ESSLLI 2010 CHP

INTRODUCTION TO CATEGORY THEORY, ALGEBRAS AND COALGEBRA

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This is the course material for our one week course at ESSLLI 2010. The course consists of two parts. The first part (approximately the first two days) presents standard introductory material which can be found in most textbooks on category theory, e. g. MacLane's book [ML]. We do not provide course notes for this part of the course. Instead we use the book of Adámek, Herrlich and Strecker [AHS] as a reference. It is freely aviable online at

`http://katmat.math.uni-bremen.de/acc/`

More specifically, the material from the course can be found in the following parts of the book: Chapter I.1 and I.6, Chapter II.7, Chapter III.11 and Chapter V.1.

For the second part of our course on Algebras and Coalgebras we use as course material a draft of a survey article on initial algebras and terminal coalgebras authored jointly with Larry Moss. This survey covers all the material presented in the course but provides many more details and references to the literature, and some of the material goes beyond what can be presented in the course.

The current draft of our survey is provided in these course notes.

Braunschweig, July 7, 2010

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Initial algebras and terminal coalgebras: a survey
– Preliminary version –

July 7, 2010

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1 Introduction

1.1 Why are initial algebras interesting?

Recursion and induction are important tools in programming. In functional programming, for example, recursion is a definition principle for functions over the (inductive) structure of data types such as natural numbers, lists or trees. And induction is the corresponding proof principle used to prove properties of programs. An important question of theoretical computer science concerns the semantics of such definitions. *Initial Algebra Semantics*, studied since the 1970's, uses the tools of category theory to unify recursion and induction at the appropriate abstract conceptual level. In this approach, the type of data on which one wants to define functions recursively and to prove properties inductively is captured by an *endofunctor* F on the category of sets (or another appropriate base category). This functor describes the signature of the data type constructor. And an initial algebra for the functor F provides a canonical minimal model of a data type with the desired constructors. Let us illustrate this by a concrete example: Consider the endofunctor on sets given by $FX = X+1$, i. e., the set construction adding a fresh element to the set X. An algebra for F is just a set equipped with a unary operation and a constant, and the initial algebra is the algebra of natural numbers $\mathbb{N}$ with the successor function and the constant 0. The abstract property of initiality of that algebra is precisely the usual principle of *recursion on natural numbers*: given an F-algebra X, i. e., a unary operation on $u : X \to X$ and a constant $x \in X$, there exists precisely one function f from the natural numbers to X with $f(0) = x$ and $f(n+1) = u(f(n))$. The existence of f is the fact that functions from $\mathbb{N}$ to X can be defined by recursion, and the uniqueness yields the proof principle of induction.

As a second example consider the set functor $FX = X \times X + 1$ (whose algebras have a binary operation and a constant) then the initial algebra is the algebra of finite binary trees, and initiality yields a tree-recursion principle.

In the present survey initial algebras are studied for all endofunctors $F : \mathcal{A} \to \mathcal{A}$. It was Jim Lambek who first studied in [L] *algebras* for F as pairs consisting of an object A of $\mathcal{A}$ and a morphism $a : FA \to A$; the corresponding *homomorphisms* from (A, a) to (A', a') are those morphisms $h : A \to A'$ in $\mathcal{A}$ which fulfil $h \cdot a = a' \cdot Fh$. If a is invertible (thus $FA \simeq A$) we call the algebra a *fixed point* of A.

The category of algebras is denoted by $\mathsf{Alg}\ F$. By an *initial algebra* μF of F is meant its initial object: this is an algebra such that for every algebra there exists a unique homomorphism from μF.

Lambek's Lemma If F has an initial algebra, then it is a fixed point.

We conclude immediately that even for $\mathcal{A} = \mathsf{Set}$ there are important examples of endofunctors that do not have an initial algebra: consider the power-set functor $\mathcal{P}$. A fundamental result of set theory known as Cantor's Theorem says that no set A is in bijective correspondence with $\mathcal{P}A$. So for all sets A, A is not a fixed point of the power set operation.

In the present paper, we study the existence and construction of initial algebras. There is a general procedure for constructing the initial algebra of F starting from the initial object 0 of $\mathcal{A}$, first used by J. Adámek in [A74]. Starting from the unique morphism $! : 0 \to F0$, we form the corresponding ω-chain:

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F(F0) \xrightarrow{F(F!)} F^3 0 \xrightarrow{F^3 !} \cdots$$

If the colimit exists and is preserved by F, then that colimit carries the initial algebra

$$\mu F = \operatorname*{colim}_{n\in\omega} F^n 0.$$

If F does not preserve that colimit, we iterate further and obtain a transfinite chain. Again, if in that chain F preserves the colimit at some stage, we obtain an initial algebra.

We illustrate the behaviour of this construction and other methods for obtaining initial algebras by numerous examples. For example the functor $FX = X \times X + 1$ (of binary algebras with a constant) yields the chain with $F^1 0 = 1$ and $F^{n+1}0 = F^n 0 \times F^n 0 + 1$. This recursion allows us to represent $F^n 0$ by the set all binary trees of depth less than n. The initial algebra $\operatorname{colim}_{n<\omega} F^n 0 = \bigcup_{n<\omega} F^n 0$ is the algebra of all finite binary trees, the binary operation is tree tupling, and the constant is the trivial singleton tree. Shortly:

$$\mu X.(X \times X + 1) = \text{ finite binary trees.}$$

1.2 Why are terminal coalgebras interesting?

A coalgebra for a functor F is the dual concept of F-algebra: it consists of an object A and a morphism $a : A \to FA$. Jan Rutten [Ru_1] presented a persuasive survey of the applications of this idea to the theory of discrete dynamical systems which are ubiquitous in computer science. For example, a sequential deterministic automaton with input set S can be described by the set A of its states together with a function

$$a : A \to A^S \times \{0, 1\}$$

whose first component

$$\frac{A \to A^S}{A \times S \to A}$$

describes the next-state function, and the second one

$$A \to \{0, 1\}$$

describes the predicate "accepting state". This is a coalgebra for the functor F given by

$$FX = X^S \times \{0, 1\}.$$

And a nondeterministic automaton is given by a function from A to $(\mathcal{P}A)^S \times \{0, 1\}$ and is thus a coalgebra for the endofunctor $F \cdot \mathcal{P}$ composed of the power-set functor and the above functor F.

For another example, consider a dynamic system with states accepting binary input and having also deadlock states (not reacting to inputs). This is given by a set A of states and a function

$$a : A \to A \times A + 1$$

assigning to every deadlock the element of 1 and to every other state the pair of the possible next states. This is a coalgebra for $FX = X \times X + 1$.

The category $\mathsf{Coalg}F$ of coalgebras has as morphisms from (A, a) to (A', a') the *coalgebra homomorphisms* which are the morphisms $h : A \to A'$ with $a' \cdot h = Fh \cdot a$. The *terminal coalgebra* νF, also called *final*[1] *coalgebra*, is (if it exists) the terminal object of this category: For every coalgebra

[1]Both terms are found in the literature, with *terminal* used in all older works and *final* in most work done after the renewal of interest in coalgebras sparked by Aczel [Ac].

there exists a unique homomorphism into νF. If a system is represented by a coalgebra (A, a), then the unique homomorphism $f : A \to \nu F$ expresses the "abstract behaviour of the states in A", as Jan Rutten demonstrated on numerous examples.

Example 1.2.1. The functor $FX = X^S \times \{0, 1\}$ has the terminal coalgebra $\mathcal{P}S^*$, the set of all formal languages over the alphabet S:

$$\nu X.(X^S \times \{0, 1\}) = \mathcal{P}S^*.$$

Given an automaton (A, a), the unique homomorphism $f : A \to \mathcal{P}S^*$ assigns to every state the language this state accepts.

Example 1.2.2. For the dynamic systems with deadlock above we have the coalgebra of all (finite and infinite) binary trees as the terminal coalgebra, shortly:

$$\nu X.(X \times X + 1) = \text{ binary trees.}$$

Given a dynamic system (A, a) the unique homomorphism $f : A \to \nu F$ assigns to every state the binary tree of all possible future developments in which the names of states are "abstracted away" and only the distinction deadlock/non-deadlock remains.

By dualizing Lambek's Lemma, we see that νF is always a fixed point of F. Thus, we can consider νF also to be an algebra for F. (Example: binary trees form an algebra for $FX = X \times X + 1$, with tree tupling and the single-node tree, analogously to μF). This algebra νF has a strong property of solvability of recursive equations. Let us illustrate this with the functor $FX = X \times X + 1$ of one binary and one nullary operation. Given a system of recursive equations

$$\begin{aligned} x_1 &= t_1 \\ x_2 &= t_2 \\ &\vdots \end{aligned}$$

where t_i is a term using the variables $x_1, x_2, \dots$ and the binary operation and constant (forming the signature of F-algebras in this case) there exists a solution in νF. This means that to every x_i we can assign a binary tree $x_i^\dagger$ such that the formal equations above become identities when the simultaneous substitution $x_i^\dagger / x_i$ is performed on the left and right sides of each equation in the system. Moreover, the solution of the system is unique, provided none of the right-hand sides is a bare variable. Algebras with this recursion property are called *completely iterative*. S. Milius proved in [M$_1$] that whenever νF exists, it is a completely iterative algebra—in fact, it can be characterized as the initial completely iterative algebra. We provide an overview of completely iterative algebra in Section 5.

Jan Rutten also discusses in [Ru$_1$] *corecursion* as an important construction principle dual to recursion, and *coinduction* as an important proof principle dual to induction. In fact, induction can be formulated abstractly as follows: given the initial algebra $(\mu F, \varphi)$ of F, then for a parallel pair $f_1, f_2 \colon \mu F \to A$ of morphisms in the base category $\mathcal{A}$ in order to prove $f_1 = f_2$ it is sufficient to present a morphism $a \colon FA \to A$ for which f_1 and f_2 are algebra homomorphisms from $(\mu F, \varphi)$ to (A, a). The coinduction is the dual principle which for the terminal coalgebra νF allows us to prove equality of morphisms $f_1, f_2 \colon A \to \nu F$.

It goes without saying that not every functor possesses a terminal coalgebra. This follows from the dual of Lambek's Lemma: the power-set functor does not have a terminal coalgebra. We study the dual of the above initial algebra construction (explicitly used by Michael Barr [Barr] for the first time): Start with the terminal object 1 of $\mathcal{A}$ and the unique morphism $! \colon F1 \to 1$ and form the ω^{op}-chain

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F(F1) \xleftarrow{F(F!)} F^3 1 \xleftarrow{F^3!} \cdots . \tag{1.1}$$

If the limit exists and is preserved by F, this is the terminal coalgebra of F:

$$\nu F = \lim_{n<\omega} F^n 1.$$

Example 1.2.3. The above functors $FX = X^S \times \{0,1\}$ and $FX = X \times X + 1$ preserve all limits of ω^{op}-chains (called ω^{op}-limits, for short), and in particular their terminal coalgebras may be obtained by taking the limit of the ω^{op}-chain in (1.1).

If F does not preserve ω^{op}-limits, one may try iterating F on 1 further (transfinite chain), see Section 3 below.

Example 1.2.4. Graphs are nothing else than coalgebras for the power-set functor $\mathcal{P}$: given $a : A \to \mathcal{P}A$ then consider, for every vertex $x \in A$, the set $a(x) \subseteq A$ as the set of all neighbours of x. But be careful: there are fewer coalgebra homomorphisms than graph homomorphisms. Given graphs (A, a) and (B, b) a coalgebra homomorphism $f : A \to B$ fulfils not only

$$\text{if } x \to x' \text{ in } A \text{ then } f(x) \to f(x') \text{ in } B$$

but also

$$\text{if } f(x) \to y' \text{ in } B \text{ then } x \to y \text{ in } A \text{ for some } y \text{ with } f(y) = y'.$$

Lambek's Lemma tells us that there exists no terminal graph. However, if $\mathcal{P}_\mathrm{f}$ is the subfunctor of $\mathcal{P}$ on all finite subsets, then coalgebras are precisely the finitely branching graphs, and a terminal coalgebra exists, see Section 2.5 for an extensive discussion. We mention this example because the limit of the ω^{op}-chain in (1.1) is *not* preserved by the functor, and so one needs a more sophisticated construction.

