Interface dynamics in discrete forward-backward diffusion equations

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The diffusive lattice ODE

$$\dot{u}_j = p_{j+1} - 2p_j + p_{j-1}, \qquad p_j = \Phi'(u_j) \tag{1}$$

with bistable nonlinearity Φ' can be regarded as a *microscopic regularization* of the ill-posed PDE

$$\partial_\tau U = \partial_\xi^2 P, \qquad P = \Phi'(U), \tag{2}$$

provided that the macroscopic variables are introduced by the *hyperbolic scaling*. The latter reads

$$\tau = \varepsilon^2 t, \qquad \xi = \varepsilon j, \qquad u_j(t) = U(\varepsilon^2 t, \varepsilon j)$$

with $\varepsilon > 0$ being the small scaling parameter. Other notably regularizations of (2) are the *Cahn-Hillard model* and the *viscous approximation*, which add $-\varepsilon^2 \partial_\xi^4 U$ and $+\varepsilon^2 \partial_\xi^2 \partial_\tau U$, respectively, to the right hand side of $(2)_1$.

Hysteretic interface motion. A key dynamical feature of any regularization of (2) are *phase interfaces*. These curves separate space-time regions in which U attains values in different *phases*, that means in either one of the two connected components of $\{u : \Phi''(u) > 0\}$; see figures 1 and 2 for illustration.

Heuristic arguments as well as numerical simulations of (1), see §2 in [3], indicate that the effective lattice dynamics for $\varepsilon \to 0$ can – for a wide class of initial data – be described by a hysteretic free boundary problem. In the case of a single interface located at $\xi_*(\tau)$, this model combines *bulk diffusion*

$$\partial_\tau U = \partial_\xi^2 P \qquad \text{for} \qquad \xi \neq \xi_*(\tau)$$

with the *Stefan condition*

$$\frac{\mathrm{d}\xi_*}{\mathrm{d}\tau} |[U]| + |[\partial_\xi P]| = |[P]| = 0$$

and the *hysteretic flow rule*

$$\frac{\mathrm{d}\xi_*}{\mathrm{d}\tau} |[U]| < 0 \quad \Longrightarrow \quad P = p^*, \qquad \frac{\mathrm{d}\xi_*}{\mathrm{d}\tau} |[U]| > 0 \quad \Longrightarrow \quad P = p_*,$$

where $|[\cdot]|$ denotes the jump across the interface. The same equations can – at least on a formal level – also be derived from the viscous approximation, see [1]. The sharp interface limit of the Cahn-Hilliard equation, however, is different as it replaces the hysteric flow rule by $P = p_{\mathrm{mx}}$, where p_{mx} represents the Maxwell line.

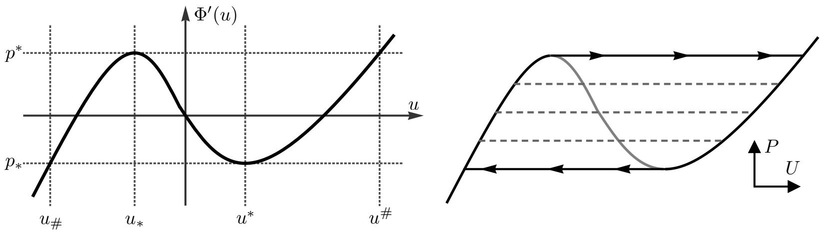

FIGURE 1. *Left panel.* Bistable derivative Φ' of a general double-well potential Φ. *Right panel.* The hysteresis loop for phase interfaces: solid and dashed lines represent moving and standing interfaces, respectively; there arrows indicate the temporal jump when U undergoes a phase transition at fixed position ξ.

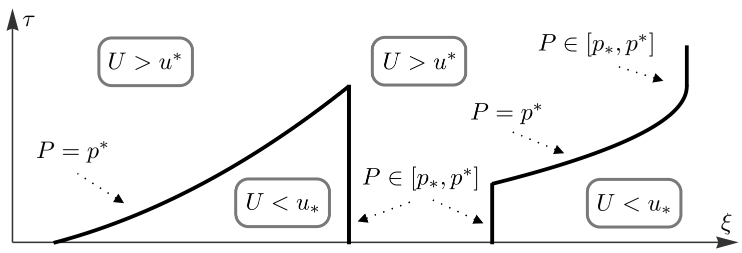

FIGURE 2. Cartoon of three phase interfaces: The first one (moving) and the second one (standing) annihilate each other in a collision. The third interface illustrates both pinning and depinning.

Rigorous analysis for a special case. Due to the existence of multiple time scales, the rigorous justification of the above limit model is – in the case of moving interfaces – currently out of reach; for standing interfaces, see [2]. For the piecewise quadratic double-well potential

$$\Phi(u) = \tfrac{1}{2} \min \big\{(1-u)^2,\, (1+u)^2\big\}, \qquad \Phi'(u) = u - \operatorname{sgn} u, \tag{3}$$

however, the analysis of (1) is considerably simpler and allows us to prove the following results:

(1) *Existence of single-interface solutions*: A certain class of microscopic single-interface states is invariant under the flow of (1)+(3). In particular, the lattice dynamics generates a single phase interface which moves since the lattice data u_j undergo a phase transition (crossing of the spinodal state $u = 0$) one after another.
(2) *Justification of the limit model*: For macroscopic single-interface data, the lattice solutions converge as $\varepsilon \to 0$ to the unique solution of the hysteretic free boundary problem.

The first thesis is a direct consequence of elementary ODE arguments and implies the representation formula

$$p_j(t) = \sum_{i\in\mathbb{Z}} g_{j-i}(t)p_i(0) - 2\sum_{k\geq 1} \chi_{(t_k^*,\infty)}(t)g_{j-k}(t-t_k^*). \tag{4}$$

Here, g abbreviates the discrete heat kernel and χ_I denotes the indicator function of the interval I. Moreover, t_k^* is the k^{th} phase transition time, which is, however, not given a priori but depends nonlinearly on the whole solution p via $\lim_{t\nearrow t_k^*} p_k(t) = +1$.

The representation formula (4) is crucial for passing to the limit $\varepsilon \to 0$ as it allows us to exploit the temporal and spation decay properties of the discrete heat kernel. In particular, assuming that the initial data are sufficiently nice we can derive upper bounds for the macroscopic interface speed as well as macroscopic compactness results for the scaled lattice data in the space of Hölder continuous functions. The convergence result then follows by combining standard arguments with a direct justification of the hysteretic flow rule and a uniqueness result from [4]. The details can be found in §3 of [3].

References

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