CSE maths refresher course

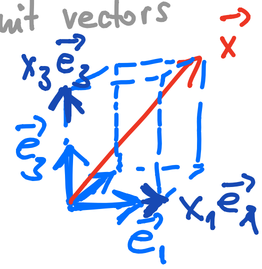
Fri 11th Oct

Fourier series

Projections in $L_2([0,2\pi])$ ← infinitely dimensional function space

rep. projection in $\mathbb{R}^d$ ← d-dimensional Euklidean space

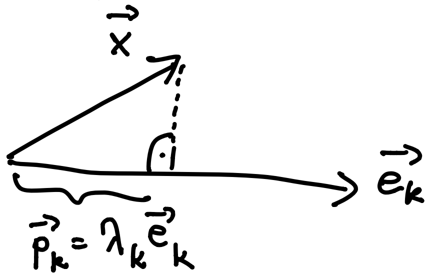
$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_d \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \sum_{k=1}^{d} x_k \underbrace{\vec{e}_k}_{\text{unit vectors}}$$



$\vec{p}_k = x_k \vec{e}_k$ is the projection of $\vec{x}$ onto $\vec{e}_k$

λ_k? $\quad \vec{e}_k \perp \vec{x} - \lambda_k \vec{e}_k$

$\vec{e}_k \cdot (\vec{x} - \lambda_k \vec{e}_k) = 0 \iff \vec{e}_k \cdot \vec{x} = \lambda_k \vec{e}_k \cdot \vec{e}_k$



$$\vec{p}_k = \lambda_k \vec{e}_k = \boxed{\frac{\langle \vec{x}, \vec{e}_k \rangle_{\mathbb{R}^d}}{\| e_k \|_2^2} \vec{e}_k} = x_k \vec{e}_k$$

again in a more general form: Let $\{\vec{v}_1, \dots, \vec{v}_d\}$ be an orthonormal system

↑ orthogonal $\vec{v}_k \perp \vec{v}_\ell$ if $k \neq \ell$ ← length 1, $\|\vec{v}_k\|_2 = 1$

linear combination $\vec{x} = \lambda_1 \vec{v}_1 + \dots + \lambda_d \vec{v}_d \quad ?\ \lambda_1, \dots, \lambda_d$

multiply by $\cdot \vec{v}_k$: $\langle \vec{x}, \vec{v}_k \rangle_{\mathbb{R}^d} = \lambda_1 \langle \vec{v}_1, \vec{v}_k \rangle_{\mathbb{R}^d} + \dots + \lambda_d \langle \vec{v}_d, \vec{v}_k \rangle_{\mathbb{R}^d}$

$= \lambda_1 \delta_{1,k} + \dots + \lambda_d \delta_{d,k} = \lambda_k$

with Kronecker symbol

$$\delta_{k,\ell} = \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{if } k = \ell \end{cases}$$

i.e. $\langle \vec{x}, \vec{v}_k \rangle_{\mathbb{R}^d} = \lambda_k$

$$\vec{x} = \langle \vec{x}, \vec{v}_1 \rangle_{\mathbb{R}^d} \vec{v}_1 + \dots + \langle \vec{x}, \vec{v}_d \rangle_{\mathbb{R}^d} \vec{v}_d$$

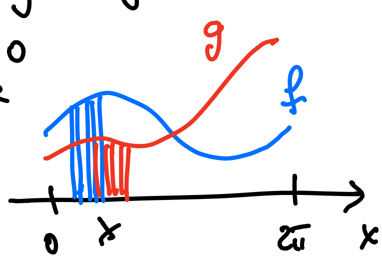
$$\boxed{\vec{x} = \sum_{k=1}^{d} \underbrace{\langle \vec{x}, \vec{v}_k \rangle \vec{v}_k}_{\text{projection of } \vec{x} \text{ onto } \vec{v}_k}}$$