As we shall see, there are *several different* constructions of the terminal coalgebra for $\mathcal{P}_\mathrm{f}$. For other functors on other categories, there are yet other constructions.

1.3 Algebras versus Coalgebras

Although *algebra* and *coalgebra* are dual terms, and although this duality persists to the level of *initial algebra* and *terminal coalgebra*, it is by no means the case that $\mathsf{Alg}\,F$ is dual to $\mathsf{Coalg}\,F$. There are easy examples of this, the simplest being the identity endofunctor of a poset $\{x, y, z\}$ with $x \le y$, $x \le z$. This poset has an initial object x but no terminal object. Thus the identity functor has an initial algebra but no terminal coalgebra.

What is true is the following: every functor $F \colon \mathcal{A} \to \mathcal{A}$ defines a functor $F^{op} \colon \mathcal{A}^{op} \to \mathcal{A}^{op}$, by the same rule as F. The category of algebras for F (in $\mathcal{A}$) is dual to the category of coalgebras for F^{op} (in $\mathcal{A}^{op}$). Shortly,

$$(\mathsf{Alg} F)^{op} = \mathsf{Coalg} F^{op}.$$

We present in Figure 1 a conceptual comparison between two sets of ideas. As it indicates, the coalgebraic concepts on the right are interesting partly because they are the structures used in the mathematics of *transition* and *observation*, as opposed to *operations*. Terminal coalgebras in this sense are

algebra for a functor	coalgebra for a functor
initial algebra	terminal coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence relation
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a terminal coalgebra
Foundation Axiom	Anti-Foundation Axiom
iterative conception of set	coiterative conception of set
set with operations	set with transitions and observations
useful in syntax	useful in semantics
bottom-up	top-down

Figure 1: The conceptual comparison

like the most abstract collections of "transitions" or "observations". We know that this is very vague, and so we hope that the examples throughout this paper will help. However, we also would like to mention other sources, such as Moss [Mo_4] (this is the source of the chart, and includes much conceptual discussion related to the set-theoretic topics), and also Rutten [Ru_1], a highly recommendable general source on coalgebra. For more recent surveys see [GU] or [JA].

The questions of interest in this survey include: what are general conditions which guarantee the existence of an initial algebra or a terminal coalgebra? We are also interested in *representations* of terminal coalgebras. The reason for this is that the existence theorems themselves frequently are fairly abstract, and so concrete representations make the terminal coalgebras more intuitive.

At this point, we want to mention the main categories and functors of interest in our study.

We begin with Set, the category of sets and functions. We are interested in the *polynomial functors* obtained from the identity functor and the constant functors by products and coproducts (including exponents from a fixed set B).

Another functor which we shall study is the *discrete probability measure functor* $\mathcal{D}$, where $\mathcal{D}(X)$ is the set of finite functions from X to $[0,1]$ which sum to 1. The definition of D on morphisms $f : X \to Y$ is

$$\mathcal{D}f(\mu)(y) \;=\; \mu(f^{-1}(y)) \;=\; \sum_{x \in f^{-1}(y)} \mu(x).$$

We already mentioned the power set functor $\mathcal{P}$; this gives the set of subsets of X. The action of $\mathcal{P}$ on functions is $f : X \to Y$ is

$$\mathcal{P}f : A \mapsto f[A] = \{f(x) : x \in A\}$$

There are also a few refinements of $\mathcal{P}$, including $\mathcal{P}_{\mathrm{f}}$, the *finite power set* functor. More generally, for each cardinal number κ , we have a functor $\mathcal{P}_\kappa$, where $\mathcal{P}_\kappa(X)$ is the set of subsets of X of size $< \kappa$. (So $\mathcal{P}_{\mathrm{f}}$ is $\mathcal{P}_\omega$.)

Part of our interest in these refinements comes from the fact that whereas $\mathcal{P}$ itself has no terminal coalgebra, every functor built from all of the functors above *except* $\mathcal{P}$ using composition *does* have a terminal coalgebra. Here is another example, again indicating the difference between the categories of algebras and coalgebras for a functor.

Example 1.3.1. On Set, let FX be the power set functor, but modified so that $F(\emptyset) = \emptyset$. Then $(\emptyset, id)$ is the initial algebra. But there is no terminal coalgebra: the only fixed point is $\emptyset = F(\emptyset)$, and this is not terminal, since it is not the codomain of any coalgebra morphism from a non-empty coalgebra.

Other categories include the posets and monotone maps, and CPO (the complete partial orders and continuous maps). Further, we shall consider Met, the category of metric spaces with distances bounded by 1 and non-expanding maps, and also the full subcategory CMS of such complete metric spaces. Both of these categories have a "power-set-like" operation, and this will be of special interest. We are also intersected in the categories Set^S of S-sorted sets (for a set S).

The aim of this survey. We have tried and collected interesting results about terminal coalgebras (and initial algebras) scattered throughout the literature. We found some results not quite complete and we completed them. We usually indicate the idea of the proof, but otherwise we provide references to where proofs can be found.

2 Finitary Iteration

Ininitial algebras and terminal coalgebras are fixed points, and so we ground our discussion by relating it to one of the most well-known fixed point results in mathematics. This is a result often attributed to Stephen Kleene, a theorem on least fixed points of monotone, continuous operations on posets with suprema of countable chains. Generalizing from orders to categories yields a theorem on initial algebra, Theorem 2.1.9. Then the dual result appears in Theorem 2.3.3. This dual result is actually the first terminal coalgebra theorem in this paper.

2.1 Initial algebras

Throughout our paper we assume that a category $\mathcal{A}$ and an endofunctor F on $\mathcal{A}$ are given. The following concept of algebra and homomorphism was studied by J. Lambek [L] for the first time: an *algebra* for F (or F-*algebra*) is an object A of $\mathcal{A}$ together with a morphism $a\colon FA \to A$, a *homomorphism* between algebras (A, a) and (B, b) is a morphism $h : A \to B$ in $\mathcal{A}$ for which the square

$$\begin{tikzcd} FA \arrow[r, "a"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\ FB \arrow[r, "b"'] & B \end{tikzcd}$$

commutes. The category of F-algebras is denoted by $\mathsf{Alg}F$. Its initial object (in case it exists) is called the *initial algebra* for F. Recall also that two morphisms $f : A \to B$ and $g : B \to A$ are *inverses* if $g \cdot f = id_A$ and $f \cdot g = id_B$. And an *isomorphism* is a morphism with an inverse. An algebra for which $a\colon FA \to A$ is an isomorphism is called a *fixed point* of F.

Lemma 2.1.1 (Lambek [L]). *If an initial F-algebra exists, then it is a fixed point of F.*

Proof. Let (A, a) be initial. Since (FA, Fa) is an algebra, we have a homomorphism h:

$$\begin{tikzcd} FA \arrow[r, "a"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\ FFA \arrow[r, "Fa"] \arrow[d, "Fa"'] & FA \arrow[d, "a"] \\ FA \arrow[r, "a"'] & A \end{tikzcd}$$

Now $a \cdot h$ is an endomorphism of (A, a) in $\mathsf{Alg}F$. Thus $a \cdot h = id_A$ by initiality. Then $h \cdot a = Fa \cdot Fh = id_{FA}$. Thus $h = a^{-1}$. ☐

Example 2.1.2. The power set functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ does not have an initial algebra: Cantor's Theorem tells us that for all sets A, there is no map of A onto $\mathcal{P}(A)$. Therefore, there exists no fixed point of $\mathcal{P}$ on Set.

Notation 2.1.3. An initial object of $\mathcal{A}$ is denoted by 0. If F has an initial algebra, we denote it by

$$\mu F \qquad \text{or} \qquad \mu X.F(X).$$

More precisely: initial algebras are unique up to isomorphism, and if we choose one, the underlying object is denoted by μF.

In Example 2.2 we saw a trivial negative example, let us mention a trivial affirmative one: If F preserves initial objects then $\mu F = 0$.

In what follows we investigate μF in the "remaining" interesting cases: for functors not preserving 0 but having fixed points.

Our opening result is a classical theorem on fixed points of monotone functions on complete partial orders. A *poset* is a pair $A = (A, \leq)$ with $\leq$ a partial order on A. In case $\leq$ is reflexive and transitive but not necessarily antisymmetric $(A, \leq)$ is called a preorder. A *monotone* function from one poset to another is a function preserving the order: $a \leq b$ implies $f(a) \leq f(b)$.

Definition 2.1.4. A poset $(A, \leq)$ is a *complete partial order* (or, *cpo*, for short) if every countable chain (also called ω-chain) has a least upper bound and A has a least element; we usually use $\perp$ for this element. A map $f : A \to B$ is *continuous* if it preserves joins of ω-chains.

Theorem 2.1.5 (Kleene)**.** *Let $\mathcal{A}$ be a cpo. Then every continuous endofunction F has the least fixed point*

$$\mu F = \sup_{n \in \omega} F^n(\perp).$$

Proof. First, an induction on $n < \omega$ shows that $F^n(\perp) \leq F^{n+1}(\perp)$. So $\{F^n(\perp) : n < \omega\}$ is a chain. Write μF for its join. By continuity, $F(\mu F) = \bigvee_n F(F^n(\perp))$. It is easy to check that $\bigvee_n F^n(\perp) = \bigvee_n F^{n+1}(\perp)$, and so $F(\mu F) = \mu F$. Thus, we have a fixed point of F. If $Fx \leq x$, then we show by induction on n that $F^n(\perp) \leq x$; hence $\mu F \leq x$ as well. ☐

Remark 2.1.6. More generally, a pre-fixed point of F is an element x with $Fx \leq x$; the argument above proves that μF is the least pre-fixed point of F.

order-theoretic concept	category-theoretic generalization
preorder $(A, \leq)$	category $\mathcal{A}$
$x \leq y$ and $y \leq x$	A and B are isomorphic objects
least element 0	initial object 0
monotone $F : A \to A$	functor $F : \mathcal{A} \to \mathcal{A}$
pre-fixed point: $Fx \leq x$	F-algebra: $f : FA \to A$
ω-chain	functor from $(\omega, \leq)$ to $\mathcal{A}$
F is continuous	F preserves ω-colimits
least pre-fixed point: $Fx \leq x$	initial F-algebra: $\varphi : F(\mu F) \to \mu F$

Figure 2: Generalizing Kleene's Theorem to categories

In this paper, we are not really interested in Kleene's Theorem but in generalizations of it, and in dualizations of those generalizations, etc. Figure 2 shows how we generalize the order-theoretic concepts in Kleene's Theorem to the level of categories. In each line, the order-theoretic concept on the left is a special case of the category-theoretic concept to its right. (To see this, recall that a pre-order $(P, \leq)$ is exactly a category in which every homset is either empty or singleton set.) Of special interest is the generalization of the completeness condition on the poset and the continuity condition on the function. We assume that every ω-chain, that is, a functor from ω to $\mathcal{A}$ has a colimit, and in particular we consider the *initial ω-chain* shown in (2.1).

Definition 2.1.7. By the *initial ω-chain* of an endofunctor F is meant the ω-chain

$$0 \xrightarrow{\ !\ } F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} \cdots \xrightarrow{F^{n-1}!} F^n 0 \xrightarrow{F^n!} F^{n+1} 0 \xrightarrow{F^{n+1}!} \cdots \tag{2.1}$$

Here $! : 0 \to F0$ is the unique morphism given by initiality. This diagram gives a functor from $(\omega, \leq)$ to $\mathcal{A}$.

