CSE Refresher Course

08. Oct. 2024

Systems of linear equations

we are interested in systems as

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned} \quad \text{OR} \quad \begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 5 \end{aligned}$$

Types of solution

Such systems can have the following solutions:

1. No solution

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 - x_2 &= 2 \end{aligned} \quad \text{contradiction}$$

2. A unique solution

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 0 \end{aligned}$$

$x_1 = 2 - x_2$ ⇝ $g(x) = 2 - x$

$x_1 = x_2$

$y = x$ ⇝ $f(x) = x$

⇝ $x_1 = x_2 = 1$

$$\mathbb{L} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{(1,1)\}$$

3. Infinitely many solutions

$$\begin{aligned} 3x_1 - 6x_2 &= 2 \\ -x_1 + 2x_2 &= 0 \end{aligned}$$

⇝ Infinitely many solutions

$x_1 = 2t,\ x_2 = t$ for all $t \in \mathbb{R}$

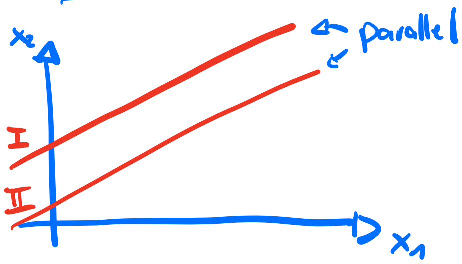
$$\mathbb{L} = \left\{ \begin{pmatrix} 2t \\ t \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R} \right\}$$

Geometrical interpretation

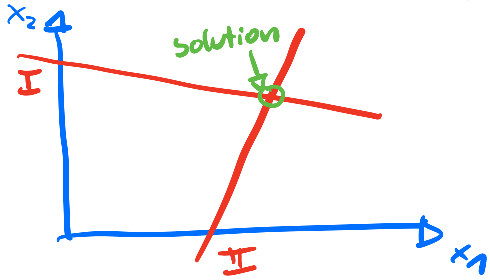
(for 2 unknowns)

Each equation describes a line in $\mathbb{R}^2$

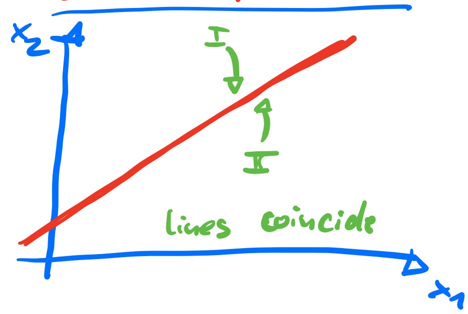
1. no solution



2 only one solution



3. ∞-many solution



⇝ For three unknowns: each equation describes a plane in $\mathbb{R}^3$

⇝ For more unknowns: Hyperplanes in $\mathbb{R}^n$

Finding solutions

use Gauss elimination. Make the notation efficient with coefficient-matrices.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ 2x_1 + 2x_2 - x_3 &= 1 \\ 3x_1 + 4x_2 + 2x_3 &= 2 \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 2 & -1 & 1 \\ 3 & 4 & 2 & 2 \end{array}\right)$$

Start bringing them in triangular shape

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 2 & -1 & 1 \\ 3 & 4 & 2 & 2 \end{array}\right) \begin{array}{l} \\ |-2\mathrm{I} \\ |-3\mathrm{I} \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & -5 & -3 \\ 0 & 1 & -4 & -4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & -5 & -3 \end{array}\right)$$

Backward substitution

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & -5 & -3 \end{array}\right) \begin{array}{l} \\ \\ |:(-5) \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & \frac{3}{5} \end{array}\right) \begin{array}{l} |-2\cdot\mathrm{III} \\ |+4\cdot\mathrm{III} \\ \\ \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & \frac{4}{5} \\ 0 & 1 & 0 & -\frac{8}{5} \\ 0 & 0 & 1 & \frac{3}{5} \end{array}\right) \begin{array}{l} |-\mathrm{II} \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{12}{5} \\ 0 & 1 & 0 & -\frac{8}{5} \\ 0 & 0 & 1 & \frac{3}{5} \end{array}\right)$$

