Vector spaces

Previously

A vectorspace over a field $\mathbb{K}$ is a set V and a field $\mathbb{K}$, equipped with two operations:

1. Vector addition „+": $V \times V \to V$ which fulfills for all $\vec{u}, \vec{v}, \vec{w} \in V$:
   - $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ „associative"
   - $\exists\, \vec{0} \in V: \ \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ „neutral element"
   - $\exists\, (-\vec{v}) \in V: \ \vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ „inverse element"
   - $\vec{v} + \vec{u} = \vec{u} + \vec{v}$ „commutative"

2. Scalar multiplication „·": $\mathbb{K} \times V \to V$ which fulfills for all $\vec{u}, \vec{v} \in V$, $\alpha, \beta \in \mathbb{K}$:
   - $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$

   distributive laws

- $(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
- $(\alpha\cdot\beta)\vec{v} = \alpha\cdot(\beta\cdot\vec{v})$
- $\mathbb{1}\cdot\vec{v} = \vec{v}$ (where $\mathbb{1}$ is the neutral element in $\mathbb{K}$)

Example 1

space of n-tuples in $\mathbb{R}^n$:

Elements in $\mathbb{R}^n$: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, Addition: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$,

Multiplication: $\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$

Multiplication in $\mathbb{R}$

Multiplication in $\mathbb{R}^n$

↳ You can check that all axioms are fulfilled.

Example 2

Space of all functions from $\mathbb{R}$ to $\mathbb{R}$, i.e:

$$\mathbb{K} = \mathbb{R}, \quad V := \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ is a function} \}$$

Addition: $(f+g)(x) := f(x) + g(x)$

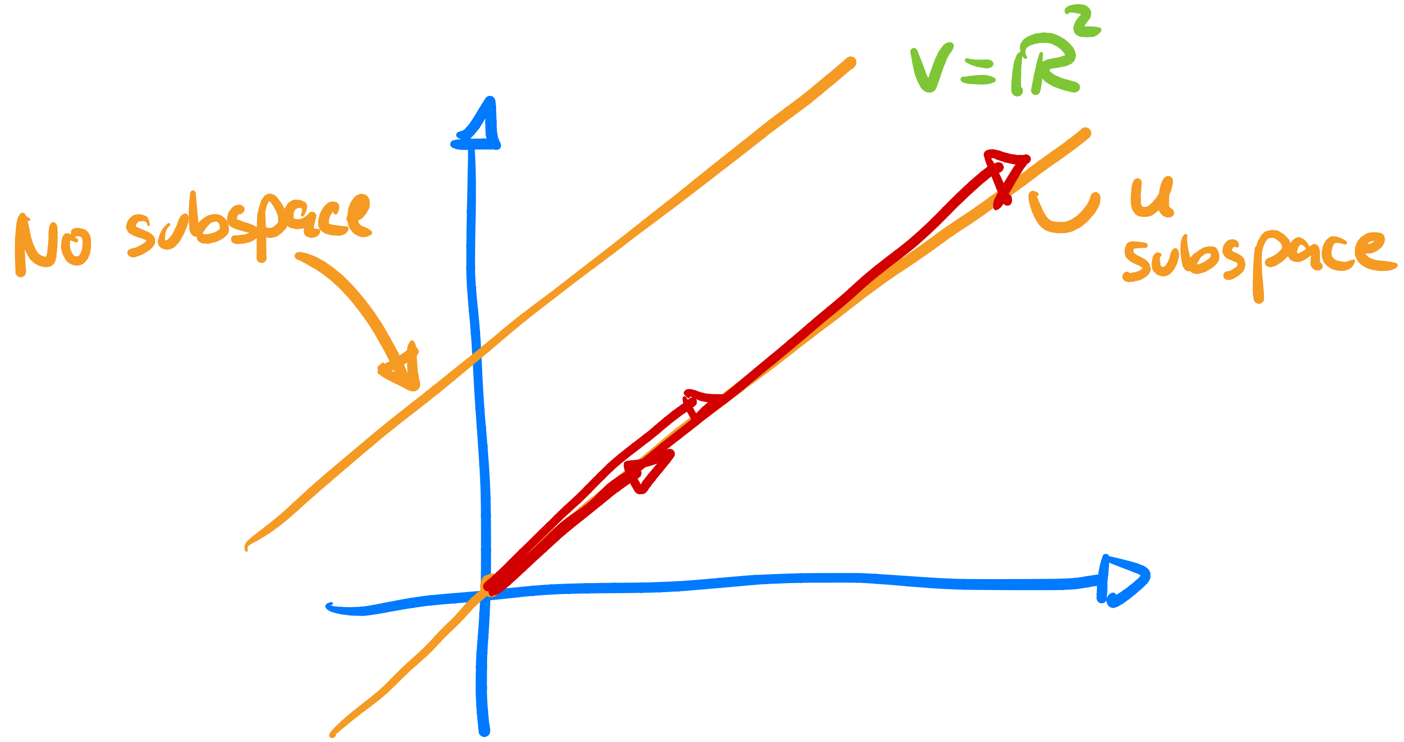
Multiplication: $(\alpha f)(x) := \alpha \cdot f(x)$

