Technische Universität Braunschweig
Institute for Partial Differential Equations

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Formulary Mathematics for Engineers B
Ordinary Differential Equations

Product form

$$y' = f(t,y) = g(t) \cdot h(y)$$

ODE in homogeneous variables

$$y' = f(t,y) = g\left(\frac{y}{t}\right) \text{ , substitution } u = \frac{y}{t}$$

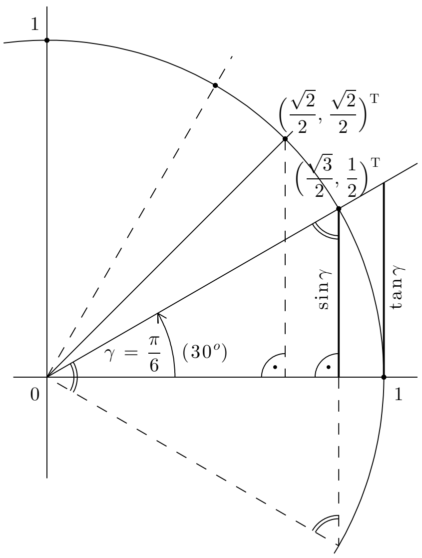
Bernoulli differential equation

$$y' + a(t)y = p(t)y^n \text{ , substitution } u = \frac{1}{y^{n-1}}$$

Euler's identity

$$\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$$

Trigonometric functions



Trigonometric relations

$$1 = \sin^2 x + \cos^2 x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin x + \sin y = 2 \cdot \sin\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right)$$

$$\cos t = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}\right)$$

$$\sin t = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}\right)$$

Exakt differential equation

$$A(x,y)\ \mathrm{d}x + B(x,y)\ \mathrm{d}y = 0 \text{ with } A_{,y} = B_{,x}$$

Euler's ODE

$$b_n t^n y^{(n)} + \ldots + b_1 t y' + b_0 y = 0 \text{ , approach } y = t^\alpha$$

Lipschitz continuity regarding y

$$\exists L: \ |f(t,y) - f(t,z)| \le L \cdot |y - z|$$

Peano's theorem $y' = f(t,y)$

$$f \text{ continuous } \Rightarrow \text{ local existence of } y = y(t)$$

Theorem of Picard-Lindelöf $y' = f(t,y)$

$$f \text{ Lipschitz-continuous regarding } y \Rightarrow \text{ local uniqueness}$$

Substantial derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, v(t)) = \frac{\partial}{\partial t}\Phi + \frac{\partial}{\partial v}\Phi \cdot \frac{\mathrm{d}v}{\mathrm{d}t}$$

Linear differential operator

$$\mathcal{L}\{y\} = y^{(n)} + a_{n-1}(t) y^{(n-1)}(t) + \ldots + a_0(t) y(t)$$

Wronskian determinant

$$W = W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

System answer of the one-mass oscillator

$$\text{amplitude } A(\alpha) = \frac{1}{\sqrt{(k-\alpha^2)^2 + \alpha^2 d^2}}$$

Important frequencies

$$\omega = \sqrt{k - \frac{d^2}{4}} \text{ , } \alpha_{\mathrm{res}} = \sqrt{k - \frac{d^2}{2}}$$

Generalized eigenvector h with eigenvector **v**

$$(A-\lambda I)\mathbf{h} = \mathbf{v}$$

Variation of the constants

$$\begin{pmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p(t) \end{pmatrix}$$

Variation of the constants for systems

$$\mathbf{q}_h(t) = VD(t)\mathbf{c} \ , \ VD(t)\mathbf{c}'(t) = \mathbf{p}(t)$$

Laplace transform $\mathcal{T}(f):[0,\infty)\to\mathbb{C}$ of $f:[0,\infty)\to\mathbb{C}$

$$\mathcal{T}(f)(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\ \mathrm{d}t$$

Laplace transforms

$f(t)$	$\mathcal{T}(f)(s)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\mathrm{e}^{bt}\sin(at)$	$\frac{a}{(s-b)^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Multiplications theorem

$$\mathcal{T}(tf(t))(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}(f)(s)$$

Differentiation theorem

$$\mathcal{T}(f')(s) = s\mathcal{T}(f)(s) - f(0)$$

Damping theorem

$$\mathcal{T}(\mathrm{e}^{at}f(t))(s) = \mathcal{T}(f)(s-a)$$

Translation theorem for $a \geq 0$

$$\mathcal{T}(H(t-a)f(t-a))(s) = \mathrm{e}^{-as}\ \mathcal{T}(f(t))(s)$$

Partial fraction decomposition

$$\frac{p(x)}{(x-x_0)(x-x_1)} = \frac{A}{x-x_0} + \frac{B}{x-x_1} \ ,$$

$$\frac{p(x)}{(x-x_0)^k} = \frac{A_1}{x-x_0} + \frac{A_2}{(x-x_0)^2} + \dots + \frac{A_k}{(x-x_0)^k}$$

L_2 **scalar product** of $f,g:[a,b]\to\mathbb{C}$

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\ \mathrm{d}x$$

Kronecker symbol

$$\delta_{k\ell} = \begin{cases} 0 \text{ if } k \neq \ell \\ 1 \text{ if } k = \ell \end{cases}$$

p-q-formulary for $x^2+px+q=0$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Dynamical systems

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}) \ , \ \mathbf{q}\in\mathbb{R}^n \ , \ \mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$$

Stationary point $\mathbf{q}^*\in\mathbb{R}^n$ with $\mathbf{f}(\mathbf{q}^*) = \mathbf{0}$, $\mathbf{q}(t) = \mathbf{q}^*$ solves $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$

Attractive stationary point $\mathbf{q}^*$

$$\exists\varepsilon>0:\ ||\mathbf{q}_0-\mathbf{q}^*||<\varepsilon \ \Rightarrow\ \lim_{t\to\infty}\mathbf{q}(t) = \mathbf{q}^*$$

Stable stationary point $\mathbf{q}^*$

$$\forall\varepsilon>0\ \exists\delta>0:\ ||\mathbf{q}_0-\mathbf{q}^*||<\delta \ \Rightarrow\ ||\mathbf{q}(t)-\mathbf{q}^*||<\varepsilon$$

Asymptotically stable stationary point $\mathbf{q}^*$ stable and attractive