⇝ solution set $\mathbb{L} = \{(\frac{12}{5}, -\frac{8}{5}, \frac{3}{5})\}$

Example for a system with complex coefficients:

$$(1+i)z_1 - z_2 = i$$
$$(1-i)z_1 + (i+1)z_2 = 1$$

$$\left(\begin{array}{cc|c} 1+i & -1 & i \\ 1-i & i+1 & 1 \end{array}\right) \quad \begin{array}{l} \\ \mid +\mathrm{I} \end{array}$$

$$\left(\begin{array}{cc|c} 1+i & -1 & i \\ 2 & i & 1+i \end{array}\right) \quad \begin{array}{l} \mid \cdot (1-i) \\ \\ \end{array}$$

$$\left(\begin{array}{cc|c} 2 & i-1 & i+1 \\ 2 & i & 1+i \end{array}\right) \quad \begin{array}{l} \\ \mid -\mathrm{I} \end{array}$$

could have also done this with less steps

$$\left(\begin{array}{cc|c} 2 & i-1 & i+1 \\ 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} \mid -(i-1)\cdot \mathrm{II} \\ \\ \end{array}$$

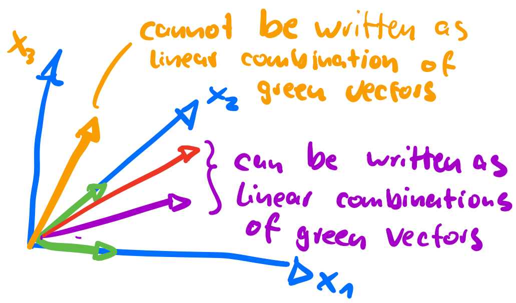
← Backward substitution

$$\left(\begin{array}{cc|c} 2 & 0 & i+1 \\ 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} \mid : 2 \\ \\ \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{i+1}{2} \\ 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} \rightsquigarrow x_1 = \frac{i+1}{2} \\ \rightsquigarrow x_2 = 0 \end{array}$$

$$\rightsquigarrow \mathbb{L} = \left\{ \left(\tfrac{i+1}{2}, 0 \right) \right\}$$

Linear independence



A set of vectors $\vec{v}_1, \dots, \vec{v}_n$ are called linear independent if

$$\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0} \implies \alpha_1 = \dots = \alpha_n = 0$$

Example

Are the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}$ linear independent?

Condition for that:

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ Has to follow that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ for linear independence

Check that with linear systems of equations:
(compute $\alpha_1, \alpha_2, \alpha_3$)

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \mid -\frac{3}{2} \cdot \mathrm{II} \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \rightarrow \alpha_1 - \alpha_3 = 0 \\ \rightsquigarrow \alpha_2 + 4\alpha_3 = 0 \\ 0 = 0 \end{array}$$

$\rightsquigarrow L = \{(t, -2t, t) \mid t \in \mathbb{R}\}$

" $\alpha_3 = t$ "

$\rightsquigarrow$ More solutions than $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$ exist

$\rightsquigarrow$ Linear dependent!

Example:

Determine a, such that $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are linear independent!

$$\left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \mid -2\mathrm{I} \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & 1-2a & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \mid -(1-2a)\mathrm{III} \\ \\ \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & 0 & 1+2a & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1+2a & 0 \end{array}\right)$$

$\rightsquigarrow$ choosing $a = -\frac{1}{2}$ leads to linear dependence
(no unique solution of system possible)

$\rightsquigarrow$ choose $a \neq \frac{1}{2}$: divide III by $(1+2a)$:

$$\left(\begin{array}{ccc|c} 1 & a & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \mid +\mathrm{III} \\ \mid -\mathrm{III} \\ \\ \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \mid -a\cdot\mathrm{II} \\ \\ \\ \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$A \qquad 0$

$\hookrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\rightarrow$ linear independent!