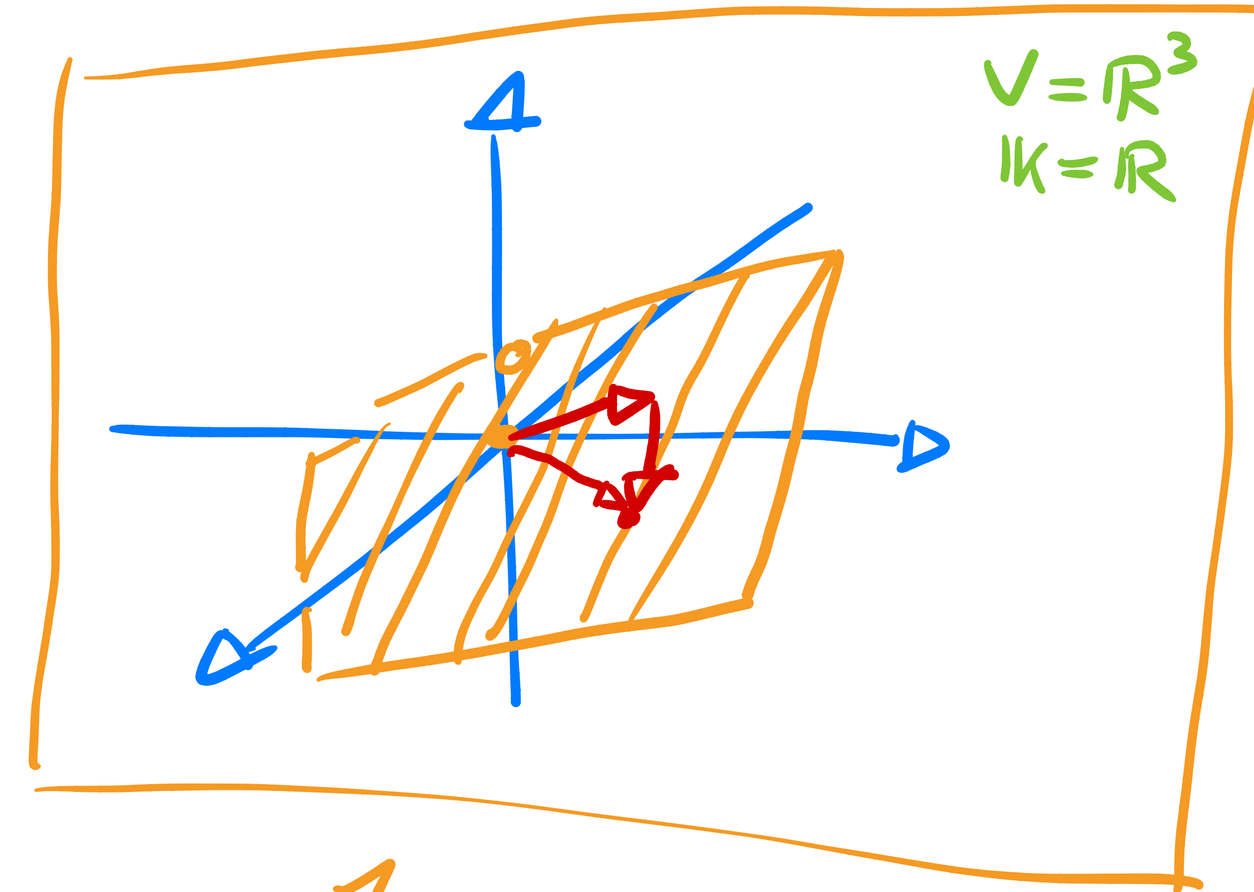
Wed, 09. October 2024

Subspaces (vector space inside of a vector space)

Let V be a vectorspace over $\mathbb{K}$ and „+", „·" the corresponding operations on V. Then $U \subseteq V$ is called a subspace of V if

- $U \neq \emptyset$ (sometimes $\vec{0} \in U$)
- $\vec{u} + \vec{v} \in U \quad \forall \vec{v}, \vec{u} \in U$
- $\alpha \vec{u} \in U \quad \forall \vec{u} \in V, \alpha \in \mathbb{K}$





Example 1

Consider $V=\mathbb{R}^3$ over $\mathbb{K}=\mathbb{R}$. Then

Plane in $\mathbb{R}^3$

$$U=\{(x,y,z)^T\in\mathbb{R}^3 \mid x-2y+3z=0\}$$

is a subspace. Verify that:

- $U\neq\emptyset$ since $(x,y,z)^T=(0,0,0)^T\in U$ ✓

  $(0-2\cdot 0+3\cdot 0=0)$

- Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \in U$, i.e. it holds $x - 2y + 3z = 0$
$\tilde{x} - 2\hat{y} + 3\tilde{z} = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \hat{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} x + \hat{x} \\ y + \tilde{y} \\ z + \tilde{z} \end{pmatrix} \in U$$

since $(x + \tilde{x}) - 2 \cdot (y + \tilde{y}) + 3(z + \tilde{z})$

$$= \underbrace{(x - 2y + 3z)}_{=0} + \underbrace{(\tilde{x} - 2\tilde{y} + 3\tilde{z})}_{=0} = 0$$

- Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U$, $\alpha \in \mathbb{R}$, i.e. it holds $\boxed{x - 2y + 3z = 0}$

Furthermore, we have

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} \in U$$ since $(\alpha x) - 2(\alpha y) + 3(\alpha z)$

$$= \alpha \underbrace{(x - 2y + 3z)}_{=0} = \alpha \cdot 0 = 0$$

Example 2

$V = \mathbb{R}^3$, $\mathbb{K} = \mathbb{R}$. Show that

$$U = \{ (x, y, z)^T \in \mathbb{R}^3 \mid x - 3y + z = 1 \}$$

is not a subspace.

In order to show that, we show that $\vec{0} \notin U$. In fact it holds that

$$0 - 3 \cdot 0 + 0 = 0 \neq 1 \qquad ↯ \quad \vec{0} \notin U.$$

$\rightsquigarrow$ U is not a subspace

Example 3

Let V be the vector space of all real functions $f: \mathbb{R} \to \mathbb{R}$. Then

$$U = \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ continious} \} = C^0(\mathbb{R}) = C^0(\mathbb{R}, \mathbb{R})$$

is a subspace.

- $U \neq \emptyset$ since $f(x) = 0$ is continious
- Since the sum of of two continious functions is continious again, we have

$$(f+g)(x) \in U \quad \text{for } f(x), g(x) \in U$$

- Since

$$(\alpha f)(x) \in U \quad \text{for } f(x) \in U \text{ and } \alpha \in \mathbb{R}$$

the third property is also fulfilled.

$\rightarrow$ U is a subspace
(and therefore a Vectorspace as well)

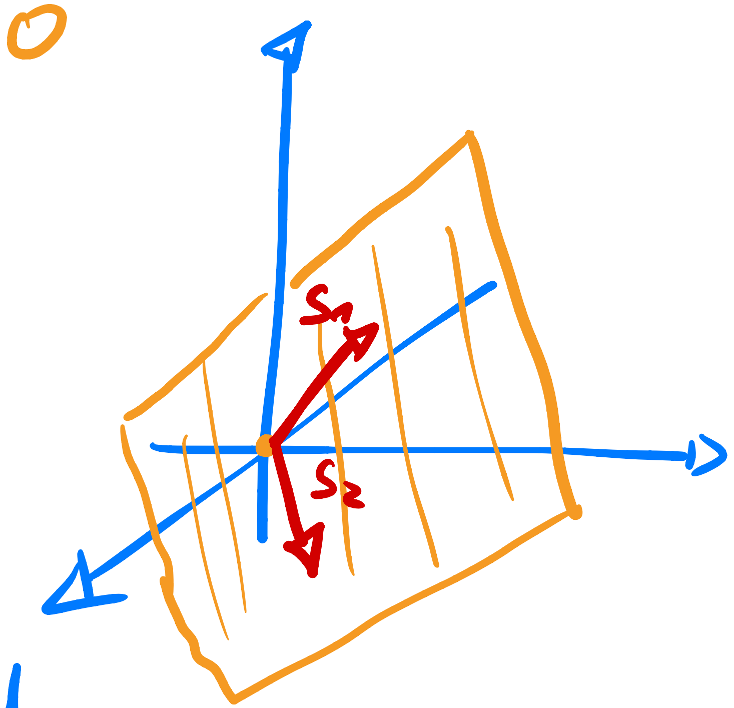
Basis of Vector Spaces

Motivation Consider a plane as a subspace of $\mathbb{R}^3$.