We must mention that later in the paper we shall need transfinite iterations of the initial sequence. But in this section, we only consider the finite iterations as in (2.1), along with the notions of *cocone* and *colimit*, our next points.

A *cocone* over the initial ω-chain is an object A of $\mathcal{A}$ together with a family of morphisms $a_n : F^n 0 \to A$ such that $a_n = a_{n+1} \cdot F^n!$ for all $n < \omega$. A *colimit* of the initial ω-chain is a cocone $(C, c_n : F^n 0 \to C)$ over it with the universal property that if $(A, \alpha_n : F^n 1 \to A)$ is any cocone over the initial ω-chain, then there is a unique *factorizing morphism* $f : C \to A$ such that for all $n < \omega$, $a_n = f \cdot c_n$.

Construction 2.1.8. Every F-algebra (A, α) induces a canonical cocone $\alpha_n : F^n 0 \to A$ over the initial ω-chain as follows: $\alpha_0 \colon 0 \to A$ is unique (since 0 is initial) and $\alpha_{n+1} = \alpha \cdot F\alpha_n \colon F(F^n 0) \to A$. The cocone property, $\alpha_n = \alpha_{n+1} \cdot F^n!$, is easy to verify by induction. We call the cocone (A, α_n) the *cocone induced by A*.

Let $c_n \colon F^n 0 \to \mu F$ be the colimit of the initial ω-chain. Applying F to each object and morphism in (2.1) yields another ω-chain

$$F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \cdots \xrightarrow{F^n!} F^{n+1} 0 \xrightarrow{F^{n+1}!} F^{n+2} 0 \xrightarrow{F^{n+2}!} \cdots \tag{2.2}$$

which obviously has the same colimit as (2.1).

This leads to the following result:

Theorem 2.1.9. *[A74] Let $\mathcal{A}$ be a category with initial object 0 and with colimits of ω-chains. If $F : \mathcal{A} \to \mathcal{A}$ preserves ω-colimits, then it has the initial algebra*

$$\mu F = \operatorname*{colim}_{n<\omega} F^n 0.$$

Proof. If $c_n \colon F^n 0 \to \mu F$ denotes the colimit cocone of (2.1), then we can define a unique morphism $\varphi \colon F(\mu F) \to \mu F$ by

$$\varphi \cdot F c_n = c_{n+1} \colon F(F^n 0) \to \mu F \quad (n < \omega).$$

To check the initiality, let (A, α) be any F-algebra and consider the cocone $\alpha_n : F^n 0 \to A$ induced by A. The unique morphism $h \colon \mu F \to A$ with $h \cdot c_n = \alpha_n$ ($n < \omega$) is an F-algebra homomorphism: the equality $h \cdot \varphi = \alpha \cdot Fh$ follows, since $(Fc_n)_{n<\omega}$ is a colimit cocone, from each Fc_n merging the parallel pair $h \cdot \varphi, \alpha \cdot Fh : F(\mu F) \to A$. Conversely, if $k \colon \mu F \to A$ is a homomorphism, then $k \cdot c_n = \alpha_n = h \cdot c_n$ for all $n < \omega$; indeed, since $c_{n+1} = \varphi \cdot F c_n$, this is easy to prove by induction. Thus, $k = h$ as desired. □

Remark 2.1.10. To obtain an initial algebra of an endofunctor F, it is not really necessary that F preserve all ω-colimits. It is sufficient to assume that F preserve the colimit of its initial ω-chain (2.1) (see the proof of Theorem 2.1.9 just above).

2.2 Examples of Initial Algebras

We present examples of initial algebras on Set obtained by finite iteration as in Theorem 2.1.9. We discuss these at some length because the same functors will appear throughout the paper.

Example 2.2.1. The functor $FX = X + 1$. First, we consider this as a functor on Set. As such, it has an initial algebra obtained as the colimit of the initial chain

$$0 \longrightarrow 1 \longrightarrow 1+1 \longrightarrow 1+1+1 \longrightarrow \cdots$$

So we may identify its n-th term with the natural number n. The colimit is the set $\mathbb{N}$ of natural numbers, with the algebra structure

$$\varphi : \mathbb{N} + 1 \to \mathbb{N} \tag{2.3}$$

the isomorphism taking the right-hand summand 1 to $0 \in \mathbb{N}$, and taking the left summand $\mathbb{N}$ to itself via the successor function.

Here are some additional details on these points. Under our identification, $F^n! : n \to n+1$ is $i \mapsto i+1$ for $0 \leq i < n$. The colimit is the disjoint union of the factors, identifying points merged by functions on the chain. In this case, this disjoint union is $\{(n, i) : i < n\}$, and we identify (n, i) with $(n+k, i+k)$ for all k. Then the colimit map $c_n : n \to \mathbb{N}$ is given by $c_n(i) = n - i - 1$. To determine the initial algebra structure $\varphi : \mathbb{N} + 1 \to \mathbb{N}$, we make two calculations using the fact that for all n, $\varphi \cdot Fc_n = c_{n+1}$. Taking $n = 0$, we see that $\varphi \cdot Fc_0 = c_1$. Thus $0 = c_1(0) = \varphi(Fc_0(0)) = \varphi(*)$, where $*$ is the unique element of 1. And for all n, we have $n + 1 = c_{n+2}(0) = \varphi(Fc_{n+1}(0)) = \varphi(n)$.

Observe that classical induction on $\mathbb{N}$ is precisely the statement that the algebra above is μF. Indeed, in order to define a function $g : \mathbb{N} \to A$, one needs only specify an element $a \in A$ giving the value $f(0)$, and an endofunction of A for the recursion. These two data are precisely an algebra structure $A + 1 \to A$ for the functor $FX = X + 1$. The function they define is exactly the unique algebra morphism $g : \mathbb{N} \to A$.

Example 2.2.2. We consider the same functor $FX = X + 1$ but now on the category Pos of posets and order-preserving functions. Here coproducts are disjoints unions of partially ordered sets, i. e., with elements of different coproduct components incomparable. So our endofunctor takes a poset and adds a fresh element incomparable to the elements of X. Then $F^n 0$ is $\{ 0, \ldots, n-1 \}$ discretely ordered, and $\mu F = \mathbb{N}$ with the same structure as in (2.3). There is also the functor $FX = X_\perp$ adding a new least element to a poset. Here, $F^n 0$ is the linear order $0 < 1 < 2 < \cdots n-1$ and μF is $\mathbb{N}$ with the usual order and the algebra structure φ with $\varphi(\perp) = 0$ and $\varphi(n) = n + 1$.

Similarly, we have the endofunctor $FX = X^\top$ adding a new top element having μF the set of natural numbers with the reverse of the usual order: $0 > 1 > 2 > \cdots$.

Example 2.2.3. Continuing with the same example functor $FX = X + 1$, we turn to the category $\mathsf{CPO}_\perp$ of cpos (with a least element $\perp$) and continuous functions preserving the least element. The initial object is the one-point cpo $0 = \{\perp\}$. Notice that coproducts in $\mathsf{CPO}_\perp$ are disjoint unions with the least elements merged. So we have $FX \cong X$, whence $\mu F \cong 0 = \{\perp\}$. However, let 2 be the two-chain, then for $FX = X + 2$ we see that $F^n 0$ is the flat cpo $\{0, 1, \ldots, n-1\}_\perp$, i. .e, n uncomparable elements with the least element $\perp$ added, and $\mu F = \mathbb{N}_\perp$ is a flat cpo, too. Next let $FX = X_\perp$ be the endofunctor adding a fresh least element to a cpo. Then $F^n 0$ is the cpo $0 < 1 < 2 < \cdots n$, and the colimit is $\mathbb{N}^\top$: the set $\mathbb{N} \cup \{\top\}$ with the usual order and $\top$ above all.

Example 2.2.4. Let MS be the category of 1-bounded metric spaces and *non-expanding maps*. That is, objects are sets endowed with a *metric* $d : X \times X \to [0, 1]$ which satisfies

(i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(x, z)$ (the triangle inequality)

Morphisms are non-expanding functions, i. e., functions $f : X \to X'$ such that for all $x, y \in X$,

$$d'(f(x), f(y)) \leq d(x, y).$$

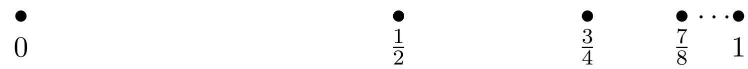
Coproducts are disjoint unions with distance 1 between points in different summands. The functor $FX = X + 1$ now has as an initial algebra the set $\mathbb{N}$ of natural numbers, but as a discrete space: distinct points have distance 1.

Example 2.2.5. We consider the full subcategory

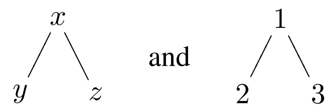
$$\mathsf{CMS}$$

of MS of *complete* metric spaces; i.e., such that every Cauchy sequence has a limit. This subcategory is closed under the products and coproducts in MS, and it contains 0. Thus, once again $\mu X.(X + 1) = \mathbb{N}$ with the discrete metric.

The situation changes when we scale the metric by $\frac{1}{2}$ (or by any other constant between 0 and 1). Let $\frac{1}{2} : \mathsf{MS} \to \mathsf{MS}$ scale a space by $\frac{1}{2}$. We now consider $GX = \frac{1}{2}X + 1$. This time the initial algebra is a Cauchy sequence together with its limit point. As a subset of $[0, 1]$, it may be identified with the points below:



Remark 2.2.6. In the next example, and at many later points, we use trees to describe algebras and coalgebras of special interest. Let us recall that a *tree* is a directed graph with a distinguished node called the *root* from which every other node can be reached by a unique directed path. We always identify isomorphic trees. Thus

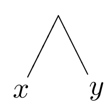


are for us the same tree.

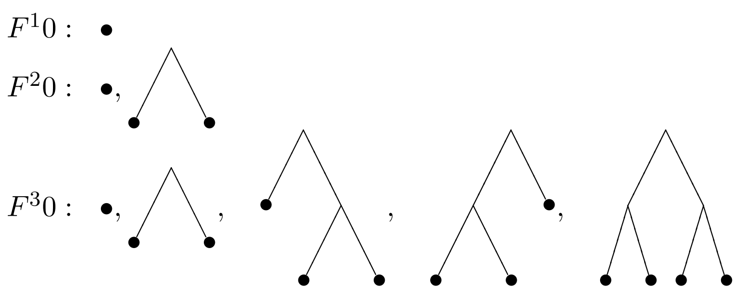
We distinguish between *unordered trees*, defined as above, and *ordered trees*. An ordered tree comes with a linear order on the children of every node. In pictures, this linear order is the left-to-right order. Thus the tree above on the left would represent the ordered tree with the order $y < z$ on the children of x. When we say just "tree" we always mean an unordered tree.

A node is a *leaf* if it has no children. A tree is *binary* if every node which is not a leaf has precisely two children.

Example 2.2.7. Let $FX = (X \times X) + 1$. When dealing with this functor and related ones, it is often useful to adopt a graphical notation. We shall draw $(x, y) \in X \times X \hookrightarrow FX$ as



We start with $F^0 0 = \emptyset$. Now we picture the elements of $F^i 0$ for $i = 1$, 2, and 3:



Then the carrier of the initial algebra μF may be taken to be the union $\bigcup_{n<\omega} F^n 0$:

$$\mu X.(X \times X) = \text{ all finite binary ordered trees.}$$

The structure map

$$\varphi : (\mu F \times \mu F) + 1 \to \mu F$$

is given by tree tupling as the left-hand component, and the unique element of 1 is taken to the single node tree.

