+1)-1 \leq n$.

Then $$L_{g,[k+1]}(n)$$

$$\overset{(DE)}{=} \max_{l \in \{0,\dots,k\}} \{1 + L_{g,[k+1]/l}(g(n))\}$$

$$\overset{\text{(Example above)}}{=} \max_{l \in \{0,\dots,k\}} \{1 + L_{g,[l]}(g(n))\}$$

$$\overset{(IH + *)}{=} \max_{l \in \{0,\dots,k\}} \{1 + l\} = 1 + k.$$

(*) To apply (IH), we need $l-1 \leq g(n)$. This holds by $l \leq k \leq n \leq g(n)$.

(2) Claim: $L_{g,\mathbb{N}}(n) = n+1$.

Proof:

$$
\begin{aligned}
& L_{g,\mathbb{N}}(n) \\
\text{(DE)} \quad &= \max_{l \in \mathbb{N}_{\leq n}} \{1 + L_{g,\mathbb{N}/l}(g(n))\} \\
\text{(Example above)} \quad &= \max_{l \in \mathbb{N}_{\leq n}} \{1 + L_{g,[l]}(g(n))\} \\
\text{(Example (1))} \quad &= \max_{l \in \mathbb{N}_{\leq n}} \{1 + l\} = 1 + n.
\end{aligned}
$$

$\square$

Reflecting NWQOs:

Problem: The descent equation quickly leads to expressions

$A/a_0/a_1/\ldots/a_n$

that are unstructured in that

they do not have nice definitions via + and ×.

Solution: Reflection is a technique to over-approximate these sets without leaving the polynomial nwqos.

Example: We already saw this in our informal reasoning. We understood the suffix as a sequence in T_h, although $[h] \neq T_h$.

What we used was the insight that the bad sequences are never longer than in T_h.

Definition:

- An nwqo reflection is a mapping $h: A \to B$ between nwqos so that
  - $h(a) \leq_B h(b)$ implies $a \leq_A b$ f.a. $a, b \in A$
  - $|h(a)|_B \leq |a|_A$ f.a. $a \in A$.

The order may be finer (less comparable elements)
and the norm may go down.

- We write $h: A \hookrightarrow B$ and say <u>B reflects A</u>.
  We write $A \hookrightarrow B$ if there is an nwqo reflection.
- Reflection is a quasi order, it is reflexive and <u>transitive</u>:
  $h: A \hookrightarrow B$ and $h': B \hookrightarrow C$ imply $h' \circ h: A \hookrightarrow C$.
- Every nwqo A reflects its substructures X, $id: X \hookrightarrow A$.
  In particular, $\Gamma_0 \hookrightarrow A$ for all A,
  $\Gamma_1 \hookrightarrow A$ for all $A \neq \emptyset$.

<u>Examples:</u>

- $[k] \hookrightarrow \Gamma_k$
- $\Gamma_k \not\hookrightarrow [k]$ for $k \geq 2$
- $[k] \hookrightarrow \mathbb{N}$ $\left.\right\}$ reflect substructures
- $\Gamma_k \hookrightarrow \Gamma_{k+1}$ $\left.\right\}$
- $\mathbb{N} \not\hookrightarrow [k]$
  $\Gamma_{k+1} \not\hookrightarrow \Gamma_k$.

Reflections are defined so that they preserve bad sequences.
Let $h: A \hookrightarrow B$ and $a_0 . a_1 \ldots$ be a sequence in A.
Then $h(a_0) . h(a_1) \ldots$ is a sequence in B.
Moreover,
- if $a_0 . a_1 \ldots$ is bad, so is $h(a_0) . h(a_1) \ldots$
- if $a_0 . a_1 \ldots$ is (g, n)-controlled, so is $h(a_0) . h(a_1) \ldots$

Proposition (Monotonicity of $L_{g,A}(n)$ w.r.t. $\hookrightarrow$):

If $A \hookrightarrow B$, then $L_{g,A}(n) \leq L_{g,B}(n)$, for all g, n.