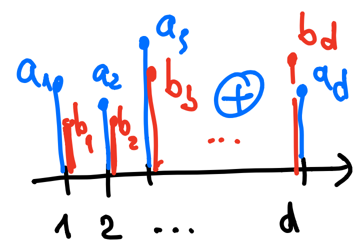
rep. Lebesgue function space $L_2([0,2\pi])$

contains all "functions" $f: [0, 2\pi] \to \mathbb{R}$ or $\mathbb{C}$ mit $\int_0^{2\pi} |f(x)|^2 dx < \infty$ ← energy

Scalar product $\langle f, g \rangle_{L_2([0,2\pi])} = \int_0^{2\pi} f(x) \overline{g(x)}\, dx$

$\mathbb{C}^d$: $\langle \vec{a}, \vec{b} \rangle = \sum_{k=1}^{d} a_k \overline{b_k}$



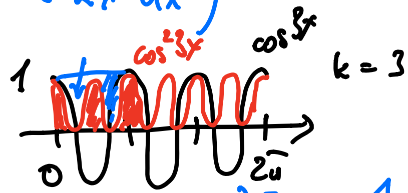


norm $\|f\|_{L_2([0,2\pi])} = \sqrt{\langle f, f \rangle_{L_2([0,2\pi])}}$ ← compare

$= \left(\int_0^{2\pi} |f(x)|^2 dx \right)^{1/2}$

$= \left(\int_0 |f(x)| \, dx \right)$

Exp. $f(x) = \cos kx$, $k \in \mathbb{Z}$, $k \neq 0$, $\|f\|_{L_2([0,2\pi])} = \left(\int_0^{2\pi} \cos^2 kx \, dx \right)^{1/2} = \sqrt{\pi}$



$\cos^2 3x + \sin^2 3x = 1$

projection of $g \in R$ onto $f(x) = \cos kx$

$$\lambda f(x) = \frac{\langle g, f \rangle_{L_2([0,2\pi])}}{\|f\|^2_{L_2([0,2\pi])}} f(x)$$

$$= \underbrace{\frac{1}{\pi} \int_0^{2\pi} g(x) \cos kx \, dx}_{\lambda \text{ coefficient}} \cdot \cos kx$$

how strongly $f(x) = \cos kx$ is contained in g.

Real Fourier series

Joseph Fourier (1768 Auxerre – 1830 Paris)

Theorem: The functions $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{\pi}} \sin kx : k = 1, 2, \dots \right\}$ form an orthonormal system in $L_2([0,2\pi])$.

Proof: $\left\langle \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{\pi}} \cos \ell x \right\rangle_{L_2([0,2\pi])} =$

$$= \frac{1}{\pi} \int_0^{2\pi} \cos kx \cdot \cos \ell x \, dx = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\cos(kx + \ell x)}_{\cos kx \cos \ell x - \cancel{\sin kx \sin \ell x}} + \underbrace{\cos(kx - \ell x)}_{\cos kx \cos \ell + \cancel{\sin kx \sin \ell x}} \, dx$$

$$= \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \cos(k+\ell)x \, dx}_{=0 \text{ due to } k, \ell \geq 1} + \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \cos(k-\ell) x \, dx}_{\substack{=0 \text{ for } k-\ell \neq 0 \\ = 2\pi \text{ for } k-\ell = 0}} = \delta_{k,\ell}$$

and $\left\langle \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{2\pi}} \right\rangle_{L_2([0,2\pi])} = \frac{1}{\pi\sqrt{2}} \int_0^{2\pi} \cos kx \, dx = 0$

and etc. for all over combinations. □

Theorem: The functions $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{\pi}} \sin kx : k = 1, 2, \dots \right\}$ span a linear hull, which is dense in $L_2([0,2\pi])$. (without proof)

i.e. every function $f \in L_2([0,2\pi])$ can be approximated by a function from the linear hull with arbitrary accuracy.