Example 2.2.8. We return to the category MS introduced in Example 2.2.4. In this category, products $X \times X'$ are cartesian products with the *maximum metric*:

$$d((x,x'),(y,y')) \quad = \quad \max\{d(x,y), d'(x',y')\}.$$

Thus, $FX = X \times X + 1$ can be considered as an endofunctor of MS. Its initial chain is

$$0 \longrightarrow 1 \longrightarrow (1 \times 1) + 1 \longrightarrow ((1 \times 1) + 1) \times ((1 \times 1) + 1) + 1 \longrightarrow \cdots$$

the same as that of Example 2.2.1, but with each set $F^n 0$ taken to be the *discrete space*: all distances between distinct points are 1. The colimit is then, not surprisingly, μF of all binary trees equipped with the discrete metric.

Example 2.2.9. Let us now study the same functor $FX = X \times X + 1$, but now on the category CMS of complete metric spaces. As we have seen the colimit of the initial ω^{op}-chain for this functor on MS is the set of all finite binary trees with the discrete metric. The same goes for CMS.

The situation changes when we consider the scaled functor as in Example 2.2.5. This would be the functor

$$GX \quad = \quad \frac{1}{2}(X \times X) + 1.$$

Here $G^{n+1}0$ consists of pairs of ordered trees $(t_1, t_2) \in G^n 0 \times G^n 0$ and of the single node tree, whose distance from any (t_1, t_2) is 1. And the distance between (t_1, t_2) and (s_1, s_2) is the maximum of the distances $d(t_i, s_i)$, $i = 1, 2$, in $G^n 0$ scaled by $\frac{1}{2}$. From this it follows that $G^n 0$ can be described as the set of all binary trees of depth less than $n$with the metric

$$d(t,u) = \begin{cases} 2^{-k} & \text{if } t \neq u \\ 0 & \text{if } t = u \end{cases} \tag{2.4}$$

for the least number k such that t and u have the same cuttings at level k.

The colimit of the initial ω-chain $G^n 0$ (of isometric embeddings) in MS is, then, the space of all finite binary ordered trees with the above metric (2.4).

However, this space is not complete. In fact, every *infinite binary tree* t yields a Cauchy sequence $t_0, t_1, \ldots$, where t_k cuts t at level k. It is not difficult to see that the colimit in CMS of $G^n 0$ is

$$\mu G \quad = \quad \text{the set of all (finite and infinite) ordered binary trees, with the metric (2.4) above}$$

We shall see a more general reason for this in Section 2.9 below.

Example 2.2.10. The functor $FX = M \times X$, where M is a fixed object in one of our categories.

(i) As an endofunctor on Set, $\mu F = \emptyset$. The same holds in Pos, MS, and CMS.

(ii) We next consider the situation in $\mathsf{CPO}_\perp$. For a cpo M the initial chain

$$\{\perp\} \longrightarrow M \times \{\perp\} \longrightarrow M \times M \times \{\perp\} \longrightarrow \cdots$$

The tuples in each factor are ordered componentwise. We write M^ω for the set of sequences of elements of M. The colimit in Pos is the set of elements in M^ω which have all but finitely many components equal to $\perp$. This is not a cpo: every infinite word $(m_0, m_1, \ldots,) \in M^\omega$ is a join of a chain:

$$(m_0, m_1, \ldots) \quad = \quad \coprod_k (m_0, m_1, \ldots, m_{k-1}, \perp, \perp, \ldots).$$

With a litte more work, it can be shown that

$$\mu X.(M \times X) \quad = \quad M^\omega.$$

The algebra structure $M \times M^\omega \to M^\omega$ adjoins a new head to a sequence in the evident manner.

Example 2.2.11. Polynomial functors. Let Σ be a (possibly non-finitary) signature, that is, a set together with arities, which are cardinals assigned to members of Σ. So Σ is a family Σ_k indexed by cardinal numbers k.

We denote by H_Σ the corresponding *polynomial functor*

$$H_\Sigma X = \coprod \Sigma_k \times X^k$$

where the coproduct ranges over all cardinals k with $\Sigma_k \neq \emptyset$. An algebra for H_Σ is a *general* Σ*-algebra*: a set A equipped with a family of functions (called *operations*) $f_\sigma : A^k \to A$ indexed by $\sigma \in \Sigma_k$.

As an example we consider the algebras of Σ-trees. By a Σ*-tree* is meant an ordered tree with nodes labelled in Σ in such a way that every node with n children is labelled by an n-ary symbol. We can form the Σ-algebra

$$T_\Sigma = \text{ all } \Sigma\text{-trees}$$

whose Σ-operations are given by tree tupling.

For finitary signatures Σ (i. e., $\Sigma_k \neq \emptyset$ holds for finite cardinal numbers only), the initial chain yields

$$\mu H_\Sigma = F_\Sigma = \text{ the algebra of finite } \Sigma\text{-trees,}$$

(for infinitary signatures we describe μH_Σ in Example 3.1.7 below). In fact, we can identify $H^1_\Sigma 0 = H_\Sigma \emptyset = \Sigma_0$ with the set of one-point trees labelled by an element of Σ_0, and

$$H^2_\Sigma 0 = H_\Sigma(\Sigma_0) = \Sigma_0 + \coprod_{k>0} \Sigma_k \times \Sigma_0^k$$

with the set of all Σ-trees of depth at most 1; more generally, $H^n_\Sigma 0$ with the set of Σ-trees of height less than n. We obtain the chain of inclusion maps

$$\emptyset \hookrightarrow H^1_\Sigma 0 \hookrightarrow H^2_\Sigma 0 \hookrightarrow \cdots$$

whose colimit (union) is F_Σ.

Example 2.2.12. For any cardinal κ, $\mathcal{P}_\kappa : \mathsf{Set} \to \mathsf{Set}$ is defined by taking a set to the collection of all subsets of size $< \kappa$, and on functions by image sets. (That is, for $a \subseteq X$ and $f : X \to Y$, $\mathcal{P}_\kappa f(a) = f[a] = \{f(x) : x \in a\}$.) When $\kappa = \omega$, we write $\mathcal{P}_\mathrm{f}$ in lieu of $\mathcal{P}_\omega$. For $\kappa \leq \omega$, $\mathcal{P}_\kappa$ preserves ω-colimits, and so we may obtain an initial algebra by the initial sequence.

(1) The finite power set endofunctor $\mathcal{P}_\mathrm{f} : \mathsf{Set} \to \mathsf{Set}$ will be of special interest in this paper. Its initial sequence is

$$\emptyset \longrightarrow \{\emptyset\} \longrightarrow \{\emptyset, \{\emptyset\}\} \longrightarrow \cdots$$

This sequence is also written

$$V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow$$

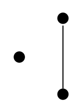
where $V_0 = \emptyset$, and $V_{n+1} = \mathcal{P}V_n$.

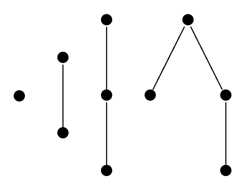
Hence $\mu\mathcal{P}_{\mathrm{f}} = V_\omega$ where $V_\omega = \bigcup_{n<\omega} V_n$. The algebra structure is the identity: observe that $\mathcal{P}_{\mathrm{f}}(V_\omega) = V_\omega$. V_ω is also called the set of *hereditarily finite (well-founded) sets*. These are the sets which are well-founded and have a finite transitive closure. (That is, all elements are finite, all elements of elements are finite, etc.)

(2) An alternative important representation of $\mu\mathcal{P}_{\mathrm{f}}$ is by finite *extensional trees*. Recall from Remark 2.2.6 that trees are considered up to isomorphism.) A tree is *extensional* if for every pair of children of a given node the two corresponding subtrees are different. Every unordered tree has an *extensional quotient* obtained by successively identifying children of a given node with the same subtrees. (See also Section 2.5 for more on extensional and strongly extensional quotients of trees.)

Observe that if $\mathcal{P}_{\mathrm{f}}(\emptyset)$ is represented by the singleton tree, then we have a natural representation of $\mathcal{P}^n_{\mathrm{f}}(\emptyset)$ by all finite extensional trees of depth less than n: the tree representing a set $\{\, x_1, \ldots, x_n \,\}$ has n children (representing x_i):

$\mathcal{P}^1_{\mathrm{f}}(\emptyset) = \{\emptyset\}$ is represented by

$\mathcal{P}^2_{\mathrm{f}}(\emptyset) = \{\emptyset, \{\emptyset\}\}$ is represented by 

$\mathcal{P}^3_{\mathrm{f}}(\emptyset) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ is represented by 

Etc. Therefore, we obtain

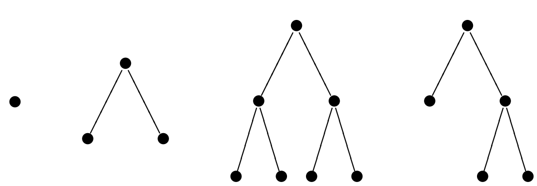
$$\mu\mathcal{P}_{\mathrm{f}} = \text{ all finite extensional trees}$$

with the algebra operations $\mathcal{P}_{\mathrm{f}}(\mu\mathcal{P}_{\mathrm{f}}) \to \mu\mathcal{P}_{\mathrm{f}}$ given by tree tupling.

(3) We analogously have the functor $\mathcal{P}_3$ of all subsets of at most two elements.

$$\mu\mathcal{P}_3 = \text{ all finite extensional trees with branching degree at most 2.}$$

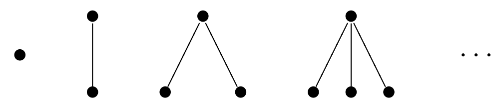
Alternatively, $\mu\mathcal{P}_3$ is given by all binary (unordered) trees: we represent a subset $\{\, a, b \,\}$ by the binary tree with children a and b. Thus, $\mathcal{P}^3_3(\emptyset)$ is now represented by the trees corresponding to $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ as follows:



It is not surprising that we have two (quite natural) representations of $\mu\mathcal{P}_3$: recall that initial algebras are only unique up to isomorphism.

Example 2.2.13. A *bag* is a pair (X, b), where X is a set and $b : X \to \mathbb{N}$ has the property that for all but finitely many x, $b(x) = 0$. (These are also called *finite multisets*.) The *size* of (X, b) is $\sum_{x \in X} b(x)$. A morphism of bags $m : (X, b) \to (Y, c)$ is a function $m \colon X \to Y$ with $c(y) = \sum_{m(x)=y} b(x)$ for all $y \in Y$.

We have a functor $\mathcal{B} : \mathsf{Set} \to \mathsf{Set}$ taking X to the bags on X. It preserves ω-colimits, and so Theorem 2.1.9 applies. The set $\mathcal{B}\emptyset$ has a single element which we represent as the single-node tree. Given a representation of $\mathcal{B}^n\emptyset$ by (unordered) trees, represent $\mathcal{B}^{n+1}\emptyset$ as follows: every bag consisting of trees $t_1, \dots, t_n \in \mathcal{B}^n\emptyset$ with multiplicities $k_1, \dots, k_n$ is represented by the tree having k_i children given by t_i for $i = 1, \dots, n$. Thus, $\mathcal{B}\mathcal{B}\emptyset$ are all trees



and $\mathcal{B}^3\emptyset$ are all finite trees of depth at most 2, etc.

The colimit of the initial sequence is again the union. The initial algebra $\mu\mathcal{B}$ can be described as the algebra of all finite (unordered) trees.