This monotonicity was used with $[h] \hookrightarrow T_h$ to deduce $L_{g,[h]}(n) \leq L_{g,T_h}(n)$ in the informal reasoning from the beginning.

Remark (Reflection is a Precongruence):

$A \hookrightarrow A'$ and $B \hookrightarrow B'$ imply
$$A + A' \hookrightarrow B + B'$$
$$A \times A' \hookrightarrow B \times B'.$$

Inductive Residual Computation

Goal: Compute / reflect A/a by induction on the structure of the polynomial nwqo A.

Base case:

$$\mathbb{N}/h \hookrightarrow T_h \qquad \text{by} \quad \mathbb{N}/h = [h] \hookrightarrow T_h.$$

Disjoint sums:

$$(A+B)/(1,a) = (A/a) + B$$
$$(A+B)/(2,b) = A + (B/b).$$

Reflection is not needed here.

Cartesian products:

$$(A \times B)/(a,b) \hookrightarrow ((A/a) \times B) + (A \times (B/b)).$$

How do we see this?

Let $g(x) = 2x$ and consider the following $(g, 3)$-controlled bad sequence in $\mathbb{N}^2$:

$(3,2).(5,1).(0,4).(17,0).(1,1).(16,0).(0,3).$

We want to reflect $\mathbb{N}^2/(3,2)$ into a polynomial nwqo.

The requirement $(3,2) \not\leq (a,b)$ on the suffix implies

$$3 \not\leq a \quad \text{or} \quad 2 \not\leq b.$$

We partition the elements in the suffix into <u>two groups</u>:

elements, where the first coordinate is in $\mathbb{N}/3$

and elements, where the second coordinate is in $\mathbb{N}/2$.

Then the suffix can be reflected in $((\mathbb{N}/3)\times\mathbb{N})+(\mathbb{N}\times(\mathbb{N}/2))$.

On the example,

$$(3,2).\begin{cases}(5,1) & (17,0)\ (1,1)\ (16,0) \quad \in \mathbb{N}\times(\mathbb{N}/2) \\ (0,4) & (0,3) \in (\mathbb{N}/3)\times\mathbb{N}.\end{cases}$$

The sequences $(5,1).(17,0).(1,1).(16,0)$ and $(0,4).(0,3)$

are indeed bad, but $\uparrow$ is not $(g, g(3))$-controlled:

$$17 \not\leq 12 = g(g(3))$$

We do not see them as disjoint sequences

but as one sequence in the disjoint sum of nwqos.

Then 17 keeps its position 3:

$$17 \leq g^2(g(3)) = 24.$$

<u>A</u> <u>Bounding</u> <u>Function</u>

<u>Goal</u>: Define a derivation relation A' on A

that describes a particular way

of reflecting A/a with $a \in A_{\leq n}$,

meaning $A/a \hookrightarrow A'$ will hold.

Moreover, for every n, $\partial_n A$ is

- a finite set of polynomial nwqos
- in polynomial normal form.

<u>Recall</u>:

$\mathbf{0} := \emptyset = T_0 = [0]$

$\mathbf{1} := T_1 \equiv [1] \equiv A^0$ for all A

$A^n := A \times \ldots \times A$, n times

$A \cdot n := A + \ldots + A$, n times

$T_n \equiv T_1 \cdot n$.

<u>Definition</u>:

$\partial_n \mathbf{0} := \emptyset$

$\partial_n \mathbf{1} := \{0\}$

$\partial_n \mathbb{N}^d := \{\mathbb{N}^{d-1} \cdot n \cdot d\}$

$\partial_n (A+B) := ((\partial_n A) + B) \cup (A + (\partial_n B))$ // + on sets:

$A + S := \{A + B \mid B \in S\}$

<u>Lemma</u>:

$\partial := \bigcup_{n \in \mathbb{N}} \partial_n$ is a well-founded relation.

<u>Lemma</u>:

Let A be a polynomial nwqo in PNF.

Let $a \in A_{\leq n}$ for some $n \in \mathbb{N}$.