A plane can be parameterized by

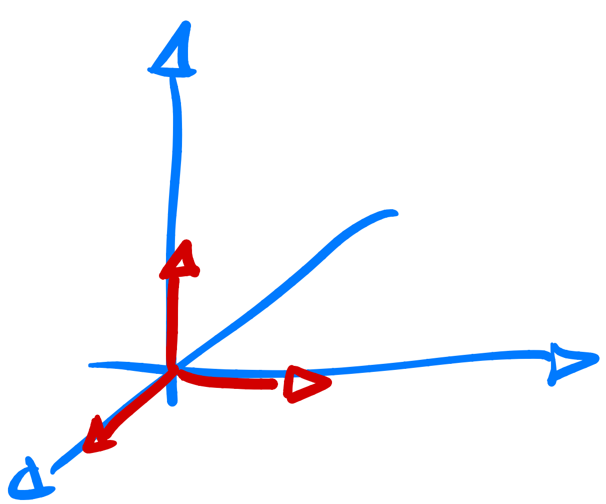
$$\vec{x} = \lambda \vec{s_1} + \mu \vec{s_2}$$

$x - 2y + 3z = 0$



The vectors $\vec{s_1}, \vec{s_2}$ span the plane and each vector which points into the plane can be represented as a linear combination of $\vec{s_1}$ and $\vec{s_2}$.

other example: $\mathbb{R}^3$



$p(x) = x^n + \dots + x + 1$

More abstract

$\mathbb{R}[x] : \{ p : \mathbb{R} \to \mathbb{R} \mid p$ polynomial of degree $n \}$

All those polynomials are linear combinations of $1, x, x^2, \dots, x^n$

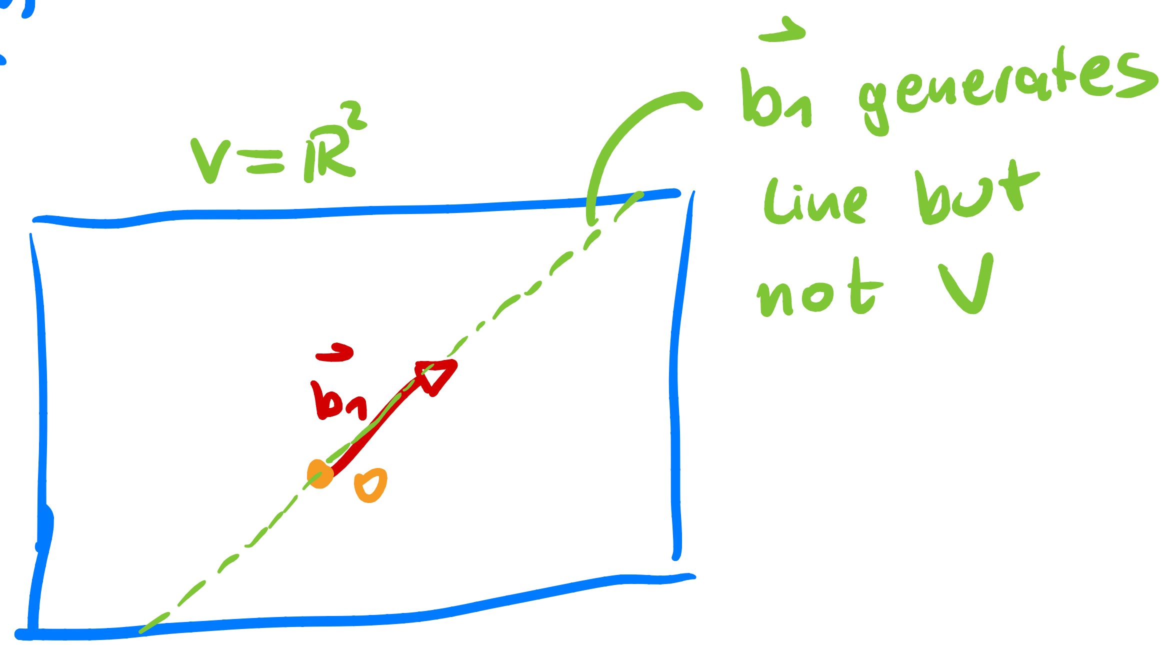
$p(x) = 3x^2 - 2x + 1$

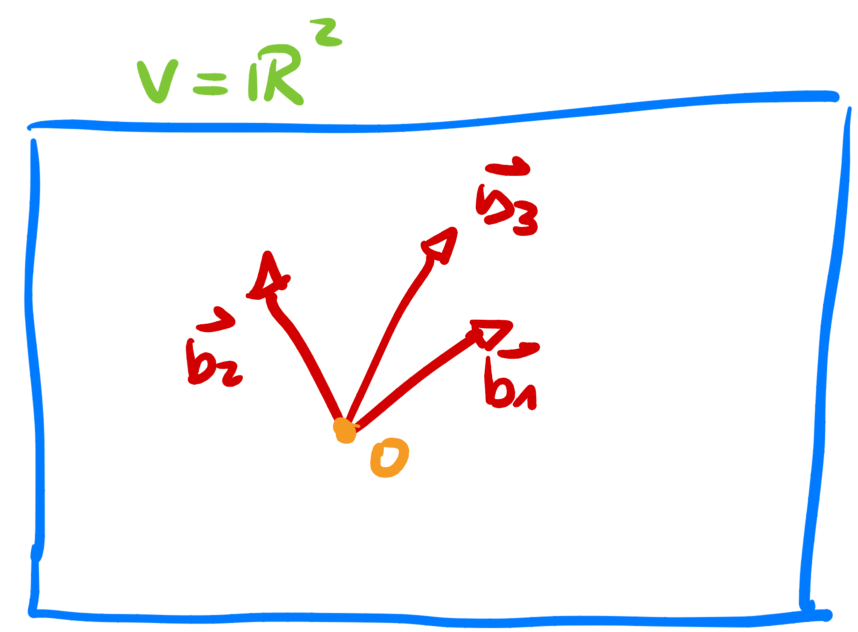
In order to formalise this, let $\mathrm{span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \{\sum_{i=1}^{n} \alpha_i \vec{v}_i \mid \alpha_i \in \mathbb{K}\}$