Example 2.2.14. Generalizing several of the above examples, we recall the concept of an *analytic functor* introduced by Andre Joyal [J1, J2]: given a group G of permutations on $k = \{0, \dots, k-1\}$, we denote by X^k/G the set of orbits under the action of G on X^n by coordinate interchange, i. e., X^n/G is the quotient of X^n modulo the equivalence $\sim_G$ with $(x_1, \dots, x_n) \sim_G (x_{p(1)}, \dots, x_{p(n)})$ for each $p \in G$. The analytic functors are precisely the coproducts of functors of the form X^k/G: in symbols, an analytic functor$F$ is of the form

$$FX = \coprod_{\sigma \in \Sigma} X^k/G_\sigma, \tag{2.5}$$

where Σ is a finitary signature, k the arity of $\sigma \in \Sigma$ and G_σ a group of permutations on k. Thus, every polynomial functor H_Σ is analytic, and $\mathcal{P}_3$ is analytic:

$$\mathcal{P}_2 X = 1 + (X \times X)/S_2,$$

where S_2 is the symmetric group of order 2. In contrast, $\mathcal{P}_f$ is not analytic. But an important example is the bag functor

$$\mathcal{B}X = \coprod_{k \in \mathbb{N}} X^k/S_k, \tag{2.6}$$

where S_k is the symmetric group of all permutations on k, is analytic.

For every analytic functor F the initial ω-chain yields a quotient of the algebra F_Σ of Example 2.2.11.

$$\mu F = F_\Sigma/\!\sim$$

where $\sim$ is the least equivalence such that for every tree $t \in F_\Sigma$, every node x of t labelled by $\sigma \in \Sigma$ and every member g of the group G_σ for σ we have $t \sim t'$, where t' is obtained from t by permuting the children of x according to g.

In fact, this is completely analogous to Example 2.2.11. We identify $F^1 0 = \Sigma_0$ with one-point trees labelled in Σ_0. And since $F^{n+1}0 = \coprod_{\sigma \in \Sigma} (F^n 0)^k / G_\sigma$, where σ is a k-ary symbol, we can identify, for every $n \in \mathbb{N}$, the set $F^n 0$ with the quotient of the set of all Σ-trees of depth less than n modulo the above equivalence $\sim$. We again obtain a chain of inclusion maps whose colimit (union) is $F_\Sigma/\!\sim$. This is illustrated by the cases $\mathcal{B}$ in Example 2.2.13 and $\mathcal{P}_3$ in Example 2.2.12(3).

The examples presented so far are probably the main ones for our paper. We mention an additional example of initial algebra obtainable by iteration via the initial chain in a category different from Set.

Example 2.2.15. Here is an example on a category which will re-appear in Section 5.1. Let BiP be the category of *bipointed* sets: objects are sets with distinguished elements $\bot$ and $\top$ that are required to be different. A morphism in BiP is a function preserving the distinguished points. There is a binary operation $\oplus$ on BiP taking $(X, \bot_X, \top_X)$ and $(Y, \bot_Y, \top_Y)$ to the disjoint union $X + Y$, identifying $\top_X$ and $\bot_Y$, and then using as distinguished points $\bot_X$ and $\top_Y$. Then $FX = X \oplus X$ extends to an obvious functor.

One example of an F-algebra is the the set D of dyadic rationals in $[0, 1]$ with 0 as $\bot$ and 1 as $\top$. The algebra structure $d\colon FD \to D$ takes x in the left-hand copy of D to $\frac{x}{2}$ and y in the right-hand copy to $\frac{y+1}{2}$. Then (D, d) is an initial F-algebra. One way to see this is to check that F preserves ω-colimits, and then to build an isomorphism between (D, d) and the algebra obtained by iteration.

Continuing with a discussion of examples of initial algebras obtained by ω-iteration on categories other than Set, we mention a result implying that for some special categories, *every* endofunctor has an initial algebra obtained that way.

Definition 2.2.16 (Freyd)**.** [Fr1] A category is called *algebraically complete* if every endofunctor has an initial algebra.

Theorem 2.2.17 (Adámek [Ad$_1$])**.** *The categories* Set_c *(countable sets and functions),* Rel_c *(countable sets and relations), and* K-Vec_c *(countably-dimensional vector spaces over a field* K *and linear functions) are algebraically complete; moreover, for every endofunctor* F*:*

$$\mu F = \operatorname*{colim}_{n<\omega} F^n 0.$$

Remark 2.2.18. (i) Every complete lattice is algebraically complete by the classical fixed-point theorem of G. Birkhoff.

(ii) Among categories with products there are essentially no other examples:

Theorem 2.2.19 (Adámek and Koubek [AK79])**.** *Every algebraically complete category with products is equivalent to a preorder.*

The following beautiful proof was provided by Peter Freyd, see [Fr1]: suppose $\mathcal{A}$ has products and is not equivalent to a preorder. That is, some hom-set $\mathcal{A}(A, B)$ has at least two elements. Then the functor

$$FX = B^{S(X)} \text{ (power of } B \text{ to the set } S(X))$$

where

$$S = \mathsf{Set}(\mathcal{A}(A, -), 2)\colon \mathcal{A} \to \mathsf{Set}^{op}$$

does not have fixed points. In fact, assuming $D \simeq FD$, we conclude that $\mathcal{A}(A, D)$ is isomorphic to

$$\mathcal{A}(A, D) \simeq \mathcal{A}(A, FD) \simeq \mathcal{A}(A, B)^{S(D)}.$$

But the cardinality of the right-hand side is at least

$$2^{S(D)} \simeq 2^{2^{\mathcal{A}(A,D)}}$$

a contradiction.

2.3 Terminal coalgebras

It is straightforward to dualize the general results of the last section. Recall from Section 1.2 that a *coalgebra* for an endofunctor F is an object A together with a morphism $a\colon A \to FA$. One dualizes initial objects 0 to terminal objects 1, ω-chains to ω^{op}-chains (that is, functors from ω^{op} to $\mathcal{A}$), colimits to limits, and the initial ω-chain of Definition 2.1.7 to the *terminal ω^{op}-chain* given by

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} F^2 1 \xleftarrow{\;F^2!\;} \cdots \xleftarrow{\;F^{n-1}!\;} F^n 1 \xleftarrow{\;F^n!\;} F^{n+1}1 \xleftarrow{\;F^{n+1}!\;} \cdots \tag{2.7}$$

Notation 2.3.1. We denote by

$$\ell_n : \lim_{i \in \omega^{op}} F^i 1 \to F^n 1$$

the projections of the limit of the terminal ω^{op}-chain of F. If F has a terminal coalgebra, we denote it by

$$\nu F \quad \text{or} \quad \nu X.F(X).$$

Construction 2.3.2. Dually to Construction 2.1.8 every coalgebra $\alpha : A \to FA$ induces a canonical cone over the terminal ω^{op}-chain of F by induction: $\alpha_0 : A \to 1$ is uniquely determined and $\alpha_{n+1} = F\alpha_n \cdot \alpha : A \to F^{n+1}1$.

It is worthwhile putting down the dual statement of Theorem 2.1.9. This was first explicitly formulated by Michael Barr [Barr]:

Theorem 2.3.3. *Let $\mathcal{A}$ be a category with terminal object 1 and with limits of ω^{op}-chains. If $F : \mathcal{A} \to \mathcal{A}$ preserves limits of ω^{op}-chains, then it has the terminal coalgebra*

$$\nu F = \lim_{n \in \omega^{op}} F^n 1.$$

Remark 2.3.4. It is sufficient to assume that F preserves the limit of its terminal chain.

We revisit the examples from Section 2.2.

Examples 2.3.5. (i) The functor $FX = X + 1$ on Set. The terminal chain is

$$1 \longleftarrow 1+1 \longleftarrow 1+1+1 \longleftarrow \cdots$$

The n-th term may be identified with $n = \{0, 1, \ldots, n-1\}$, and the connecting function $F^n! = f_n : n+1 \to n$ is then given by $f_n(i) = i$ for $i < n$, and $f_n(n) = n-1$. The limit of this chain is the set of ω-tuples $(x_0, \ldots, x_n, \ldots)$ with $f_n(x_{n+1}) = x_n$ for all n. One such tuple is $\top = (0, 1, 2, \ldots)$. Every other tuple is of the form $(0, 1, \ldots, k, k, \ldots, k, \ldots)$. Thus we describe the terminal coalgebra as

$$\nu X.(X+1) \quad = \quad \mathbb{N}^\top,$$

the set of natural numbers with an element $\top$ added. The coalgebra structure $\mathbb{N}^\top \to \mathbb{N}^\top + 1$ has $\top$ as a fixed point, sends 0 to the point in the right summand 1, and is otherwise the predecessor function on $\mathbb{N}$.

A coalgebra for $FX = X + 1$ is a set A together with a partial endofunction α (a function from A to $A + 1$, where we think of the element of 1 as an "undefined" element). Thus we get the following dual to classical induction on $\mathbb{N}$ as we have seen it in Example 2.2.1: in order to define a function $f : A \to \mathbb{N}^\top$, all we need is a partial endofunction $\alpha : A \rightharpoonup A$ on A. Then f is the unique coalgebra morphism from (A, α) to $\mathbb{N}^\top$. Explicitly, if there is some n such that $\alpha^n(a)$ is undefined, then $f(a) = n - 1$ for the least such n. Otherwise $\alpha^n(a)$ is defined for all n, and we have $f(a) = \top$.

As a concrete example of this method of *definition by corecursion* , let Σ be a set which we think of as an *alphabet*, and let

$$\Sigma^\infty \quad = \quad \Sigma^* + \Sigma^\omega$$

be the set of all finite and infinite words on Σ. We can define the *length function* $l : \Sigma^\infty \to \mathbb{N}^\top$ as soon as we specify a partial function $\alpha : \Sigma^\infty \rightharpoonup \Sigma^\infty$. We take $\alpha(w)$ to be the tail of w if w is not-empty; that is, α removes the first letter whenever possible. Then the resulting $l : \Sigma^\infty \to \mathbb{N}^\top$ is the desired length function, and $l(w) = \top$ for infinite words w.

(ii) The functor $FX = X + 1$ on Pos. Here the terminal coalgebra is $\mathbb{N}^\top$ with the discrete order. The structure is the same map as in item (i) above.

The functor $FX = X_\perp$ (cf. Example 2.2.2) has $\nu F = \mathbb{N}^\top$ with the usual order $0 < 1 < 2 < \cdots < \top$ and the same structure as before.

(iii) The functor $FX = X + 1$ on $\mathsf{CPO}_\perp$ essentially is the identity functor, whence $\nu F = 1 = \{\perp\}$. The functor $FX = X_\perp$ has as terminal coalgebra the same poset as in item (ii) above. This is not surprising: $\mathsf{CPO}_\perp$ is closed in Pos under limits (but not colimits).

(iv) On the category MS of 1-bounded metric spaces, the terminal coalgebra of the same functor $FX = X + 1$ is again $\mathbb{N}^\top$, this time with a discrete metric. The structure is the same as we have seen.

(v) On CMS, the terminal coalgebra of $FX = X + 1$ is what we saw in item (iv) just above. When we change the functor to scale the metric by $\frac{1}{2}$ in each summand, we get the same complete metric space that we had Example 2.2.5. However, the structure is the inverse. This is an example of a more general phenomenon that we shall explore in Section 2.9 below.