Then there is $A' \in \partial_n A$ so that $A/a \cong A'$.

<u>Proof</u>:

Let $A = \mathbb{N}^{d_1} + \ldots + \mathbb{N}^{d_m}$

Let $a \in A_{\leq n}$.

The existence of a implies $m \geq 1$, $A = \mathbf{0} = \emptyset$ cannot be the case.

We proceed by <u>induction on m</u>:

<u>$m = 1$</u>: $A = \mathbb{N}^d$

We proceed by <u>induction on d</u>:

<u>$d = 0$</u>: Then $A = \mathbb{N}^0 = \mathbf{1}$.

Now $A/a = \emptyset = \mathbf{0} \hookrightarrow \mathbf{0} \in \partial_n \mathbf{1}$.

<u>$d = 1$</u>: Then $A = \mathbb{N}$. We use $a \leq n$.

Then $A/a = [a] \hookrightarrow T_n \equiv T_1 \cdot n \cdot 1$
$= \mathbf{1} \cdot n \cdot 1 \in \partial_n \mathbb{N}^1$.

Assume the claim has been shown up to $d-1 \geq 1$.

<u>$d \geq 2$</u>: $A = \mathbb{N}^d = \mathbb{N} \times \mathbb{N}^{d-1}$.

Then $a = (h, b)$ with $h \in \mathbb{N}_{\leq n}$ and $b \in \mathbb{N}^{d-1}_{\leq n}$.

$$
\begin{aligned}
& A/a \\
\text{(above)} \quad & \hookrightarrow ((\mathbb{N}/h) \times \mathbb{N}^{d-1}) + (\mathbb{N} \times (\mathbb{N}^{d-1}/b)) \\
\text{(IH + precongruence)} \quad & \hookrightarrow (\mathbf{1} \cdot n \times \mathbb{N}^{d-1}) + (\mathbb{N} \times (\mathbb{N}^{d-1}/b)) \\
\text{(IH + precongruence)} \quad & \hookrightarrow \mathbb{N}^{d-1} \cdot n + \mathbb{N} \times (\mathbb{N}^{d-2} \cdot n \cdot (d-1)) \\
& \equiv \mathbb{N}^{d-1} \cdot n + \mathbb{N}^{d-1} \cdot n \cdot (d-1) \\
& = \mathbb{N}^{d-1} \cdot n \cdot d \\
& \in \partial_n \mathbb{N}^d .
\end{aligned}
$$

<u>$m > 1$</u>: Let $A = B + C$ and $a = (1, b)$ with $b \in B_{\leq n}$.

The case of $a = (2, c)$ is symmetric.

Then $A/a = (B/b) + C$.

By the induction hypothesis, $B/b \hookrightarrow B'$
for some $B' \in \partial_n B$.

By precongruence,
$(B/b) + C \hookrightarrow B' + C \in \partial_n (B + C)$.

$\square$

__Definition:__

Let A be a polynomial nwqo in PNF.

Define
$$M_{g,A}(n) := \max_{A' \in \partial_n A} \{1 + M_{g,A'}(g(n))\}.$$ // Here we need $\partial_n 0 = \emptyset$.

By the well-foundedness of ∂
and the finiteness of $\partial_n A$,
this is well-defined (König's lemma).

__Proposition:__

For any polynomial nwqo A in PNF, any g, and any n:
$$L_{g,A}(n) \leq M_{g,A}(n).$$

__Proof:__

If $A_{\leq n} = \emptyset$, then $L_{g,A}(n) = 0 \leq M_{g,A}(n)$.

Otherwise we can use the __descent equation__
to determine $a \in A_{\leq n}$ that maximizes $L_{g,A/a}(n)$,
meaning
$$L_{g,A}(n) = 1 + L_{g,A/a}(g(n))$$

Using the above lemma, we obtain $A/a \hookrightarrow A' \in \partial_n A$.

