Proof of the Length Function Theorem

Step 1: · Study properties of controlled bad sequences.
· Derive a bounding function $M_{g,A}(n)$ for the length of (g,n)-controlled bad sequences.

Step 2: · Study properties of the functions $M_{g,A}(n)$.
· Classify them in the Grzegorczyk hierarchy.

Residual NWQOs and a Descent Equation

· In the case of $\mathbb{N}$, we have $L_{g,\mathbb{N}}(n) = n+1$,
the longest (g,n)-controlled bad sequence
is $\underbrace{n, n-1, \dots, 1, 0}_{n+1 \text{ elements}}$.

· Actually, the sequence only shows $\boxed{L_{g,\mathbb{N}}(n) \geq n+1}$.
We have not argued why it is the longest.
Here is a sketch.
In every (g,n)-controlled bad sequence $k, l, m, \dots,$
once $k \leq n$ (by control) has been fixed,
the remaining elements $l, m, \dots$ have to stem
from $\{0, \dots, k-1\}$ of size k.
So the suffix, which is again bad,
is of length at most
$L_{g,\Gamma_k}(n) = k$, by the pigeonhole principle.
Choosing $k=n$ maximizes the length of the suffix,
so $\boxed{L_{g,\mathbb{N}}(n) \leq n+1}$.

Strategy:

Consider a (g,n)-controlled bad sequence $a_0 a_1 \ldots a_n$ over an nwqo A.

Distinguish between a_0 with $|a_0| \le n$ and the suffix $a_1 \ldots a_n$.

The suffix satisfies

- $a_0 \not\le a_i$ for all i (otherwise the full sequence was good)
- it is bad — " —
- it is $(g, g(n))$-controlled (otherwise the full sequence was not controlled).