A system $A|0$ is solveable if and only if $A|b$ (same system with aritrary right hand side) is solveable.

Hence $A|b$ is solveable if columns of A are linear independent

M

Matrices

A matrix is a rectangular array or table of numbers, symbols, expressions such as

$$A = \overbrace{\begin{pmatrix} 1 & 9 & -2 \\ 3 & -1 & \frac{3}{2} \end{pmatrix}}^{3} \left.\vphantom{\begin{matrix} 1 \\ 3 \end{matrix}}\right\} 2$$

Dimension of the Matrix A: $\underset{\text{no. of rows}}{2} \times \underset{\text{no. of columns}}{3}$ „A is a 2 times 3 matrix"

The set of all matrices with m rows and n columns is denoted as $\mathbb{K}^{m \times n}$

$\mathbb{K} \to$ Entries of the matrices are elements of $\mathbb{K}$.

The entry of A in the i-th row and j-th column is denoted as $A_{i,j}$ / a_{ij}

Matrix-Vector multiplication

Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$. Then we define the matrix-vector multiplication by

$$Ax = \begin{pmatrix} \sum_{i=1}^{n} a_{1i} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} x_i \end{pmatrix}$$

Dimension of matrix an vector have to fit!

Example

$$A = \begin{pmatrix} 1 & 9 & -2 \\ 3 & -1 & \frac{3}{2} \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow Ax = \begin{pmatrix} 1 \cdot 1 + 9 \cdot 1 + (-2) \cdot 2 \\ 3 \cdot 1 + (-1) \cdot 1 + \frac{3}{2} \cdot 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$\leadsto$ We can use this in order to rewrite systems of equations!

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_2 + 2x_3 &= 0 \\ x_1 \quad - x_3 &= -1 \end{aligned} \iff \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}}_{\text{coefficient matrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\text{right hand side}}$$

$\rightsquigarrow$ compare this with the coefficient notation from before

$$Ax = b \quad \rightsquigarrow \quad A \mid b$$

„matrix vector multiplication" „coefficient notation"

Matrix-Matrix multiplication

Let $X \in \mathbb{K}^{l \times m}$, $Y \in \mathbb{K}^{m \times n}$. Then the matrix product of X and Y is given as

$$XY = \begin{pmatrix} | & & | \\ X y_1 & , \dots , & X y_n \\ | & & | \end{pmatrix} \in \mathbb{K}^{\ell \times n}$$

where $y_1, \dots, y_n \in \mathbb{K}^m$ denote the columns of Y.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1-2+9 & 1+4-3 \\ 4-5+18 & 4+10-6 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 17 & 8 \end{pmatrix}$$

Some special matrices

- unit matrix / identity matrix:

$$I_{(n)} = \left.\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\} n \text{ rows} = E_{(n)}$$

$\underbrace{\qquad\qquad}_{n \text{ columns}}$

- Transposed matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \qquad [A \in \mathbb{R}^{2\times 3} \rightsquigarrow A^T \in \mathbb{R}^{3\times 2}]$$

- Adjoint matrix (in case that $B \in \mathbb{R}^{m\times n} \Rightarrow B^H = B^T$)

$$B = \begin{pmatrix} 1 & i \\ 1+i & -1 \\ 2 & 3-i \end{pmatrix} \Rightarrow B^H = (\overline{B})^T = \begin{pmatrix} 1 & 1-i & 2 \\ -i & -1 & 3+i \end{pmatrix}$$

- Inverse matrix

Let $A \in \mathbb{K}^{n\times n}$. Then the inverse matrix is defined as the matrix which satisfies $A^{-1}A = AA^{-1} = I_n$

Does A^{-1} exist?

