A toy model for hysteretic phase transitions

Michael Herrmann

(joint work with Michael Helmers, University of Bonn)

The one-dimensional lattice ODE

$$\dot{u}_j = p_{j+1} - 2\,p_j + p_{j-1} \quad \text{with} \quad p_j = u_j - \operatorname{sign} u_j \tag{1}$$

admits solutions with propagating phase interfaces and provides a microscopic justification for macroscopic hysteresis models. Here, the two *phases* correspond to the sets $\{u < 0\}$ and $\{u > 0\}$, on which the bistable function $u \mapsto u - \operatorname{sign} u$ is strictly increasing.

Microscopic dynamics. For a finite system with $N < \infty$ particles and either periodic or Neumann boundary conditions, equation (1) can be regarded is a microscopic H^{-1}-gradient flow for u. In particular, it satisfies the energy balance

$$\dot{\mathcal{E}}(t) = -\mathcal{D}(t)\,, \quad \mathcal{E} := \tfrac{1}{2}\sum_j p_j^2\,, \quad \mathcal{D} := \sum_j (p_{j+1} - p_j)^2\,,$$

so there is a strong tendency to reach a state with small dissipation. However, due to phase transitions (one of the u_j's changes sign) there exist small time intervals with huge dissipation and strong microscopic fluctuations, see Figures 1 and 2 for an illustration.

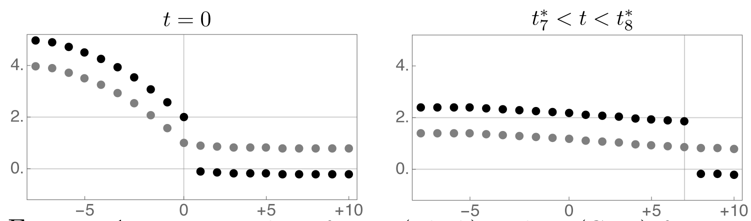

Figure 1. Two snapshots for u_j (Black) and p_j (Gray) for a numerical single-interface solution with 20 particles: The phase interface (vertical line) propagates to the right since the particles undergo a phase transition one after another.

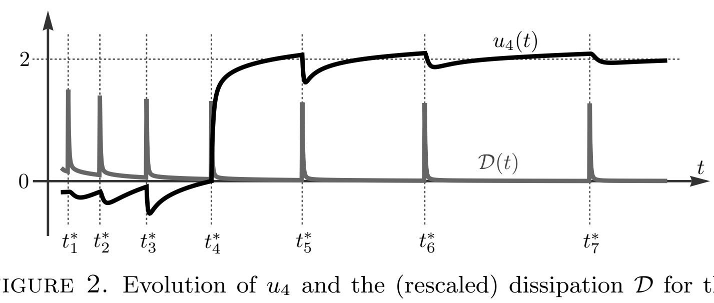

Figure 2. Evolution of u_4 and the (rescaled) dissipation $\mathcal{D}$ for the simulation from Figure 1; the particle j undergoes a phase transition at time t_j^*.

Macroscopic dynamics. The parabolic scaling limit

$$\tau := \varepsilon^2 t, \qquad \xi := \varepsilon j$$

has been investigated in [1] for a system with infinitely many particles and under certain assumptions on the microscopic initial data; the main result can be formulated as follows: The discrete p-data converge as $\varepsilon \to 0$ strongly to a limit function P, which is uniquely determined by the hysteretic free boundary problem

$$\partial_\tau \big(P(\tau, \xi) + \mu(\tau, \xi) \big) = \partial_\xi^2 P(\tau, \xi) \,, \qquad \mu(\cdot, \xi) = \mathcal{R}\big[P(\cdot, \xi) \big] \,. \tag{2}$$

Here, $\mathcal{R}$ abbreviates the hysteresis operator from Figure 3 and the limit U of the u-data satisfies $U = P + \mu$. The well-posedness of the initial value problem to (2) has been proven in [2].

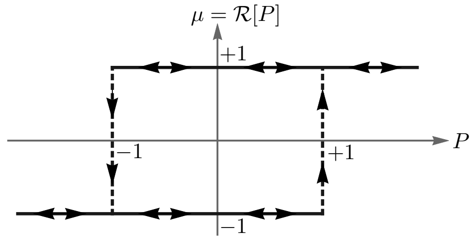

FIGURE 3. The relay operator $\mathcal{R}$ describes the hysteresis of phase interfaces in the macroscopic scaling limit.

REFERENCES

[1] M. Helmers, M. Herrmann: *Interface dynamics in discrete forward-backward diffusion equations*, SIAM Multiscale Model. Simul., vol. 11(4), pp. 1261–1297, 2013.

[2] A. Visintin: *Quasilinear parabolic P.D.E.s with discontinuous hysteresis*, Ann. Mat. Pura Appl., vol. 185(4), pp. 487–519, 2006.