We call $\{\vec{b}_1, \ldots, \vec{b}_n\}$ a basis (generating set) of a vector space V if

1. $\mathrm{span}\{\vec{b}_1, \ldots, \vec{b}_n\} = V$
   „basis generates V"
2. $\vec{b}_1, \ldots, \vec{b}_n$ are linear independent
   „basis is minimal"

<u>Intuition</u>





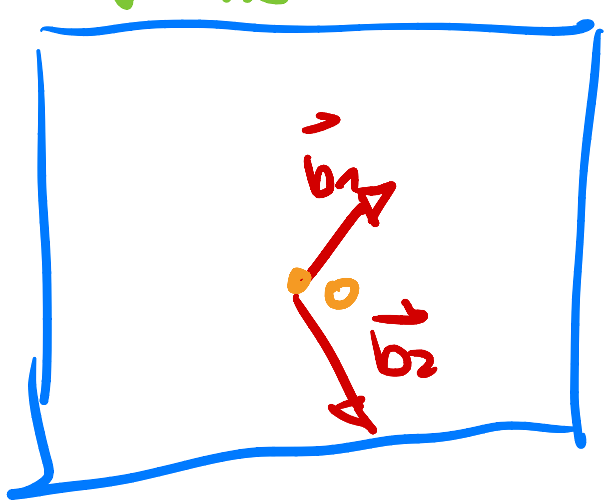
$\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ generates V, but is not minimal.

$V = \mathbb{R}^2$

$\leadsto$ Right amount of vectors (vectors are not linear dependent) $\}$ $\{\vec{b_1}, \vec{b_2}\}$ is basis

$\leadsto$ $\{\vec{b_1}, \vec{b_2}\}$ spans the Vetorspace

$V = \mathbb{R}^2$



$\leadsto$ same is true here

↳ New choice of vectors also forms a basis

$\leadsto$ Basis are not unique !

Example 1

Is
$$\left\{ \overset{\vec{b_1}}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \end{pmatrix}}, \overset{\vec{b_2}}{\begin{pmatrix} -1 \\ 1 \\ -2 \\ -3 \\ 3 \end{pmatrix}}, \overset{\vec{b_3}}{\begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \\ -3 \end{pmatrix}}, \overset{\vec{b_4}}{\begin{pmatrix} 2 \\ 10 \\ 14 \\ 10 \\ 10 \end{pmatrix}} \right\}$$
a basis of $V = \{\vec{b_1}, \vec{b_2}, \vec{b_3}, \vec{b_4}\}$?

We have to check two properties from the definition:

1. $V = \text{span}\{\vec{b_1}, \vec{b_2}, \vec{b_3}, \vec{b_4}\}$ is fulfilled by definition

2. Are the vectors linear independent?

$$\begin{matrix} 1 & \boxed{-1} & \boxed{1} & 2 \\ 2 & 1 & -1 & 10 \\ 3 & -2 & 2 & 14 \\ 4 & -3 & 3 & 10 \\ 0 & 3 & -3 & 10 \end{matrix}$$

Columns are linear dependent!

$\vec{b_2} = (-1) \cdot \vec{b_3}$

$\rightsquigarrow \{\vec{b_1}, \vec{b_2}, \vec{b_3}, \vec{b_4}\}$ not a basis!

But what if we only take $\{\vec{b_1}, \vec{b_2}, \vec{b_4}\}$ as possible basis?

[$V = \text{span}\{\vec{b_1}, \vec{b_2}, \vec{b_4}\}$ is fulfilled]

Chek for linear independence:

$$
\left(\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
2 & 1 & 10 & 0 \\
3 & -2 & 14 & 0 \\
4 & -3 & 10 & 0 \\
0 & 3 & 10 & 0
\end{array}\right)
\begin{array}{l}
\\
\mid -2\cdot I \\
\mid -3\cdot I \\
\mid \cdot 4\cdot I \\
\\
\end{array}
\rightsquigarrow
\left(\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
0 & 3 & 6 & 0 \\
0 & 1 & 8 & 0 \\
0 & 1 & 2 & 0 \\
0 & 3 & 10 & 0
\end{array}\right)
\begin{array}{l}
\\
\\
\mid -\frac{1}{3}\cdot II \\
\mid -\frac{1}{3} II \\
\mid - II
\end{array}
\rightsquigarrow
\left(\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
0 & 3 & 6 & 0 \\
0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0
\end{array}\right)
\begin{array}{l}
\\
\\
\\
\\
\mid -\frac{4}{6}\cdot III
\end{array}
$$

$$
\rightsquigarrow
\left(\begin{array}{ccc|c}
1 & -1 & 2 & 0 \\
0 & 3 & 6 & 0 \\
0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

$\xrightarrow[\text{subst.}]{\text{Backward}}$ $\alpha_1, \alpha_2, \alpha_4 = 0$ in $\alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \alpha_4 \vec{b}_4 = 0$

$\rightsquigarrow$ $\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$ are basis!

Example 2

Let $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{C}$). Then we introduce the vector space of square integrable functions on $[0,1]$ as

$$L^2([0,1]) := \left\{ f:[0,1]\to \mathbb{K} \;\middle|\; \int_0^1 |f(x)|^2\,dx < \infty \right\}$$

We want to show that this is a vector space, i.e. we show that it is a subspace of the vector space of all function.