$$\left(\begin{aligned} Ax &= b \\ \underbrace{A^{-1}A}_{=I}x &= A^{-1}b \\ x &= A^{-1}b \end{aligned}\right)$$

$$AA^{-1} = I_n$$

$$\rightsquigarrow \quad A \begin{pmatrix} | & & | \\ A^{-1}_1 & \cdots & A^{-1}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$\rightsquigarrow \quad \underbrace{A A^{-1}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \ldots, \; A A^{-1}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\text{multiple execution of Gauss}}$$

$\rightsquigarrow$ Each system of equations can be used to compute the i-th column of A^{-1}

Instead of solving the same system with only different right hand sides, we solve it once with multiple right hand sides at the same time.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \mid -3\,\mathrm{I} \end{array} \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \begin{array}{l} \mid +\,\mathrm{II} \\ \\ \end{array} \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \begin{array}{l} \\ \mid :(-2) \end{array}$$

$$\leadsto \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

1st column of A^{-1} — second column of A^{-1}

$$\leadsto A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Determinants

Lets compute the inverse matrix again for general $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Assume $a \neq 0$ otherwises swap rows

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \begin{array}{l} \\ \mid - \frac{c}{a} \cdot \mathrm{I} \end{array} \leadsto \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{da-cb}{a} & -\frac{c}{a} & 1 \end{array}\right) \begin{array}{l} \mid : a \\ \mid : \frac{da-cb}{a} \end{array}$$

$$\leadsto \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{da-cb} & \frac{a}{da-cb} \end{array}\right) \begin{array}{l} \mid - \frac{b}{a} \cdot \mathrm{II} \\ \\ \end{array} \leadsto \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-cb} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$\hookrightarrow A^{-1} = \frac{1}{\boxed{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\neq 0$

$\leadsto$ can determine the inverse of $A \in \mathbb{R}^{2\times 2}$, if $ad-bc \neq 0$

$\leadsto$ Define $\det(A) := ad-cb$ $(= |A|)$. Then A is invertible if $\det(A) \neq 0$

Determinant $\qquad A \in \mathbb{R}^{2\times 2}$

We can extend this intuition from 2×2 matrices to general $n\times n$ matrices as well:

- $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

- $$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{33}$$
$$- \left(a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12} \right)$$

<u>General rule for $A \in \mathbb{K}^{n\times n}$:</u>

Let $\tilde{A}_{ij}$ the submatrix which we get after ignoring the i-th row and j-th column, i.e. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $\tilde{A}_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$, $\tilde{A}_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$

Using this, the determinant of $A \in \mathbb{K}^{n\times n}$ is given as

$$\det(A) = \sum_{i=1}^{n} (-1)^{(i+1)} A_{ji} \det\left(\tilde{A}_{ij}\right) = \sum_{i=1}^{n} (-1)^{(i+1)} A_{ij} \det\left(\tilde{A}_{ji}\right)$$

for j an arbitrary integer $1 \leq j \leq n$.

Example

$$\det \begin{array}{c} \begin{matrix} j=1 \\ \downarrow \end{matrix} \hspace{5em} \\ \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 3 \end{pmatrix} \end{array} = (+1)\cdot \det\begin{pmatrix} 1 & 7 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \cancel{- 0\cdot\det\begin{pmatrix} 2 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}}$$

$$\cancel{+ 0\cdot\det\begin{pmatrix} 2 & 3 & 3 \\ 1 & 7 & -1 \\ 0 & 0 & 3 \end{pmatrix}} - 1\cdot\det\begin{pmatrix} 2 & 3 & 3 \\ 1 & 7 & -1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 7 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} - \det\begin{pmatrix} 2 & 3 & 3 \\ 1 & 7 & -1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$= 1\cdot 2\cdot 3 - \left(2\cdot\det\begin{pmatrix} 7 & -1 \\ 2 & 2 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\right)$$

$$= 6 - (2\cdot 16 - 1\cdot 0) = 6 - 32 = \underline{\underline{-26}}$$

$\rightsquigarrow$ Matrix is invertible

Example

check whether $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ are linear independent.

