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A new Riccati ADI method

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For the *unique* low-rank residual approximate solution of the Riccati equation *solves with* $(A^* - \mu I_n)$ and the *explicitly known residual factors* are sufficient.

Approximate the solution $X \approx ZYZ^*$ of the algebraic Riccati equation $\mathcal{R}(X) = A^*X + XA + C^*C - XBB^*X = 0$ such that the residual remains of low rank, more precisely $\operatorname{rank}(\mathcal{R}(ZYZ^*)) = \operatorname{rank}(C^*C)$. For a concise presentation here we consider $C^* \in \mathbb{C}^{n\times 1}$.

Direct approach

Let $Z_k \in \mathbb{C}^{n\times k}$ be a basis of the rational Krylov subspace $\mathcal{K}_k(A^*, C^*, s)$ with a set of shifts $s \subset \mathbb{C} \setminus \Lambda(A^*)$ satisfying $s \cap -\bar{s} = \emptyset$. Consider the rational Arnoldi decomposition

$$A^* \begin{bmatrix} C^* & Z_k \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ I_k \end{bmatrix}}_{=:\underline{K_k}} = \begin{bmatrix} C^* & Z_k \end{bmatrix} \underbrace{\begin{bmatrix} h_k \\ \underline{H_{-k}} \end{bmatrix}}_{=:\underline{H_k}}.$$

Let $X_k = Z_k Y_k Z_k^*$ be the approximate solution with rank one residual. Then

$$\begin{aligned} \mathcal{R}(X_k) &= A^*X_k + X_kA + C^*C - X_kBB^*X_k \\ &= \begin{bmatrix} C^* & Z_k \end{bmatrix} \left(\underline{H_k} Y_k \underline{K_k}^* + \underline{K_k} Y_k \underline{H_k}^* + e_1e_1^* - \underline{K_k} Y_k S_k Y_k \underline{K_k}^* \right) \begin{bmatrix} C \\ Z_k^* \end{bmatrix} \end{aligned}$$

with $S_k := Z_k^* BB^* Z_k$. Because $\begin{bmatrix} C^* & Z_k \end{bmatrix}$ has full rank, the inner matrix must have rank one, i.e.

$$\begin{aligned} &\underline{H_k} Y_k \underline{K_k}^* + \underline{K_k} Y_k \underline{H_k}^* + e_1e_1^* - \underline{K_k} Y_k S_k Y_k \underline{K_k}^* \\ = &\begin{bmatrix} 1 & h_k Y_k \\ Y_k h_k^* & \underline{H_{-k}} Y_k + Y_k \underline{H_{-k}}^* - Y_k S_k Y_k \end{bmatrix} = \begin{bmatrix} 1 & h_k Y_k \\ Y_k h_k^* & Y_k h_k^* h_k Y_k \end{bmatrix}. \end{aligned}$$

Due to the lower right block Y_k is determined by

$$\begin{aligned} &0 = \underline{H_{-k}} Y_k + Y_k \underline{H_{-k}}^* - Y_k(S_k + h_k^* h_k) Y_k \\ \Leftrightarrow\ &0 = Y_k^{-1} \underline{H_{-k}} + \underline{H_{-k}}^* Y_k^{-1} - (S_k + h_k^* h_k). \end{aligned}$$

The residual factor is given by $R_k = \begin{bmatrix} C^* & Z_k \end{bmatrix} \begin{bmatrix} 1 \\ Y_k h_k^* \end{bmatrix}$, i.e. $\mathcal{R}(X_k) = R_k R_k^*$.

Iterative approach

Generate the Krylov basis Z_k and the rational Arnoldi decomposition iteratively. Let $s \subset \mathbb{C}^+ \setminus \Lambda(A^*)$. The basis Z_k is extended such that $Y_k = I_k$ to reduce the effort for updates.

initialize $R = C^*$, $Z = [\,]$, $h_0 = [\,]$, $\underline{H_{-0}} = [\,]$, $s_0 = [\,]$
for $j = 0, \ldots, k-1$
  choose shift $\mu = s_{j+1}$
  $W = (A^* - \mu I_n)^{-1} R$
  set $\tilde{h} = \begin{bmatrix} h_j & 1 \end{bmatrix}$, $\tilde{H}_- = \begin{bmatrix} \underline{H_{-j}} & h_j^* \\ & \mu \end{bmatrix}$, $\underline{\tilde{H}_{j+1}} = \begin{bmatrix} \tilde{h} \\ \tilde{H}_- \end{bmatrix}$
  solve $0 = \tilde{Y}\tilde{H}_- + \tilde{H}_-^*\tilde{Y} - \tilde{h}^*\tilde{h} - \begin{bmatrix} s_j & B^*W \end{bmatrix}^* \begin{bmatrix} s_j & B^*W \end{bmatrix}$
  Cholesky decomposition $\tilde{Y} = G^*G$ with $G = \begin{bmatrix} I_j & v \\ & \lambda \end{bmatrix}$
  $\underline{H_{j+1}} = \begin{bmatrix} 1 & \\ & G \end{bmatrix} \underline{\tilde{H}_{j+1}} G^{-1}$, $h_{j+1} = e_1^\mathsf{T} \underline{H_{j+1}}$
  $Z = \begin{bmatrix} Z & W \end{bmatrix} G^{-1}$
  $R = R + (Ze_{j+1})(e_{j+1}^\mathsf{T} h_{j+1}^*)$
  $s_{j+1} = \begin{bmatrix} s_j & B^*(Ze_{j+1}) \end{bmatrix}$
end for
return Z with $X_k = ZZ^*$ and the residual factor R

In a more sophisticated implementation we have:

- Direct calculation of G and G^{-1} and the structure of G is exploited to speed up computations.
- Generalized Riccati equation with E.
- Realification for complex $\mu, \bar{\mu} \in s$ with only one solve.
- Multiple system solves in parallel.

Comparison with other Riccati ADI methods

	Cayley subsp. iter. [1]	RADI [2]	this approach
uniqueness of approximation	no	no	yes
only solves with $(A^* - \mu I_n)$	yes	no	yes
explicit residual factor	no	yes	yes

References

[1] Lin, Y. and Simoncini, V. (2014), *A new subspace iteration method for the algebraic Riccati equation*, Numer. Linear Algebra Appl., 22, pages 26– 47, doi:10.1002/nla.1936

[2] Benner, P., Bujanović, Z., Kürschner, P. et al., *RADI: a low-rank ADI-type algorithm for large scale algebraic Riccati equations*, Numer. Math. (2018) 138: 301. https://doi.org/10.1007/s00211-017-0907-5

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