Example 2.3.6. The functor $FX = M \times X$ where M is a fixed (but arbitrary) object in the category. On Set, we identify each set with its product with a singleton and therefore write the terminal chain as

$$1 \longleftarrow M \longleftarrow M \times M \longleftarrow M \times M \times M \longleftarrow \cdots$$

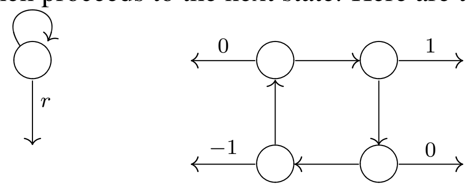
The connecting morphisms are all the projections onto the left-most factors. We obtain as a limit the set of all *streams* on M, also known as the *infinite words* on M:

$$\nu X.(M \times X) = M^\omega.$$

The coalgebra structure $M^\omega \to M \times M^\omega$ is $\langle \mathsf{head}, \mathsf{tail} \rangle$, where

$$\begin{array}{lcl} \mathsf{head}(a_1, a_2, a_3 \ldots) & = & a_1 \\ \mathsf{tail}(a_1, a_2, a_3 \ldots) & = & (a_2, a_3 \ldots) \end{array}$$

As a concrete example, take $M = \mathbb{R}$, the set of real numbers. A coalgebra for $\mathbb{R} \times X$ is then a set A (of states) and a function $\delta : A \to \mathbb{R} \times A$. We think of the coalgebra as an automaton which, in each state, yields an output number and then proceeds to the next state. Here are two examples:



One way to represent real valued streams is by taking the set A of *real analytic functions*, see [PE]. Here one considers $f : \mathbb{R} \to \mathbb{R}$ such that for every n there is an open interval around 0 such that $f'(x)$ is in the interval, and f agrees with its Taylor series. The coalgebra structure on A is given by

$$\varphi(x) \mapsto (\varphi(0), \varphi'(x))$$

for every analytic function. This coalgebra is isomorphic to the a subcoalgebra of the stream coalgebra that we saw above; to every analytic function $\varphi(x)$ one associates the stream of coefficients of the Taylor series of $\varphi(x)$, i. e., A is isomorpic to the subcoalgebra of those streams σ in $\mathbb{R}^\omega$ such that $\sum_{i=0}^{\infty} \frac{\sigma_i}{i!} x_i < \infty$. The above automata thus present analytic functions by corecursion. In the coalgebra on the left, we have a function whose value at 0 is r and equal to its own derivative: $\varphi(x) = re^x$. On the right, we obtain four functions:

$$\sin x, \cos x, -\sin x, -\cos x.$$

Example 2.3.7. For every signature Σ the terminal ω^{op}-chain for the functor H_Σ yields the terminal coalgebra

$$\nu H_\Sigma = T_\Sigma$$

of all (ordered, finite and infinite) Σ-trees. This holds not only for the finitary case but for general signatures:

In fact, let us identify $1 = H_\Sigma^0 1$ with the set consisting of the singleton tree labelled by a symbol $\perp \notin \Sigma$. Then $H_\Sigma^1 1 = \coprod \Sigma_k \times 1^k$ can be identified with the set of all trees of depth ≤ 1 whose root is labelled in Σ and whose k (for root label of arity k) leaves are labelled by $\perp$. Analogously $H_\Sigma^2 1 = \coprod \Sigma_k \times (H_\Sigma 1)^k$ are trees with a root labelled in Σ_k and the k subtrees are the above trees in $H_\Sigma^1 1$; thus all leaves of depth 2 have label $\perp$ etc. Put $\Sigma_\perp = \Sigma + \{\perp\}$, with $\perp$ nullary. We see that for every $n < \omega$ we can identify

$$H_\Sigma^n 1 = \text{all } \Sigma_\perp\text{-trees of depth at most } n \text{ with all leaves at depth } n \text{ labelled by } \perp.$$

Moreover, the connecting map

$$H_\Sigma^{n+1} \longrightarrow H_\Sigma^n 1$$

simply cuts the $\perp$-labelled leaves away and relabels all leaves of depth n by $\perp$. Now the limit of this ω^{op}-chain is

$$H_\Sigma^\omega 1 = T_\Sigma,$$

the set of all Σ-trees, with the limit cone $T_\Sigma \to H_\Sigma^n 1$ given by cutting all trees at level n and labelling all leaves of depth n by $\perp$.

We revisit this example in Section 2.9 where we use metrics on trees.

Example 2.3.8. A deterministic automaton with input set S is a set A (of states) and the next-state function $A \times S \to A$ that we can write in the curried form $A \to A^S$, plus the predicate $A \to \{0, 1\}$ of accepting states. This can be viewed as a coalgebra for $FX = \{0, 1\} \times X^S$. This is the polynomial functor of two S-ary operations. Applying the previous example we see that the terminal coalgebra consists of all labellings of nodes of the complete S-ary tree by 0 or 1. Now the complete S-ary tree can be represented by the set S^* of all words over S: the empty word ε is the root, and the children of every node $x_1 \cdots x_n$ are all the nodes $x_1 \cdots x_n y$ where y ranges over S. In this sense a binary labelling of the complete S-ary tree is simply a subset of S^*. We get that

$$\nu F = \mathcal{P} S^*$$

is the coalgebra of all formal languages on M.

Example 2.3.9. For every analytic functor F (see Example 2.2.14) the terminal coalgebra is obtained by the terminal ω^{op}-chain. In fact, it is easy to derive from the formula (2.5) that analytic functors preserve limits of ω^{op}-chains. We will see a description of this terminal coalgebra in Example 2.7.6(iii) below. The following example is a special case.

Example 2.3.10. The bag functor $\mathcal{B}$ of Example 2.2.13 has the terminal coalgebra

$$\nu \mathcal{B} = \text{ all finitely branching unordered trees.}$$

In fact, we can identify $\mathcal{B}^n 1$, analogously as in Example 2.3.7, with all unordered finitely branching trees of depth $\leq n$ whose leaves at level n are labelled by $\perp$, The limit $\nu\mathcal{B}$ is then as stated.

Example 2.3.11. The finite power-set functor $\mathcal{P}_\mathrm{f}$ is an example of a set functor which does not preserve limits of ω^{op}-chains, and, moreover, the limit of its terminal ω^{op}-chain is not the terminal $\mathcal{P}_\mathrm{f}$-coalgebra, see [AK95]. We will describe the terminal coalgebra below, see Section 2.4.

2.4 Weakly terminal coalgebras, and finitary functors

This section blends several topics. One aim is to present constructions of the terminal coalgebra for the finite power set functor $\mathcal{P}_\mathrm{f}$ (we eventually get to that in Example 2.4.19, Theorem 2.6.3 and Example 2.7.6(ii)). Unfortunately, this functor does not preserve limits of ω^{op}-chains, and the terminal coalgebra is not obtained as the limit of the terminal ω^{op}-chain, cf. Theorem 2.3.3. It would be possible to directly construct $\nu\mathcal{P}_\mathrm{f}$, but we have chosen a more general and expansive presentation. The other ideas in this section are those of a weakly terminal coalgebra, finitary functors, and congruences.

Definition 2.4.1. Let $F : \mathcal{A} \to \mathcal{A}$ be an endofunctor on an arbitrary category. An F-coalgebra A is *weakly terminal* if for every coalgebra B there is *at least one* homomorphism of B into A.

Examples 2.4.2. (i) The terminal coalgebra of the identity functor $Id : \mathsf{Set} \to \mathsf{Set}$ is the singleton coalgebra. A coalgebra $\alpha : A \to A$ is weakly terminal iff α has a fixed point.

(ii) If A is a weakly terminal coalgebra and $f : A \to B$ a coalgebra homomorphism, then B is again weakly terminal.

(iii) We next consider the finite power-set functor $\mathcal{P}_\mathrm{f} : \mathsf{Set} \to \mathsf{Set}$. The coalgebra T_Σ of all ordered finitely branching trees is weakly terminal for $\mathcal{P}_\mathrm{f}$. The corresponding coalgebra structure $\delta : T_\Sigma \to \mathcal{P}_\mathrm{f} T_\Sigma$ is defined by[2]

$$\delta(t) = \{\, t_x : x \text{ a child of the root of } t \,\} \tag{2.8}$$

Indeed, recall that every $\mathcal{P}_\mathrm{f}$-coalgebra A is considered to be a finitely branching graph: the function $\alpha : A \to \mathcal{P}_\mathrm{f} A$ gives to every node x the set $\alpha(x)$ of all neighbors. From every coalgebra we have a canonical, albeit not unique homomorphism h into T_Σ. It maps every vertex x of A to its tree unfolding obtained by breadth-first search starting in x. It is easy to see that h is a homomorphism. But it is not unique: the one-point "loop graph" has many homomorphisms; for example we can map it to the infinite binary tree and to the infinite chain. We shall see in Example 2.4.19(i) below a quotient of T_Σ which is a terminal coalgebra.

(iv) The power-set functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ does not have a weakly terminal coalgebra. Indeed, from a weakly terminal coalgebra we can always construct a terminal one, see Theorem 2.4.16.

An important source of weakly terminal coalgebras stems from presentations of set functors F as quotients of polynomial functors H_Σ. Recall that quotients of an object A in a category are (represented by) epimorphisms with domain A; here this means natural transformations with domain H_Σ having all components surjective.

Definition 2.4.3. An *epitransformation* $G \to F$ is a natural transformation having epimorhic components. By a *presentation* of a set functor F we mean a signature Σ and an epitransformation $\varepsilon : H_\Sigma \to F$.

Examples 2.4.4. (i) We shall be using a presentation of $\mathcal{P}_\mathrm{f}$ given by the signature Σ having, for each n, just one function symbol σ_n of arity n. Thus,

$$H_\Sigma X = \coprod_{n<\omega} X^n = X^*.$$

The natural transformation $\varepsilon : H_\Sigma \to \mathcal{P}_\mathrm{f}$ is given by

$$\varepsilon_X(x_1, \ldots, x_n) \quad = \quad \{x_1, \ldots, x_n\}$$

(ii) A presentation for the bag functor $\mathcal{B}$ can easily be obtained from (2.6). Take the same Σ as in the previous item and let $\varepsilon_X : H_\Sigma X \to \mathcal{B}X$ be given as the coproduct of the quotients $X^n \to X^n/S_n$, $n \in \mathbb{N}$, where S_n is the symmetric group.

(iii) The Aczel-Mendler (see [AcM]) functor $(-)^3_2$ is given by

$$(A)^3_2 \quad = \quad \{(a,b,c) \in A \times A \times A : (a=b) \text{ or } (a=c) \text{ or } (b=c)\}.$$

Note that if $(a,b,c) \in (A)^3_2$ and $f : A \to B$, then $(fa, fb, fc) \in (B)^3_2$. We therefore take the action of $(-)^3_2$ on morphisms by pointwise application, as expected. For a presentation of $(-)^3_2$ we take Σ to have three binary symbols, say σ, τ, and ρ. Then we define $\varepsilon : H_\Sigma \to (-)^3_2$ by: $\varepsilon_A(\sigma(a,b)) = (a,a,b)$, $\varepsilon_A(\tau(a,b)) = (a,b,a)$, and $\varepsilon_A(\rho(a,b)) = (b,a,a)$.

[2] From now on we often write t_x for the subtree rooted at a node x of a tree t.

Observation 2.4.5. Note that every presentation $\varepsilon : H_\Sigma \to F$ induces an embedding from the category of H_Σ-coalgebras to the category of F-coalgebras by post-composition:

$$(A \xrightarrow{\alpha} H_\Sigma A) \qquad \mapsto \qquad (A \xrightarrow{\alpha} H_\Sigma A \xrightarrow{\varepsilon_A} FA).$$

For example, the terminal coalgebra $\nu H_\Sigma = T_\Sigma$ of all Σ-trees can be understood as an F-coalgebra.

Lemma 2.4.6. *Let $\varepsilon : H_\Sigma \to F$ be a presentation of the functor $F : \mathsf{Set} \to \mathsf{Set}$. Then the F-coalgebra T_Σ is weakly terminal.*

Proof. Let $\alpha : A \to FA$ be any F-coalgebra. Since $\varepsilon_A : H_\Sigma A \to FA$ is a surjection, we can choose a morphism $m : FA \to H_\Sigma A$ such that $\varepsilon_A \cdot m = id$. So we obtain an H_Σ-coalgebra $m \cdot \alpha : A \to H_\Sigma A$. We thus have an H_Σ-coalgebra homomorphism $h : A \to T_\Sigma$, and we show that h is also an F-coalgebra homomorphism from (A, α) to $(T_\Sigma, \varepsilon_{T_\Sigma} \cdot \tau)$, where $\tau : T_\Sigma \to H_\Sigma T_\Sigma$ is the structure of the terminal coalgebra for H_Σ. Indeed, consider the diagram below:

$$\begin{tikzcd}
A \arrow[r, "m\cdot\alpha"] \arrow[rr, bend left, "\alpha"] \arrow[d, "h"'] & H_\Sigma A \arrow[r, "\varepsilon_A"] \arrow[d, "H_\Sigma h"] & FA \arrow[d, "Fh"] \\
T_\Sigma \arrow[r, "\tau"'] & H_\Sigma T_\Sigma \arrow[r, "\varepsilon_{T_\Sigma}"'] & FT_\Sigma
\end{tikzcd}$$

The top commutes because $\varepsilon_A \cdot m = id$, the square on the left by definition of h, and the one on the right by naturality of ε. Thus the outside commutes, showing h to be a morphism of F-coalgebras. $\square$

Examples 2.4.7. (i) The coalgebra T_Σ of all finitely branching trees is weakly terminal for $\mathcal{P}_\mathrm{f}$. We saw this in Example 2.4.2(iii). But we can also apply Lemma 2.4.6 to the presentation of $\mathcal{P}_\mathrm{f}$ in Example 2.4.4(i).