$$f(x) = \underbrace{\left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle_{L_2} \frac{1}{\sqrt{2\pi}}}_{\frac{a_0}{2}} + \sum_{k=1}^{\infty} \underbrace{\left\langle f, \frac{1}{\sqrt{\pi}} \cos kx \right\rangle_{L_2} \frac{1}{\sqrt{\pi}}}_{a_k} \cos kx + \sum_{k=1}^{\infty} \underbrace{\left\langle f, \frac{1}{\sqrt{\pi}} \sin kx \right\rangle_{L_2} \frac{1}{\sqrt{\pi}}}_{b_k} \sin kx$$

Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum^{\infty} a_k \cos kx + b_k \sin kx}$$

2π

Fourier coefficients $\frac{a_0}{2} = \langle f, \frac{1}{\sqrt{2\pi}} \rangle_{L_2} \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx$, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$ (*)

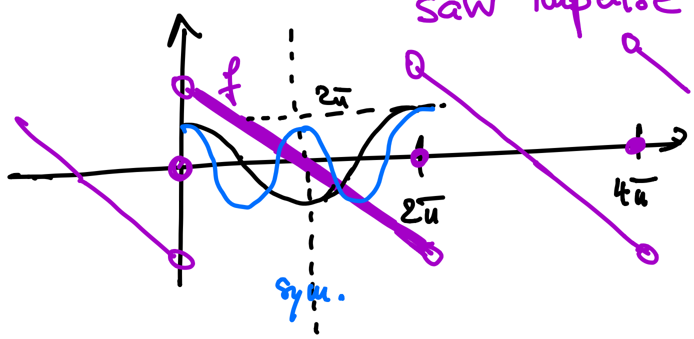
and $a_k = \langle f, \frac{1}{\sqrt{\pi}} \cos kx \rangle_{L_2} \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx$

$b_k = \langle f, \frac{1}{\sqrt{\pi}} \sin kx \rangle_{L_2} \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx$

<u>Exp.</u>

$$f(x) = \begin{cases} 0 & \text{for } x = 2l\pi \\ \frac{1}{2}(\pi - x) & \text{for } x \in (0, 2\pi) \\ 2\pi\text{ - periodically expanded} \end{cases}$$

saw impulse



$$a_k = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2}(\pi - x) \cos kx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{2}(\pi - x) \cos kx \, dx + \underbrace{\frac{1}{\pi} \int_{\pi}^{2\pi} \frac{1}{2}(\pi - x) \cos kx \, dx}_{\text{substitution } x = 2\pi - z}$$ with $\frac{dx}{dz} = -1$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi - x}{2} \cos kx \, dx - \frac{1}{\pi} \int_{\pi}^{0} \frac{z - \pi}{2} \underbrace{\cos(k(2\pi - z))}_{\cos kz} \, dz = \underline{\underline{0}}$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} \underbrace{\frac{\pi - x}{2}}_{u} \underbrace{\sin kx}_{v'} \, dx \overset{\text{i. by parts}}{=} \ldots = \frac{x - \pi}{2\pi} \frac{\cos kx}{k} \Big|_{x=0}^{2\pi} - \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos kx}{k} \, dx}_{u'v} = 0$$

with $u' = -1$, $v = -\frac{\cos kx}{k}$

$$= \underbrace{\frac{1}{2\pi}(2\pi - \pi)}_{1/2} \frac{\overset{=1}{\cos 2\pi k}}{k} - \underbrace{\frac{1}{2\pi}(0 - \pi)}_{-1/2} \frac{\overset{=1}{\cos 0}}{k}$$