- $L^2([0,1]) \neq \emptyset$ since $f(x)=0$ is an element $L^2([0,1])$ since

  $$\int_0^1 |0|^2\,dx = 0 < \infty$$

- Addition of $f, g \in L^2([0,1])$ is in $L^2([0,1])$ again, since

$$\int_0^1 |f(x)+g(x)|^2\,dx \leq \int_0^1 |f(x)|^2 + 2|f(x)g(x)| + |g(x)|^2\,dx$$

$$= \int_0^1 |f(x)|^2\,dx + 2\cdot \int_0^1 |f(x)g(x)|\,dx + \int_0^1 |g(x)|^2\,dx$$

Cauchy-Schwarz inequality

$$\leq \underbrace{\int_0^1 |f(x)|^2\,dx}_{<\infty} + 2\left[\left(\int_0^1 |f(x)|^2\,dx\right)\left(\int_0^1 |g(x)|^2\,dx\right)\right]^{\frac{1}{2}} + \int_0^1 |g(x)|^2\,dx$$

$$\underbrace{\qquad}_{<\infty} \quad \underbrace{\qquad}_{<\infty} \quad \underbrace{\qquad}_{<\infty}$$

$< \infty$ $\rightsquigarrow f+g \in L^2([0,1])$

- Multiplication of $f \in L^2([0,1])$ with $\alpha \in \mathbb{K}$ is in $L^2([0,1])$ since

$$\int_0^1 |\alpha f(x)|^2 dx = \underbrace{|\alpha|^2}_{<\infty} \cdot \underbrace{\int_0^1 |f(x)|^2 dx}_{<\infty} < \infty$$

$\rightsquigarrow \alpha f \in L^2([0,1])$

$\rightarrow$ $L^2([0,1])$ is a subspace

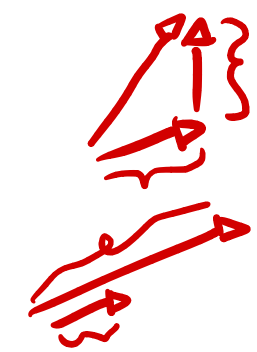
$\rightsquigarrow$ A basis of this vectorspace is given as

$$\left\{ b_1(x) = 1,\ b_{2j}(x) = \frac{1}{\sqrt{2}} \cos(2j\pi x),\ b_{2j+1}(x) = \frac{1}{\sqrt{2}} \sin(2j\pi x) \quad \text{for } j = 1, 2, \dots \right\}$$

Normed vector spaces

Let V be a $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ vector space. $\|\cdot\|: V \to \mathbb{K}$ is called a norm (on V) if it holds for all $\vec{x}, \vec{y} \in V, \alpha \in \mathbb{K}$ that

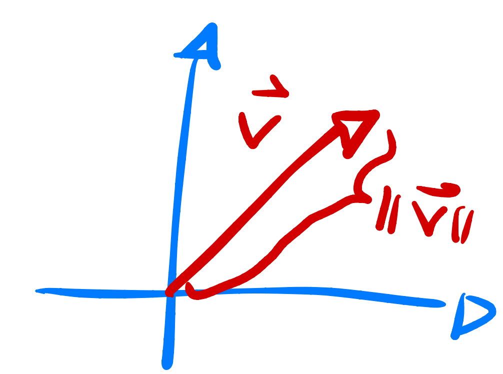
1. $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
2. $\|\alpha \vec{x}\| = |\alpha| \cdot \|\vec{x}\|$
3. $\|\vec{x}\| = 0 \Rightarrow \vec{x} = 0$ $\quad (\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = 0)$



A vector space with a norm is a normed vector space $(V, \|\cdot\|)$

Intuition

A norm measures the distance to zero / length of a vector



A few examples

1. **euclidean norm**

let $V = \mathbb{R}^n$, then $\|\vec{x}\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$ is a norm on V

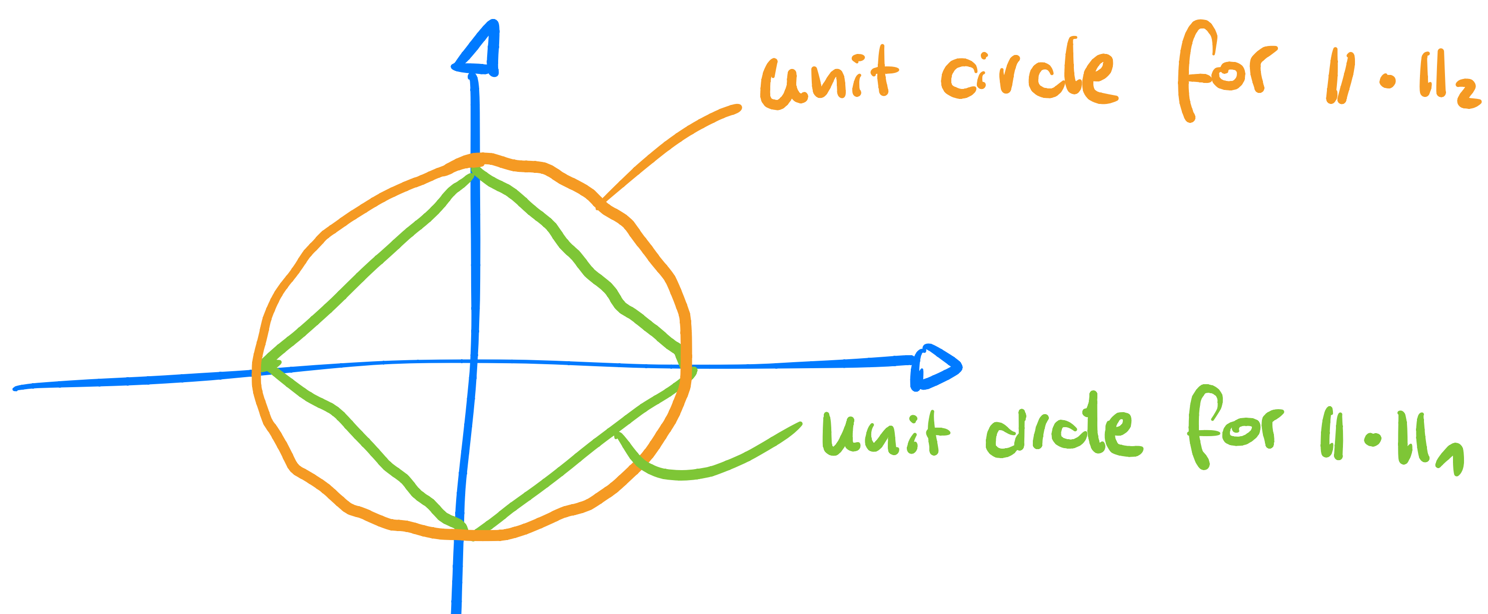
2. **Taxicab norm / Manhattan norm**

Let $V = \mathbb{R}^n$, then $\|\vec{x}\|_1 = |x_1| + |x_2| + \dots + |x_n|$

Let's compare 1 & 2:

we drawing the unit circle

$$B_1(0) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = 1 \}$$

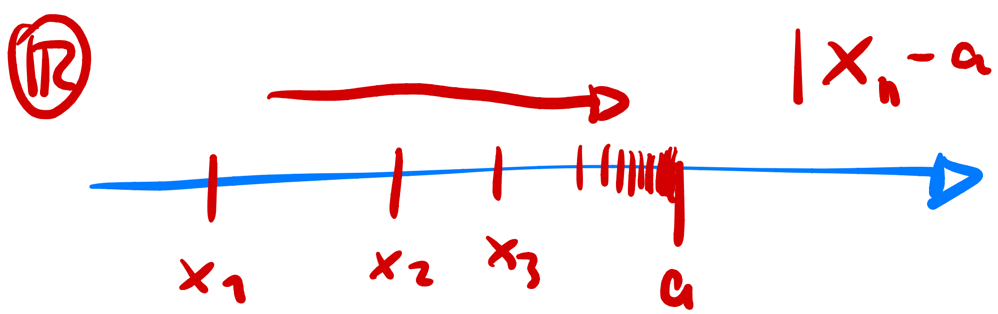


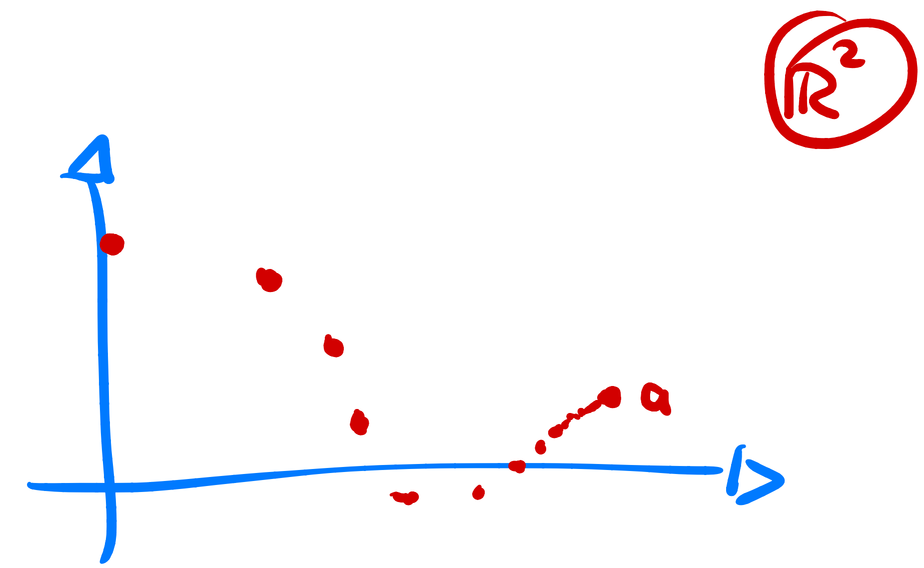
3. L^2-norm

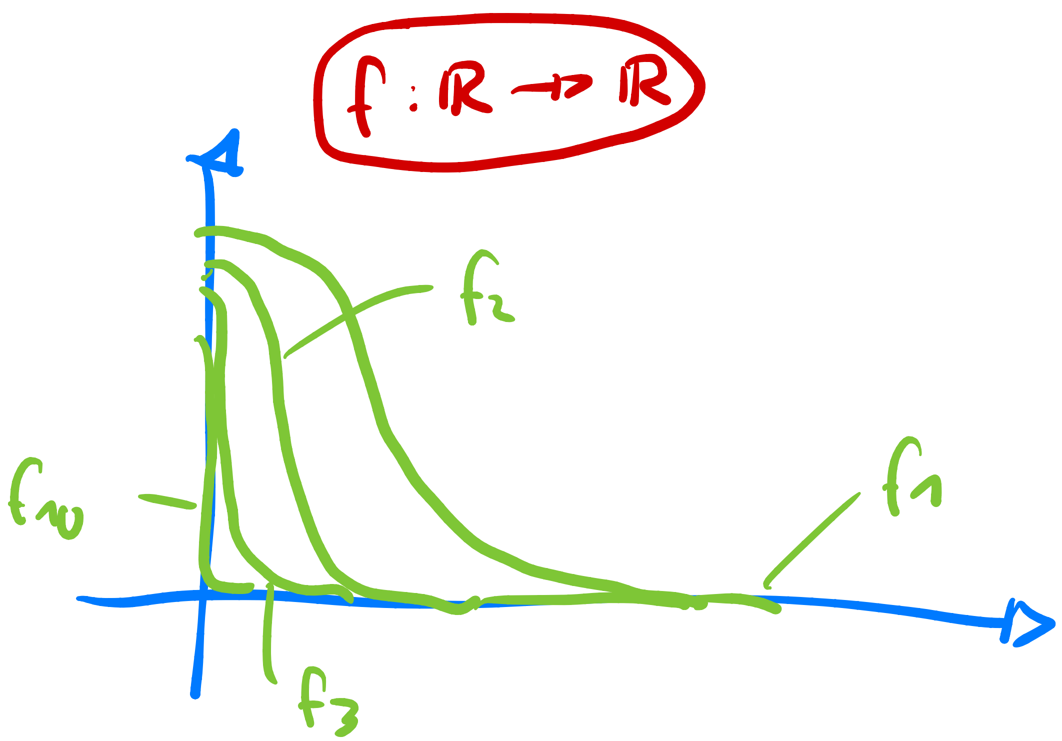
$V = L^2([0,1])$, then $\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}}$ is a norm on V.