⇝ First possibility: solve

$$\left(\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right)$$

A

⇝ second: compute

$$\det(A) = \det\begin{pmatrix} -1 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = -1 \cdot \det\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} - 1 \cdot \det\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$

$$= (-1) \cdot (-1-4) - (1-6) + (2+3)$$

$$= 15 \neq 0$$

⇝ vectors linear independent.

Survival rule for computing determinants of large matrices:

Use Gauss-elimination first!

You can use Gauss-elimination during the computation of determinants with two differences to normal:

- Swap rows or columns, the sign of the det changes
- Do not multiply a row with a scalar without adding it to another row

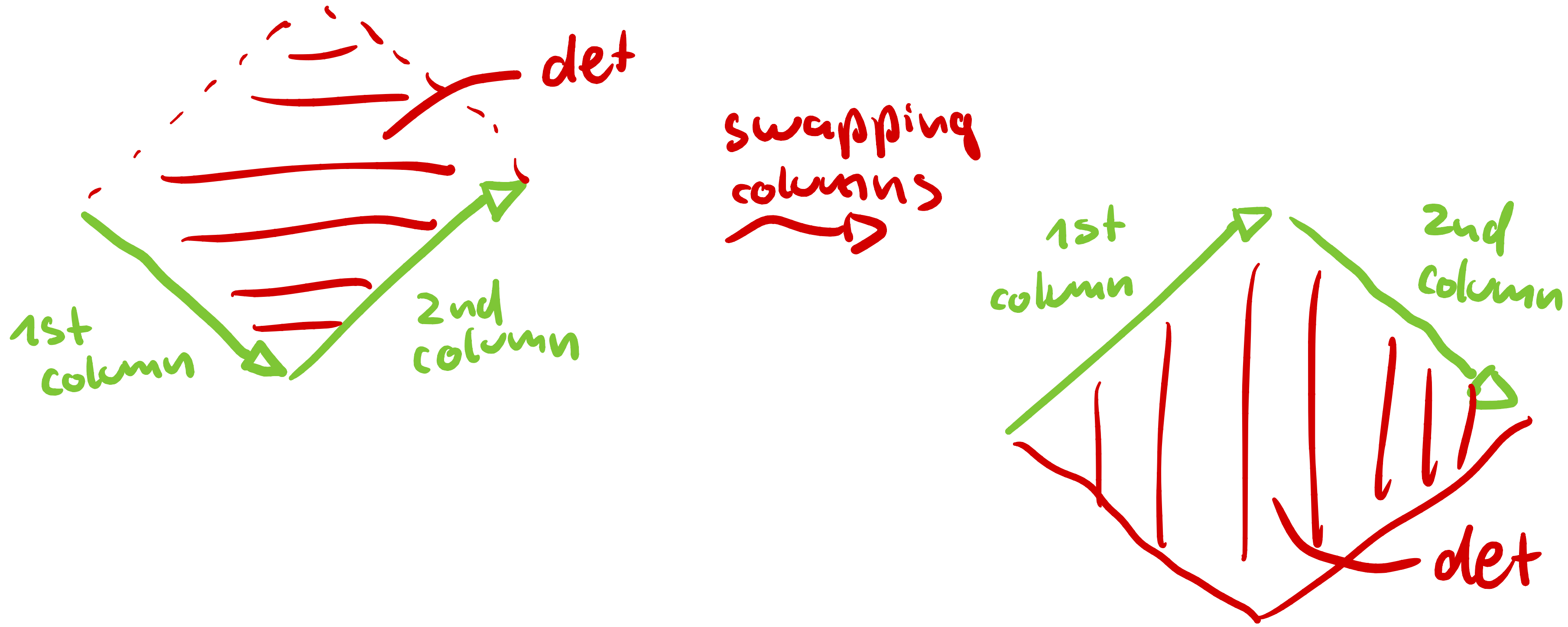
$$\det\begin{pmatrix}-1 & 1 & 3\\ 1 & -1 & 2\\ 1 & 2 & 1\end{pmatrix}\begin{array}{l} \\ +\mathrm{I} \\ +\mathrm{I}\end{array} \rightsquigarrow \det\begin{pmatrix}-1 & 1 & 3\\ 0 & 0 & 5\\ 0 & 3 & 4\end{pmatrix} = -\det\begin{pmatrix}-1 & 1 & 3\\ 0 & 3 & 4\\ 0 & 0 & 5\end{pmatrix} = -((-1)\cdot 3\cdot 5)$$