$= \frac{1}{k}$ i.e. $b_k = \frac{1}{k}$

frequencies

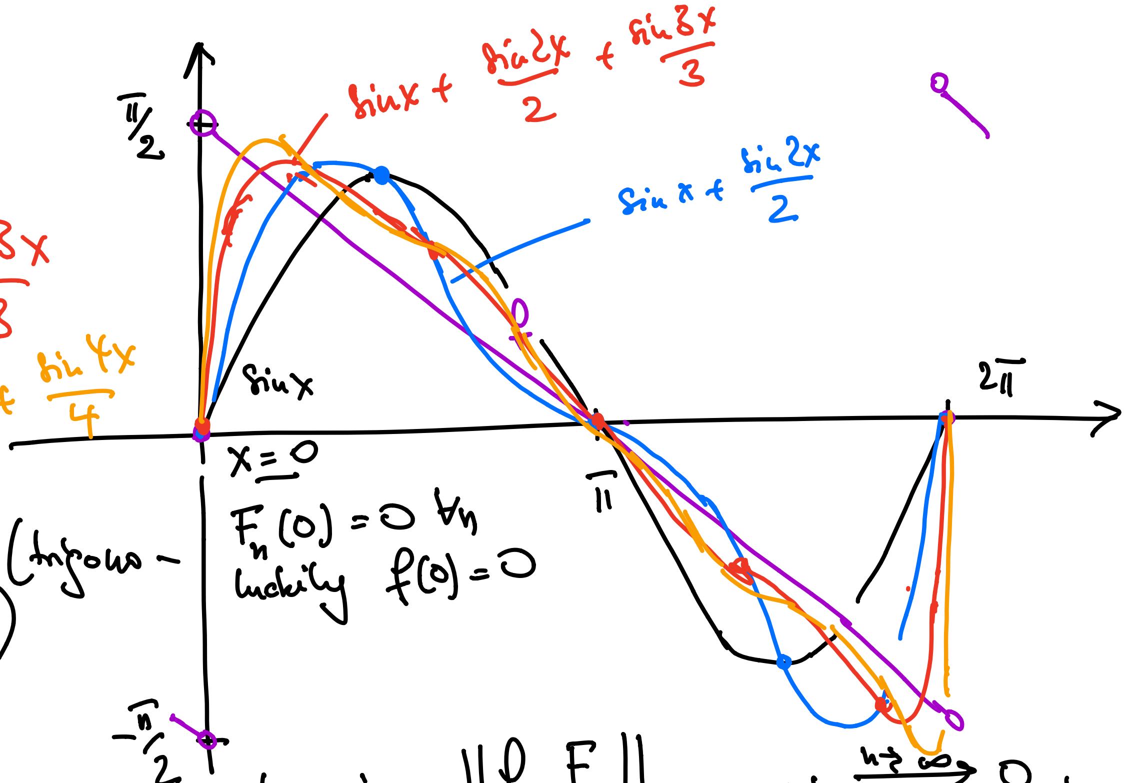
$$\Rightarrow f(x) = \sum_{k=1}^{\infty} b_k \sin kx = \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots$$

cf. divergent harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \to \infty$

partial sums

- $\sin x$
- $\sin x + \frac{\sin 2x}{2}$
- $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3}$
- ——— " ——— $+ \frac{\sin 4x}{4}$

....



$F_n(0) = 0 \ \forall n$
luckily $f(0) = 0$

Fourier polynomials (trigonometric polynomials)

$$F_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$$

converge to f in the sense of L_2, i.e. $\|f - F_n\|_{L_2([0,2\pi])} \xrightarrow{n\to\infty} 0$,

Complex Fourier series

- wikipedia : concepts
- scripts : calculus 1 & 2
- textbooks on calculus
  e.g. tu-bs.de → library
  → Springer Link
  basic idea
- So einfach ist Mathematik ...
  ODE / PDE

$f : [0, 2\pi] \to \mathbb{C}$

Theorem : The functions $\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx}, k \in \mathbb{Z} \right\}$ form an orthonormal system.

Proof: $\left\langle \frac{1}{\sqrt{2\pi}} e^{ikx}, \frac{1}{\sqrt{2\pi}} e^{i\ell x} \right\rangle_{L_2([0,2\pi])} =$

$$\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{ikx} \cdot \overline{\frac{1}{\sqrt{2\pi}} e^{i\ell x}} \, dx = \int_0^{2\pi} \frac{1}{2\pi} e^{ikx} e^{-i\ell x} \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-\ell)x} \, dx = \begin{cases} 0 & \text{for } k \neq \ell \\ \frac{1}{2\pi} \int_0^{2\pi} 1 \, dx = 1 & \text{for } k = \ell \end{cases}$$