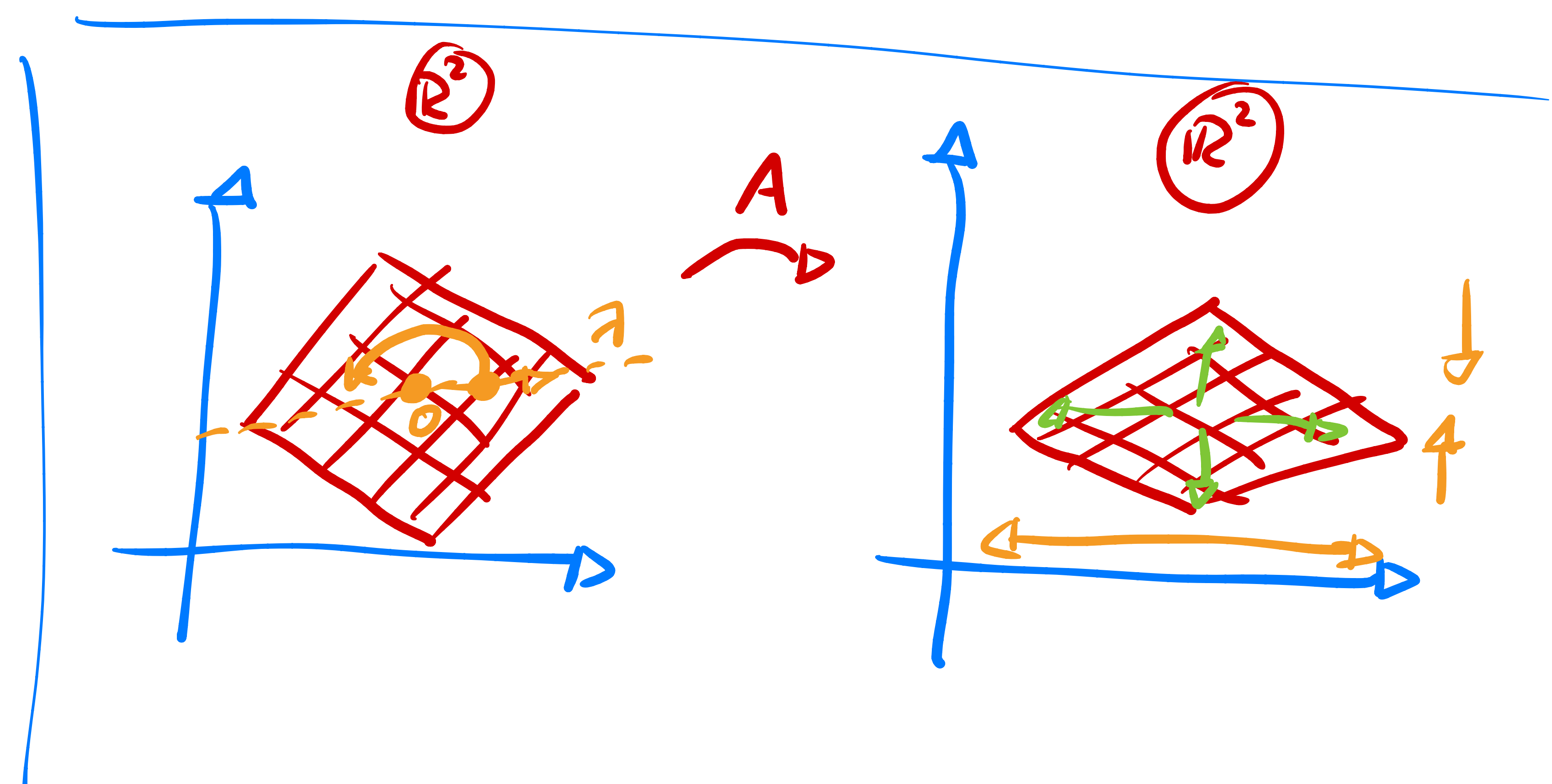
→ Definition of convergence

$|x_n - a| < \varepsilon$









Eigenvalues

Let V be a $\mathbb{K}$ vectorspace. $\lambda \in \mathbb{K}$ is called eigenvalue of a linear mapping $A: V \to V$, if it holds

$$\exists \vec{x} \in V \setminus \{0\}: \quad Ax = \lambda x$$

$\vec{x}$: is called eigenvector for λ.

Eigenvalues for matrices

Let $A \in \mathbb{R}^{m \times n}$. Then the eigenvalues can be computed as follows

$$A\vec{x} = \lambda \vec{x} \iff A\vec{x} - \lambda \vec{x} = \vec{0} \iff \underbrace{(A - \lambda I)}_{}\underbrace{\vec{x}}_{\neq \vec{0}} = \vec{0} \iff \det(A - \lambda I) = 0$$

Charateristic polynomials $\to$ $\det(\lambda I - A) = 0$

Example

Let $A = \begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$. Then $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -3 \\ -3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 9$

$$= \lambda^2 - 3\lambda - 9$$

Hence, continue to compute the zeros of the char. polynomial

$$0 = \lambda^2 - 3\lambda - 9 = \underbrace{\lambda^2 - 2 \cdot \tfrac{3}{2} \lambda + \left(\tfrac{3}{2}\right)^2}_{= \left(\lambda - \frac{3}{2}\right)^2} - \left(\tfrac{3}{2}\right)^2 - 9$$

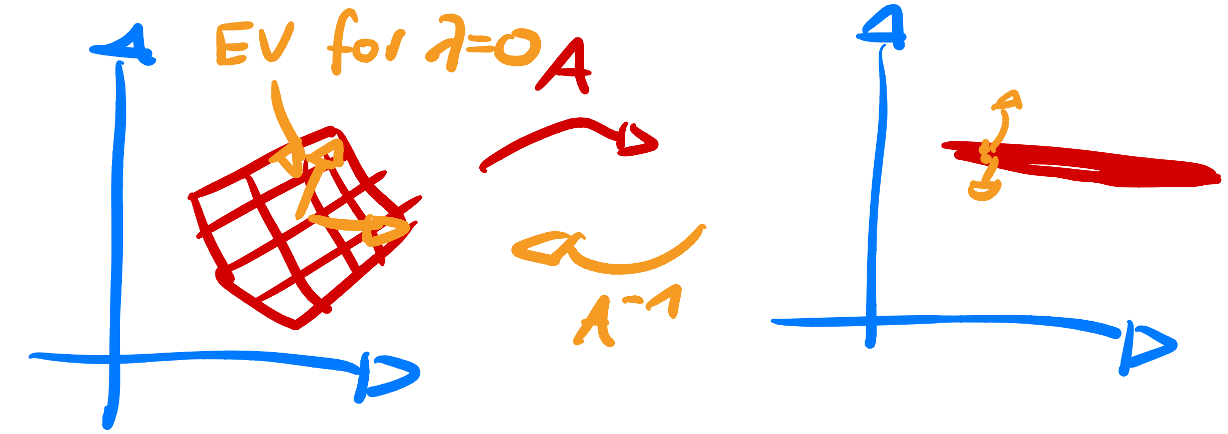
$$\Leftrightarrow \quad \left(\lambda - \tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2 - 9 \overset{!}{=} 0 \quad \Leftrightarrow \quad \lambda_{1/2} = \tfrac{3}{2} \pm \sqrt{\tfrac{45}{4}} = \tfrac{3}{2} \cdot (1 \pm \sqrt{5})$$

The eigenvalue 0

$Ax = 0$

The eigenvalue 0 exists. If 0 occurs as an eigenvalue, A is not invertible anymore

$$Ax = b \rightsquigarrow x = A^{-1} b$$



Example

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix}$$

Start with eigenvalues:

$$\underline{0} \overset{!}{=} \det(M - \lambda I) = \det \begin{pmatrix} (1-\lambda) & 0 & -1 \\ 0 & (1-\lambda) & 0 \\ -1 & 0 & (-2-\lambda) \end{pmatrix}$$

$$= (1-\lambda)^2 \cdot (-2-\lambda) - (1-\lambda)$$

$$= \underline{-\lambda^3 + 4\lambda - 3}$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_{2/3} &= \frac{-1 \pm \sqrt{13}}{2} \end{aligned}}$$