$$= \underline{\underline{15}}$$

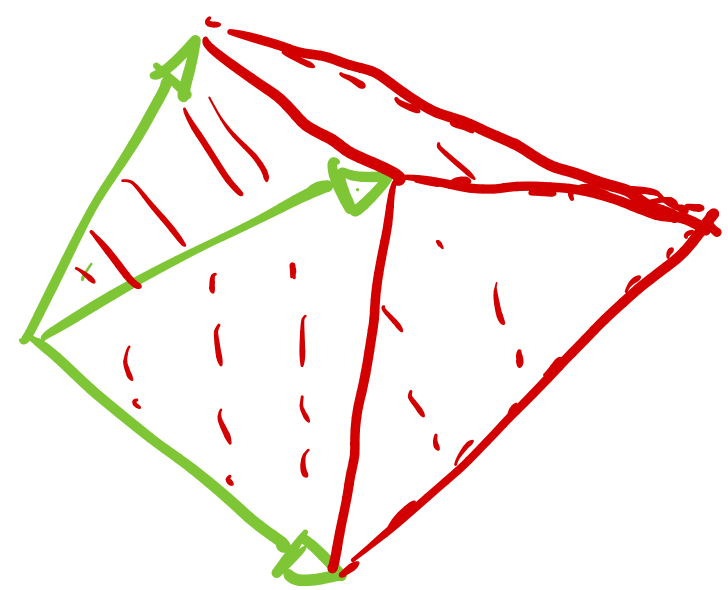
Geometric interpretation

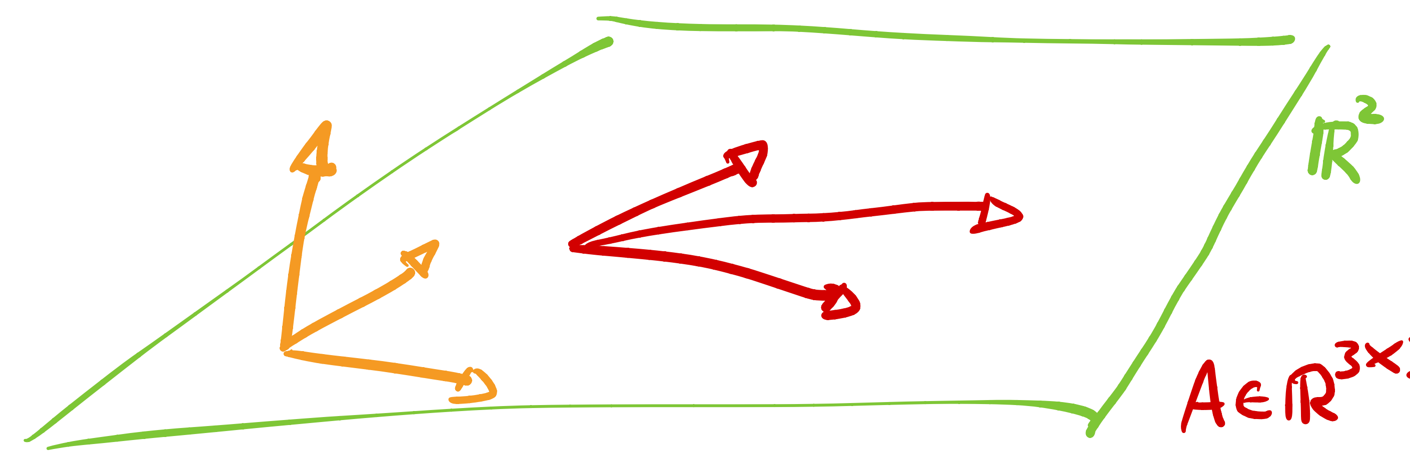
Determinants describe the area spanned by its columns!