$$e^{i(k-\ell)x} \overset{\text{comp.}}{=} \cos(k-\ell)x + i \sin(k-\ell)x$$

$$= \delta_{k,\ell} \qquad \square$$

again projection

$$f(x) = \sum_{k \in \mathbb{Z}} \left\langle f, \underbrace{\frac{1}{\sqrt{2\pi}} e^{ikx}} \right\rangle_{L_2([0,2\pi])} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{ikx}}$$

functions in the "base"

$$= \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{2\pi} \left\langle f, e^{ikx} \right\rangle}_{c_k} e^{ikx}$$

conjugate complex number

complex Fourier coefficient $\quad c_k = \frac{1}{2\pi}\int_0 f(x)\, e^{-ikx}\, dx$

Fourier coefficients

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{ikx} + c_{-k} e^{-ikx}\right)$$

compare $\quad \ldots = + c_{-2}e^{-2ix} + c_{-1}e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{2ix} + \ldots$

$$\ldots = c_0 + (c_1 e^{ix} + c_{-1}e^{-ix}) + (c_2 e^{2ix} + c_{-2}e^{-2ix}) + \ldots$$

$$= c_0 + \sum_{k=1}^{\infty} c_k(\cos kx + i\sin kx) + c_{-k}(\cos kx - i\sin kx)$$

$$= \underbrace{c_0}_{\frac{a_0}{2}} + \sum_{k=1}^{\infty} \underbrace{(c_k + c_{-k})}_{a_k}\cos kx + \underbrace{i(c_k - c_{-k})}_{b_k}\sin kx$$

$$\Rightarrow \quad \frac{a_0}{2} = c_0, \quad a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k}) \quad \text{for } k = 1,2,\ldots$$

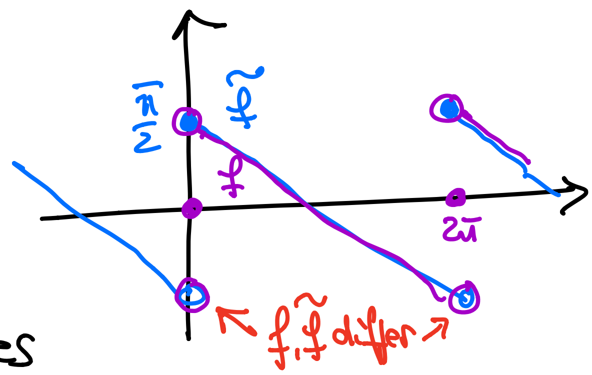
and $\quad c_k = \frac{1}{2}(a_k - ib_k), \quad c_{-k} = \frac{1}{2}(a_k + ib_k) \quad$ for $k = 1,2,\ldots$

Exp. $f(x) = \begin{cases} 0 & \text{for } x = 2\ell\pi \\ \frac{1}{2}(\pi - x) & \text{for } x\in(0,2\pi) \\ 2\pi\text{-periodically extended} \end{cases}$ with $a_k = 0$, $b_k = \frac{1}{k}$

$$\Rightarrow \quad c_0 = 0, \quad c_k = \frac{-i}{2k}, \quad c_{-k} = \frac{i}{2k} = \frac{-i}{2(-k)}, \quad f(x) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{-i}{2k} e^{ikx}$$

Remark $\quad \tilde{f}(x) = \begin{cases} \frac{\pi - x}{2} & \text{for } x\in[0,2\pi) \\ 2\pi\text{-periodically extended} \end{cases}$

$f(0) = 0, \quad \tilde{f}(0) = \frac{\pi}{2}$



$\tilde{f}(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{x}$ $\quad$ same Fourier series

not pointwise but in the sense of L_2, i.e.