Eigenvectors for $\lambda_1 = 1$

$\vec{x}$ are those vectors which fulfill $(M - \lambda_1 I)\vec{x} = \vec{0}$

$$\Longleftrightarrow \left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, solve this linear system of equations:

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 \end{array}\right) \xrightarrow{\text{Gauss}} \cdots \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \rightsquigarrow \vec{x} \in \overbrace{\left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}}^{\text{Eigenspace}} = U_1$$

Eigenvectors for $\lambda_2 = \frac{-1+\sqrt{13}}{2}$

$$(M - \lambda_2 I)\vec{x} = \vec{0} \quad \Longleftrightarrow \quad \begin{pmatrix} \frac{3-\sqrt{13}}{2} & 0 & -1 \\ 0 & \frac{3-\sqrt{13}}{2} & 0 \\ -1 & 0 & \frac{-3-\sqrt{13}}{2} \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

System of equations:

$$\left.\begin{matrix} 3-\sqrt{13} & 0 & -2 \\ 0 & 3-\sqrt{13} & 0 \\ -2 & 0 & -3-\sqrt{13} \end{matrix}\;\right|\;\begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad \overset{\text{Gauss}}{\rightsquigarrow} \cdots \rightsquigarrow \quad \left.\begin{matrix} 1 & 0 & \frac{\sqrt{13}+3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & t \end{matrix}\;\right|\;\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

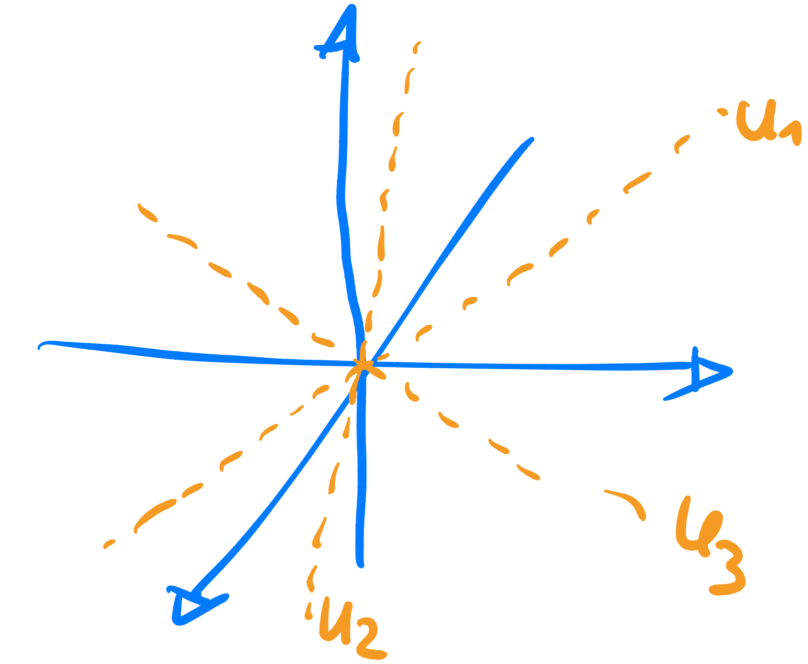
$$x_1 + 0x_2 + \underbrace{\frac{\sqrt{13}+3}{2}t}_{} = 0$$

$$\begin{pmatrix} \frac{\sqrt{13}+3}{2}(-t) \\ 0 \\ t \end{pmatrix} \quad \underbrace{\vec{x} \in \left\{ \begin{pmatrix} \frac{\sqrt{13}+3}{2}t \\ 0 \\ -t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} = U_2}_{\text{Eigenspace for } \lambda_2}$$

Eigenvectors for $\lambda_3 = \frac{-1-\sqrt{13}}{2}$

$$(M - \lambda_3 I)\vec{x} = \vec{0} \iff \begin{pmatrix} \frac{3+\sqrt{13}}{2} & 0 & -1 \\ 0 & \frac{3+\sqrt{13}}{2} & 0 \\ -1 & 0 & \frac{-3+\sqrt{13}}{2} \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss $\rightsquigarrow$ $\vec{x} \in \left\{ \begin{pmatrix} \frac{3-\sqrt{13}}{2} t \\ 0 \\ -t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} = U_3$



$\rightsquigarrow$ Notice that the eigenspaces are vectorspaces

$\rightsquigarrow$ their dimension adds up to the dimension V in case that all EVs are distinct

$\rightsquigarrow$ All eigenvectors belonging to the same eigenvalues form a subspace of V

For the previous example:

U_1, U_2, U_3 are subspaces of $\mathbb{R}^3$ with basis vectors:

$$U_1 \rightsquigarrow \vec{b}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad U_2 \rightsquigarrow \vec{b}_2 = \begin{pmatrix} \frac{\sqrt{13}+3}{2} \\ 0 \\ -1 \end{pmatrix} \quad U_3 \rightsquigarrow \vec{b}_3 = \begin{pmatrix} \frac{3-\sqrt{13}}{2} \\ 0 \\ -1 \end{pmatrix}$$

⇝ One can check that that $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ forms a basis of $\mathbb{R}^3$.

⇝ This only happens if eigenvalues of Matrix are distinct

⇝ In this case, we call the Matrix diagonisabel

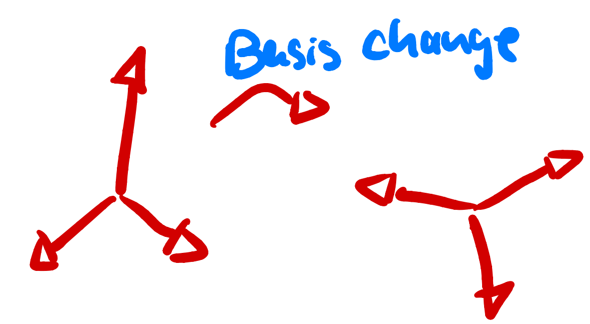
$\downarrow$

In this case, there exists a regular matrix B with $B^{-1} M B = D$

$\uparrow$ Diagonal matrix

$$B = \begin{pmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{pmatrix}$$

$$\boxed{B^{-1} \underline{M} B = D}$$



$$\downarrow$$

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$ (eigenvalues on diagonal)

Eigen decomposition

$\rightsquigarrow$ In case that not all eigenvalues are distinct: Jordan-Normal form.