(ii) The functor $\mathcal{P}_\mathsf{c}$ of countable subsets can be presented by the signature Σ with one ω-ary symbol σ and one constant c. The natural transformation ε has components $\varepsilon_X : X^\omega + 1 \to \mathcal{P}_\mathsf{c} X$ that take $(x_n)_{n<\omega}$ to $\{ x_n \mid n < \omega \}$ and the element of the right-hand component to $\emptyset \in \mathcal{P}_\mathsf{c} X$. Thus, we see that the terminal H_Σ-coalgebra of all countably branching trees is weakly terminal for $\mathcal{P}_\mathsf{c}$.

Recall that the main point of this section is to investigate terminal coalgebras for $\mathcal{P}_\mathrm{f}$. One key property that this functor enjoys is that it is *finitary*, and practically all results about $\mathcal{P}_\mathrm{f}$ in this paper generalize to the setting of finitary functors. For general functors "finitary" means preservation of filtered colimits, see Section 4.2. below. For endofunctors on Set we use the following definition (which is equivalent, see [AP04]):

For finitary endofunctors we shall prove that the limit of their terminal ω^{op}-chain is a weakly terminal coalgebra (which is in general not a terminal coalgebra).

Definition 2.4.8 ([AP04])**.** An endofunctor F of Set is called *finitary* if for each element $x \in FX$ there exists a finite subset $M \subseteq X$ such that $x \in Fi[FM]$ where $i : M \to X$ is the inclusion map.

Example 2.4.9. (i) For a signature Σ the polynomial functor H_Σ is finitary iff Σ is a finitary signature.

(ii) If F is finitary, then we have a presentation by a finitary signature Σ. Indeed, given F let Σ be the signature defined by $\Sigma_n = F(n)$ for all $n \in \mathbb{N}$. Define $\varepsilon_X : H_\Sigma X \to FX$ by assigning to $f : n \to X$ in the summand X^n corresponding to $\sigma \in \Sigma_n$ the value $\varepsilon_X(\sigma) = Ff(\sigma)$. It follows immediately that ε_n is surjective for each $n \in \mathbb{N}$, and since F is finitary, we conclude that all components are surjective.

Lemma 2.4.10 ([AT])**.** *Let* $F : \mathsf{Set} \to \mathsf{Set}$*. Then the following are equivalent:*

(i) F *is finitary.*

(ii) There is a finitary signature Σ *such that* F *is a quotient of* H_Σ*.*

Finitariness is an important concept, and we shall see later that the equivalent formulations above can also be stated in more general categories: see Sections 4.1 and 4.2.

Now let $F : \mathsf{Set} \to \mathsf{Set}$ be an endofunctor and recall the terminal ω^{op}-chain for F (see (2.7)) in Section 2.3). Let us denote its limit by

$$F^\omega 1 = \lim_{n \in \omega^{op}} F^n 1 \qquad \text{with limit projections } \ell_n : F^\omega 1 \to F^n 1 \text{ for all } n \in \omega^{op}.$$

Then we obtain a unique map $m : F(F^\omega 1) \to F^\omega 1$ having the property that for all n, the triangles below commute:

$$\begin{tikzcd} F(F^\omega 1) \arrow[rr, "m"] \arrow[dr, "F\ell_n"'] & & F^\omega 1 \arrow[dl, "\ell_{n+1}"] \\ & F^{n+1} 1 & \end{tikzcd} \tag{2.9}$$

The above limit $F^\omega 1$ is the terminal F-coalgebra whenever it is preserved by F. We shall prove below that $F^\omega 1$ is a weakly terminal coalgebra whenever the cone $F\ell_n$, $n \in \omega^{op}$, is *collectively monic*: for every pair $f, g : X \to F(F^\omega 1)$ if $F\ell_n \cdot f = F\ell_n \cdot g$ holds for all $n \in \omega^{op}$ then $f = g$. For finitary endofunctors the cone $F\ell_n$ is indeed collectively monic, see the proof of Lemma 2.4.12 below.

Lemma 2.4.11. *Let* $\ell_n : L \to L_n$ *be a limit cone of an* ω^{op}*-chain in* Set*. For every finite subset* $f : S \hookrightarrow L$ *there exists* n *such that* $\ell_n \cdot f$ *is a monomorphism.*

Proof. Recall that L may be taken to be the set of functions g with domain ω such that for all n, $g(n) \in L_n$ and such that for all n, $\ell_{n+1,n}(g(n+1)) = g(n)$. Moreover, we have $\ell_n(g) = g(n)$ for all n.

Let a and b be different members of $f[S]$. We think of a and b as functions on the natural numbers, and each ℓ_n works by applying these to the number n. Since $a \neq b$, we must have some $n = n(a,b)$ such that $\ell_n(a) = a(n) \neq b(n) = \ell_n(b)$. And for all $m > n$, we also have $\ell_m(a) \neq \ell_m(b)$. The set

$$\mathcal{S} \quad = \quad \{(a,b) \in f[S] \times f[S] : a \neq b\}$$

is also finite, and so $\max\{n(a,b) : (a,b) \in \mathcal{S}\}$ is a natural number, say k. For this k, $\ell_k \cdot f$ is injective. $\square$

Lemma 2.4.12 (Worrell [W2])**.** *Let* $F : \mathsf{Set} \to \mathsf{Set}$ *be finitary. Then* $m : F(F^\omega 1) \to F^\omega 1$ *in (2.9) is a split monomorphism.*

Proof. The desired statement holds trivially if F is constantly $\emptyset$. If not, then clearly $F1 \neq \emptyset$, thus, there exists a coalgebra $\alpha : 1 \to F1$. The canonical cone $\alpha_n : 1 \to F^n 1$ of Construction 2.3.2 induces a morphism $\alpha_\omega : 1 \to F^\omega 1$. This proves $F^\omega 1 \neq \emptyset$.

We now prove that $(F\ell_n)$ is a collectively monomorphic cone. Let $x, y \in F(F^\omega 1)$. Assuming that for all k, $F\ell_k(x) = F\ell_k(y)$, we show that $x = y$. By the assumption that F is finitary, there is a finite subset $f : S \hookrightarrow F^\omega 1$ and $x', y' \in FS$ such that $x = (Ff)x'$ and $y = (Ff)y'$. Without loss of generality we may assume that S is non-empty. We see that

$$(F\ell_k \cdot Ff)x' = F\ell_k(x) = F\ell_k(y) = (F\ell_k \cdot Ff)y'.$$

By Lemma 2.4.11, there exists some n such that $\ell_n \cdot f$ is injective. Functors on Set preserve injectivity of maps with non-empty domain because they are split monomorphisms. Thus, $F(\ell_k \cdot f)$ is injective. So $x' = y'$, and thus $x = y$.

From the definition (2.9) of m it now follows that it is monic. It is a split mono since $F(F^\omega 1) \neq \emptyset$ (indeed, consider $F\alpha_\omega : F1 \to F(F^\omega 1)$. □

Proposition 2.4.13. *For every finitary endofunctor F the coalgebra on $F^\omega 1$ given by an arbitrary splitting of m is weakly terminal.*

Proof. Let $\overline{m} : F^\omega 1 \to F(F^\omega 1)$ be a splitting of m, i. e., $\overline{m} \cdot m = id$. Let $\alpha : A \to FA$ be a coalgebra. From Construction 2.3.2 we know, we have a cone $\alpha_k : A \to F^k 1$ satisfying $\alpha_0 = !$ and $\alpha_{k+1} = F\alpha_k \cdot \alpha$. This cone then yields a unique $h : A \to F^\omega 1$ such that for all k, $\ell_k \cdot h = \alpha_k$. Notice also that $\ell_{k+1} \cdot m = F\ell_k$ implies $\ell_{k+1} = F\ell_k \cdot \overline{m}$.

We wish to prove that $Fh \cdot \alpha = \overline{m} \cdot h$. Because the cone $(F\ell_k)$ is collectively monic, it is sufficient to show that for all k, $F\ell_k \cdot Fh \cdot \alpha = F\ell_k \cdot \overline{m} \cdot h$. As the two computations below show, both compositions are α_{k+1}:

$$\begin{aligned} F\ell_k \cdot Fh \cdot \alpha &= F\alpha_k \cdot \alpha \\ &= \alpha_{k+1} \end{aligned} \qquad\qquad \begin{aligned} F\ell_k \cdot \overline{m} \cdot h &= \ell_{k+1} \cdot h \\ &= \alpha_{k+1} \end{aligned}$$

This completes the proof. □

Recall that one of the goals in this section is to construct a terminal coalgebra for the finite power set functor $\mathcal{P}_{\mathrm{f}}$. At this point, we have only a weakly terminal coalgebra. One construction involves taking a quotient of a weakly terminal coalgebra, and we turn to this next.

Definition 2.4.14. Let $\alpha : A \to FA$ be a coalgebra, and let $\overline{A}$ be a quotient of A represented by an epimorphism $e : A \to \overline{A}$. We say that e is a *congruence* if there is an (obviously unique) coalgebra structure $\overline{\alpha} : \overline{A} \to F\overline{A}$ making e a coalgebra homomorphism:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "e"'] & FA \arrow[d, "Fe"] \\ \overline{A} \arrow[r, "\overline{\alpha}"'] & F\overline{A} \end{tikzcd}$$

Since we are dealing with sets in this section, we often identify the epimorphism e with its kernel equivalence $\sim$ given by $a \sim b$ iff $e(a) = e(b)$. We write $A/{\sim}$ for the above coalgebra $\overline{A}$.

Example 2.4.15. Let G be a finitely branching graph, considered as a coalgebra for $\mathcal{P}_{\mathrm{f}}$. A congruence on G is then precisely an equivalence relation $\sim$ on G which is also a *graph bisimulation*, i. e., a relation $R \subseteq G \times G$ on the nodes such that whenever xRy, then

$$\begin{array}{l} \text{for every child } x' \text{ of } x \text{ there is some child } y' \text{ of } y \text{ such that } x'Ry', \text{ and} \\ \text{for every child } y' \text{ of } y \text{ there is some child } x' \text{ of } x \text{ such that } x'Ry'. \end{array} \tag{2.10}$$

We leave the easy proof of the fact that bisimulation equivalences are precisely the congruences to the reader.

Although we will not be using the concept of bisimulation in this section, we remind readers familiar with it that the largest bisimulation relation on a graph is always an equivalence relation, and hence a congruence of $\mathcal{P}_{\mathrm{f}}$-coalgebras.

Theorem 2.4.16. *Let F be an endofunctor of* Set. *Then for every weakly terminal coalgebra A there exists the largest congruence $\sim$, and the coalgebra $A/{\sim}$ is terminal.*

Remark 2.4.17. Notice that the statement of the above theorem holds for endofunctors on every cocomplete and cowellpowered category. The proof is a standard argument using Freyd's Adjoint Functor Theorem, we include it for completeness.