Then
$$\begin{aligned}
& L_{g,A}(n) \\
\text{(Choice of } a) &= 1 + L_{g,A/a}(g(n)) \\
\text{(Monotonicity rules)} &\leq 1 + L_{g,A'}(g(n)) \\
\text{(Induction hypothesis)} &\leq 1 + M_{g,A'}(g(n)) \\
\text{(Def. of } M_{g,A}(n)) &\leq 1 + M_{g,A}(n).
\end{aligned}$$

This is a well-founded induction along ∂_n, the base case is 0, covered initially.

$\square$

Step 2

Goal: Classify $M_{g,A}(n)$ in the Grzegorczyk hierarchy.

Approach: Study an isomorphism between polynomial nwqos and their maximal order types (ordinals below ω^ω)

Maximal Order Types

- Well-order $(A, \leq)$:
  - total
  - every subset has a minimal element.
- Ordinal numbers: $0, 1, 2, \ldots,$
  $\omega, \omega+1, \omega+2, \ldots,$
  $\omega+\omega = 2\omega, 2\omega+1, 2\omega+2, \ldots,$
  $\vdots$
  $\omega \cdot \omega = \omega^2, \omega^2+1, \omega^2+2, \ldots$

Definition:
An ordinal is an equivalence class of well-orders under order isomorphism.

It only matters how long you can count,
the names of elements are not important.
It is still helpful to have a concrete representation.
Consider

$$i : (A, \leq) \longrightarrow (\mathcal{P}(A), \subseteq)$$
$$a \longmapsto A_{<a} = \{ b \in A \mid b < a \}$$

The ordinals themselves are well-ordered:
$X \leq Y$, if X is isomorphic to an initial segment of Y.
The isomorphism then says that every ordinal can be written as the well-ordered set of smaller ordinals:

$3 \equiv \{0, 1, 2\}$ $\quad$ $3 < 4$ by $i(3) \subseteq i(4)$

$4 \equiv \{0, 1, 2, 3\}$ $\quad$ $\omega \equiv \mathbb{N}$.

One can also use membership to define the ordering.
This yields the standard definition of ordinals.

Definition (von Neumann):

A set α is an ordinal, if

- it is transitive, meaning $\beta \in \alpha$ implies $\beta \subseteq \alpha$,
- $\in$ is a strict well-order on α.

This definition leads to the following representation, often called von Neumann ordinals:

$$0 \equiv \emptyset \qquad 2 \equiv \{\emptyset, \{\emptyset\}\} \qquad \ldots$$
$$1 \equiv \{\emptyset\} \qquad 3 \equiv \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

The order type of a well-order is then the ordinal (understood as an equivalence class) that contains the well-order.

We will lift the notion of order type to sets and to well-quasi orders.

Intermezzo: Transfinite Induction

With the axiom of choice, every set can be well-ordered.
The proof technique of transfinite induction can then be used to establish properties for all elements in the set, in a way very similar to induction.

Theorem (Transfinite Induction):

Let $(W, \leq)$ be well-ordered. Let $I(x) := x\downarrow \setminus \{x\}$. Let $P \subseteq W$.
If $\forall x \in W. (I(x) \subseteq P \Rightarrow x \in P)$, then $P = W$.

Proof:

Towards a contradiction, assume $P \subsetneq W$.

Then $W \setminus P \neq \emptyset$ and thus has a $\leq$-minimal element a.

Since a is $\leq$-minimal, $I(a) \subseteq P$.

By the assumption, $a \in P$. $\lightning$ $\square$

How does this look in practice?

still well-founded induction:

normal induction:

- Base case: $\alpha = 0$, show $\alpha \in P$.
- Inductive step 1: successor ordinal $\alpha + 1$ (like $3 \rightsquigarrow 4$),
  show $\alpha + 1 \in P$ assuming $\alpha \in P$.

Inductive step 2: limit ordinal λ:
$\exists \alpha < \lambda \wedge \forall \alpha < \lambda\, \exists \beta.\ \alpha < \beta < \lambda$,

show $\lambda \in P$ assuming $\alpha \in P$
for all $\alpha < \lambda$