$$f \overset{L_2}{=} g \iff \|f - g\|_{L_2([0,2\pi])} = \left(\int_0^{2\pi} |f(x) - g(x)|^2\, dx\right)^{1/2} = 0$$

single points do not affect the integral

Fourier series converge in the sense of L_2, typically not pointwise ∇

derivatives, $f \in C^1([0,2\pi])$, i.e. once differentiable function f

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$\left(a_k \xrightarrow{k\to\infty} 0,\ b_k \xrightarrow{k\to\infty} 0 \right)$$

$$f'(x) = \sum_{k=-\infty}^{\infty} ik\, c_k e^{ikx} = \sum_{k=1}^{\infty} -k a_k \sin kx + k b_k \cos kx$$

$$\left(k a_k \xrightarrow{k\to\infty} 0,\ k b_k \xrightarrow{k\to\infty} 0 \right)$$

The smoother the function f is,
the faster decay the Fourier coefficients.

Frequencies

regard only one summand of Fourier series $g(x) = a_k \cos \underline{k}x + b_k \sin \underline{k}x$

harmonic oscillation, $a_k, b_k \in \mathbb{R}$ — frequency k

ansatz $g(x) = A_k \cos(kx - \varphi_k)$, amplitude A_k, phase φ_k

$$A_k \cos(kx - \varphi_k) = \underbrace{A_k \cos \varphi_k}_{a_k} \cos kx + \underbrace{A_k \sin \varphi_k}_{b_k} \sin kx$$

$$a_k = A_k \cos \varphi_k, \quad b_k = A_k \sin \varphi_k$$

$$A_k = \sqrt{a_k^2 + b_k^2} = \sqrt{(A_k \cos\varphi_k)^2 + (A_k \sin \varphi_k)^2} = A_k \sqrt{\cos^2\varphi_k + \sin^2 \varphi_k}$$

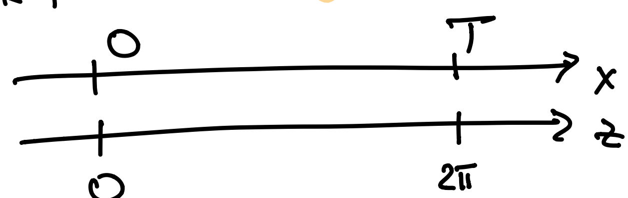
$$\tan \varphi_k = \frac{\sin \varphi_k}{\cos \varphi_k} = \frac{b_k}{a_k} \quad \text{or} \quad \varphi_k = \arctan \frac{b_k}{a_k}$$

More general periodic functions

g might be a function of period T, $g(x) = g(x+T) \ \forall x$

Search for $$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{2\pi k x}{T} + b_k \sin \frac{2\pi k x}{T}$$

coordinate transformation



$$z = \frac{2\pi}{T} x$$

$x = 0 \mapsto z = 0$, $x = T \mapsto z = 2\pi$

$f(z) = g(x)$ is 2π-periodic function

Fourier coefficients of f: $\tilde{a}_k(f) = \frac{1}{\pi} \int^{2\pi} f(z) \cos kz \, dz$

Fourier coefficients of f

substitution $x = \frac{T}{2\pi} z$, $dx = \frac{T}{2\pi} dz$

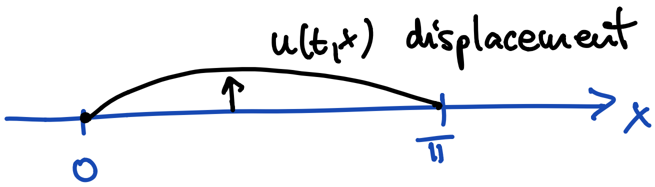
$$a_k(g) = \frac{1}{\cancel{\pi}} \int_0^T g(x) \cos \frac{2\pi k x}{T} \cdot \frac{2\cancel{\pi}}{T} \, dx$$

$$= \frac{2}{T} \int_0^T g(x) \cos \frac{2\pi k x}{T} \, dx$$

analogously $b_k(g) = \frac{2}{T} \int_0^T g(x) \sin \frac{2\pi k x}{T} \, dx$

Fourier series in eigen-oscillations

string



$$\underset{\text{density}}{\rho} \underbrace{\frac{\partial^2}{\partial t^2} u(t,x)}_{\text{acceleration}} = \underset{\text{tension}}{P} \frac{\partial^2}{\partial x^2} u(t,x)$$