in $\mathbb{R}^2$

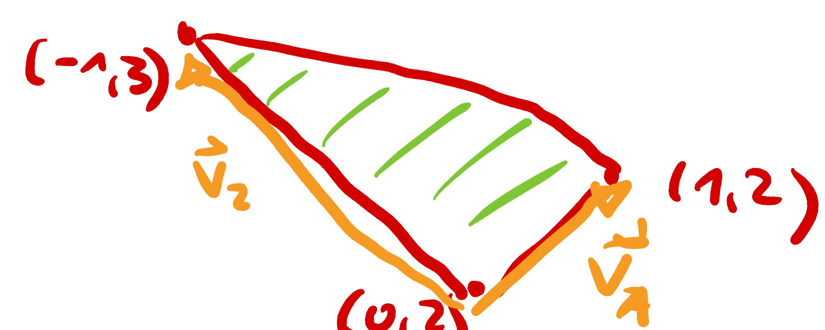


in $\mathbb{R}^3$





Task: compute the area enclosed by the triange with the corners $(1,2), (0,2), (-1,3)$.



⇝ need vectors which describe the edges:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\text{Area} = \frac{1}{2} \cdot \left| \det(\vec{v}_1, \vec{v}_2) \right| = \frac{1}{2} \cdot \left| \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right| = \frac{1}{2} \cdot 1 = \underline{\underline{\frac{1}{2}}}$$

Summary:

↳ Following properties are equivalent:

- A invertible (regular/non-singular)
- $\det(A) \neq 0$
- The rows/columns of A are linear independent
- $A|b$ can be solved uniquely for each right hand side b
- $Ax = 0$ has only the solution $x = 0$

Vector spaces

A vectorspace over a field $\mathbb{K}$ is a set V and a field $\mathbb{K}$, equipped with two operations:

1. Vector addition „$+$": $V \times V \to V$ which fulfills for all $\vec{u}, \vec{v}, \vec{w} \in V$:
   - $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ „associative"
   - $\exists\, \vec{0} \in V: \ \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ „neutral element"
   - $\exists\, (-\vec{v}) \in V: \ \vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ „inverse element"
   - $\vec{v} + \vec{u} = \vec{u} + \vec{v}$ „commutative"

2. Scalar multiplication „$\cdot$": $\mathbb{K} \times V \to V$ which fulfills for all $\vec{u}, \vec{v} \in V$, $\alpha, \beta \in \mathbb{K}$:
   - $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$

distributive laws

- $(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
- $(\alpha\cdot\beta)\vec{v} = \alpha\cdot(\beta\cdot\vec{v})$
- $\mathbb{1}\cdot\vec{v} = \vec{v}$ (where $\mathbb{1}$ is the netral element in $\mathbb{K}$)

Example 1

space of n-tuples in $\mathbb{R}^n$:

Elements in $\mathbb{R}^n$: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, Addition: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$,

Multiplication: $\underbrace{\alpha\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\text{Multiplication in } \mathbb{R}^n} := \begin{pmatrix} \overbrace{\alpha x_1}^{\text{Multiplication in } \mathbb{R}} \\ \vdots \\ \alpha x_n \end{pmatrix}$

↳ You can check that all axioms are fulfilled.

Example 2

Space of all functions from $\mathbb{R}$ to $\mathbb{R}$, i.e:

$$\mathbb{K} = \mathbb{R}, \quad V := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a function} \}$$

Addition: $(f+g)(x) := f(x) + g(x)$

Multiplication: $(\alpha f)(x) := \alpha \cdot f(x)$