boundary condition $u(t,0) = u(t,\pi) = 0$

look for harmonic oscillations (eigen-oscillations) $u(t,x) = \cos \omega t \cdot V(x)$ one frequency

$$\frac{\partial^2}{\partial t^2} u = -\omega^2 \cos \omega t \cdot V(x), \quad \frac{\partial^2}{\partial x^2} u = \cos \omega t \cdot V''(x)$$

$$-\rho \omega^2 \cancel{\cos \omega t} \, V(x) = P \cancel{\cos \omega t} \cdot V''(x) \quad \text{with } V(0) = V(\pi) = 0$$

$$\underbrace{-\frac{\rho \omega^2}{P}}_{\text{eigenvalue}} V(x) = \underbrace{\frac{d^2}{dx^2}}_{\text{linear differential operator}} V(x)$$

eigenvalue problem for $\frac{d^2}{dx^2}$

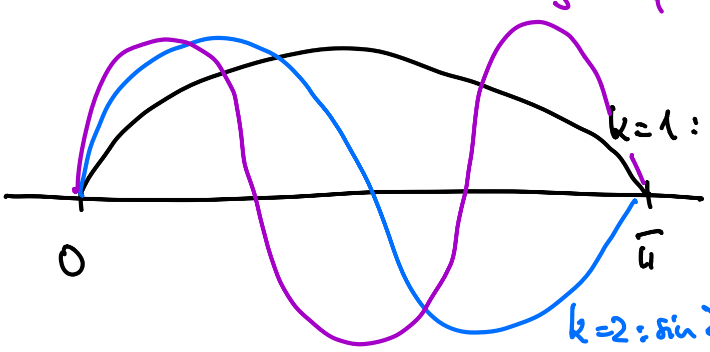
eigen-form (eigen-functions $V_k(x) = \sin kx$, $k = 1, 2, 3, \dots$

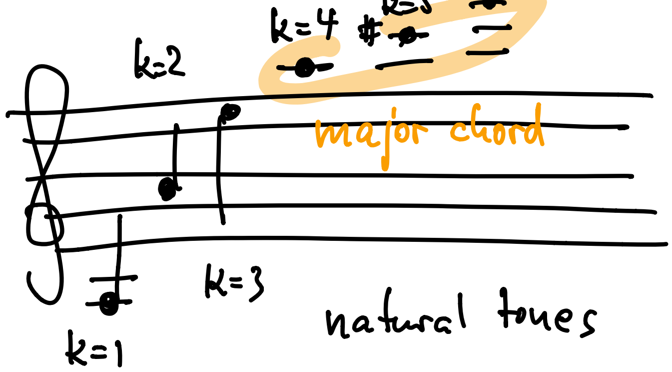
eigenvalues $k^2 = \frac{\rho \omega^2}{P}$ or eigen-frequencies $\omega_k = \sqrt{\frac{P}{\rho}} \cdot k$

one eigen-oscillation $u_k(t,x) = \cos \omega_k t \cdot \sin kx$

Fourier-series of u is $u(t,x) = \sum_{k=1}^{\infty} \gamma_k \cos \omega_k t \cdot \sin kx$

Superposition / linear combination of eigen-oscillating eigenforms of a string





Rem: standard Fourier Series

$$f(x) = \frac{a_0}{2} \cdot 1 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

eigen-forms

of an infinitely long string under tension with a 2π-periodic deformation.

resp. eigen-value problem

$$\lambda \, V(x) = \frac{d^2}{dx^2} V(x)$$

for 2π-periodic and continuous function V

i.e. boundary condition

$$V(0) = V(2\pi), \quad V'(0) = V'(2\pi)$$

Solutions

$\sin kx$, $k = 1, 2, 3, \dots$, $\lambda = -k^2$

$\cos kx$, $k = 1, 2, 3, \dots$, $\lambda = -k^2$

1, $\lambda = 0$

Have a good start to your CSE studies.