Proof. Let $\alpha : A \to FA$ be a weakly terminal coalgebra. Since Set fulfils both conditions in Remark 2.4.17, there exists the greatest congruence $e : A \to \overline{A}$: take the join of all congruences in the complete lattice of all quotients of A; this is easily seen to be a congruence. The unique $\overline{\alpha} : \overline{A} \to F\overline{A}$ for which e is a coalgebra homomorphism is, obviously, a weakly terminal coalgebra (since A is, cf. Example 2.4.2(ii)). It remains to prove that if $f, g : B \to A$ are two coalgebra homomorphisms, then $f = g$. To this end, take the coequalizer $k : \overline{A} \to \widetilde{A}$ of f and g. Then since f and g are coalgebra homomorphisms, $Fk \cdot \overline{\alpha}$ merges them, and consequently there exists a unique coalgebra structure $\widetilde{\alpha} : \widetilde{A} \to F\widetilde{A}$ such that k is a coalgebra homomorphism. Thus, $k \cdot e : A \to \widetilde{A}$ is a congruence on A, but the choice of e as the largest congruence implies that k is an isomorphism. Therefore $f = g$, as desired. □

Example 2.4.18. For every finitary functor F with a presentation $H_\Sigma \to F$ we see that νF is the quotient of the weakly terminal coalgebra T_Σ of all Σ-trees (cf. Lemma 2.4.6) modulo the largest congruence $\sim$. In Section 2.7 below we shall provide a more explicit description of $\sim$.

Examples 2.4.19. We now have several descriptions of the terminal coalgebra for $\mathcal{P}_{\mathrm{f}}$.

(i) As a quotient $T_\Sigma/{\approx}$ of the coalgebra of all ordered finitely branching trees modulo the greatest congruence $\approx$, cf. Examples 2.4.7 and 2.4.2.

(ii) As a quotient of the coalgebra D of all unordered finitely branching trees (with the same structure as in (2.8)) modulo the greatest congruence. Indeed, this coalgebra D is a weakly terminal $\mathcal{P}_{\mathrm{f}}$-coalgebra (cf. Example 2.4.2(ii)) since the canonical quotient map $q : T_\Sigma \to D$ is obviously a coalgebra homomorphism.

(iii) As a subcoalgebra of the coalgebra $\mathcal{P}_{\mathrm{f}}^{\omega}1$. In fact, Worrel [W2] has shown that every finitary set functor F has a terminal coalgebra which is a subcoalgebra of $F^{\omega}1$ (cf. Theorem 2.6.3 below).

For $\mathcal{P}_{\mathrm{f}}$ we are interested in sharpening Theorem 2.4.16 by having a more explicit formulation of the largest congruence $\approx$ on T_Σ. That is, our work up until now has worked with $\approx$ as a relation on the whole of T_Σ. But to see whether $t \approx u$ for two particular trees t and u, we would prefer to look only at t, u, and their subtrees. The corresponding description in Corollary 2.4.23 is due to Barr [Barr].

An operator on the CPO of equivalence relations on T_Σ. We mentioned the weakly terminal $\mathcal{P}_{\mathrm{f}}$-coalgebra T_Σ of all ordered finitely-branching trees. Let $(\mathsf{Eq}(T_\Sigma), \leq)$ be the CPO of all equivalence relations on T_Σ, with $\leq$ the reverse inclusion: $R \leq S$ iff $S \supseteq R$. The least element $\perp$ is $T_\Sigma \times T_\Sigma$. Our operator f is defined so that an equivalence relation R is a congruence of the coalgebra T_Σ iff it is a fixed point of f (cf. Example 2.4.15). Let $f : \mathsf{Eq}(T_\Sigma) \to \mathsf{Eq}(T_\Sigma)$ be given by assigning to a relation R the relation $f(R)$ defined by

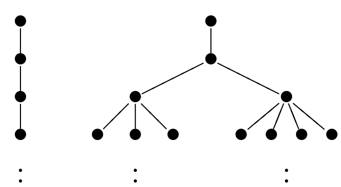
$$t\; f(R)\; u \qquad \text{iff} \qquad \begin{array}{l}\text{for all children } x \text{ of the root of } t \text{ there is some}\\ \text{child } y \text{ of the root of } u \text{ such that } t_x\; R\; u_y\text{; and vice-versa.}\end{array} \tag{2.11}$$

Lemma 2.4.20. *The operator f is continuous, and μf is the greatest congruence $\approx$ on the $\mathcal{P}_{\mathrm{f}}$-coalgebra T_Σ.*

Proof. We must verify that f is ω-continuous, and this is where the fact that T_Σ consists of *finitely branching* trees enters. Suppose that $R_0 \supseteq R_1 \supseteq \cdots R_n \supseteq \cdots$, and write R^* for $\bigcap_n R_n$ and S^* for $\bigcap_n f(R_n)$. Clearly, f is monotone, and so $f(R^*)$ is contained in S^*. We must show that if $t\; S^*\; u$, then $t\; f(R^*)\; u$. Let x be a child of the root of t. For all n, there is some child y of the root of u (depending on n) so that $x\; R_n\; y$. Since u is finitely branching, there is some fixed y so that for infinitely many n, $x\; R_n\; y$. But since the R_n are non-increasing, we have some y such that for *all* n, $x\; R_n\; y$. This concludes our verification of the continuity of f. By Kleene's Theorem 2.1.5, we obtain the least fixed point of f in ω steps: $\mu f = \bigcap_n f^n(\perp)$.

That this is the greatest congruence comes from the definition of the order on $\mathsf{Eq}(T_\Sigma)$ as reverse inclusion and the fact that an equivalence R on T_Σ is a congruence iff $f(R) = R$. $\square$

Henceforth, we denote the congruence μf by $\approx$. As an example, the two trees shown below are related by $\approx$:



(2.12)

Indeed, every finitely branching tree without leaves is again in the $\approx$ relation to these two trees. The easiest way to see this is to check by induction on n that for all trees t and u without leaves, $t\; f^n(\perp)\; u$.

We conclude this section with a statement of the representation of a terminal coalgebra for $\mathcal{P}_{\mathrm{f}}$, analogous to that of Barr [Barr]. To get started, recall the concept of an *extensional quotient* of an unordered tree from Example 2.2.12. Analogously, for an ordered tree t, we take the extensional quotient of the tree obtained from t by forgetting the ordering of children.

Notation 2.4.21. For any tree t we write $\partial_n t$ for the extensional quotient of the "cutting" of t at a height of n. Formally, this is the set of points whose distance from the root of t is at most n, considered as an induced subgraph of t. For example, $\partial_0 t$ is a one-point tree.

Proposition 2.4.22. *The congruence $\approx$ merges two trees t and s iff*

$$\partial_n t = \partial_n u \qquad \textit{for all } n < \omega. \tag{2.13}$$

Proof. Since $\approx$ is $\mu f = \bigcap_n f^n(\perp)$, we only need to verify by induction on n that $f^n(\perp)$ relates t and u iff $\partial_n t = \partial_n u$. The case $n = 0$ is clear: $\perp$ is $T_\Sigma \times T_\Sigma$, and $\partial_0 t = \partial_0 u$ is the root only tree. For the induction step we use (2.11): $f^{n+1}(\perp)$ relates t und u iff for every child x of the root of t there exists a child y of the root of u with $\partial_n t_x = \partial_n u_y$, and vice versa. Since the subtrees rooted at children of the root of $\partial_{n+1} t$ are precisely the trees $\partial_n t_x$, and analogously for $\partial_{n+1} u$, we conclude that $f^{n+1}(\perp)$ relates t and u iff $\partial_{n+1} t = \partial_{n+1} u$. $\square$

Corollary 2.4.23 (Barr [Barr])**.** *The quotient of the coalgebra D of all finitely branching trees modulo the congruence defined in (2.13) is terminal for $\mathcal{P}_\mathrm{f}$.*

2.5 A terminal coalgebra for $\mathcal{P}_\mathrm{f}$ using strongly extensional trees

We have already seen several descriptions of a terminal coalgebra for $\mathcal{P}_\mathrm{f} : \mathsf{Set} \to \mathsf{Set}$ in Section 2.4. This section gives another description, using *strongly extensional trees*. It is due to James Worrell [W2].

Recall that a tree t is extensional if distinct children of the same node define different subtrees, see Example 2.2.12(2).

Definition 2.5.1. A *tree bisimulation* between two trees t and u is a bisimulation (cf. Example 2.4.15) R between the sets of nodes such that the roots of t and u are related by R, and whenever two nodes are related by R, then they have the same distance from the root.

Two trees are called *tree bisimilar* if there is a tree bisimulation between them.

A tree t is called *strongly extensional* if every tree bisimulation on it is a subrelation of the identity. More explicitly, t is strongly extensional iff distinct children x and y of the same node define subtrees t_x and t_y which are not tree bisimilar.

Remark 2.5.2. (i) In (2.12) we see two extensional trees: the left-hand one is strongly extensional, the right-hand one is not; indeed, consider the relation relating all nodes of the same depths.

(ii) Every strongly extensional tree is clearly extensional.

(iii) It is trivial to prove that every composition of tree bisimulations is again a tree bisimulation. In addition, the opposite of every tree bisimulation is a tree bisimulation: if R is a tree bisimulation from t to u, then R^{op} is a tree bisimulation from u to t.

(iv) Observe that the notion of tree bisimulation is different from the usual graph bisimulation. For example, the picture below

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \arrow[d] \\ & & \bullet \end{tikzcd} \tag{2.14}$$

depicts a strongly extensional tree but there is a graph bisimulation relating the two leaves.

Proposition 2.5.3. *Let t be a finite tree. Then t is extensional iff t is strongly extensional.*

Proof. Let t be extensional, and let R be any tree bisimulation on t. We claim that for all n, if $x \mathrel{R} y$ and x and y have height n in t, then the corresponding subtrees t_x and t_y are equal. The proof is by induction on n. For $n = 0$, the result is obvious because the nodes of height 0 are leaves. Assume our result for n, and let x and y be related by R and of height $n + 1$. Then by the induction hypothesis and extensionality of t, for every child x' of x there is a *unique* child y' of y and $t_{x'} = t_{y'}$; and vice-versa. This implies that $t_x = t_y$.

It now follows that if t is an extensional tree, then t must be strongly extensional. $\square$

Definition 2.5.4. The *strongly extensional quotient* $\bar{t}$ of a tree t is the quotient tree obtained from t via its largest tree bisimulation.

Observe that every tree t is tree bisimilar to its extensional and strongly extensional quotients; the canonical quotient maps obviously are tree bisimulations.

Lemma 2.5.5. *If t and u are strongly extensional trees related by a tree bisimulation, then they are equal.*

Proof. Suppose we have a tree bisimilation R between t and u. Then $R^{op} \cdot R$ is a tree bisimulation on t, whence $R^{op} \cdot R \subseteq \Delta$ by strong extensionality. But every node of t is related to at least one node of u (use induction on the distance of nodes from the root). Thus $R^{op} \cdot R = \Delta$. Similarly, $R \cdot R^{op} = \Delta$. Thus, R (is a function and it) is an isomorphism of trees, and we identify such trees, see Remark 2.2.6. $\square$

Now recall from Example 2.4.19(ii) the coalgebra D of all (unordered) finitely branching trees. We write $\approx$ for the greatest congruence on D. This is defined by (2.13).

Theorem 2.5.6 (Worrell [W2])**.** *The coalgebra of all finitely branching strongly extensional trees (as a subcoalgebra of D) is a terminal coalgebra for $\mathcal{P}_{\mathrm{f}}$.*

Proof. The set $D_0 \subseteq D$ of all strongly extensional trees clearly forms a subcoalgebra. We prove that this is isomorphic to the terminal coalgebra $D/{\approx}$. For that we need to verify that given trees $t, u \in D$ then

$$t \approx u \qquad \text{iff} \qquad \bar{t} = \bar{u}.$$

Then the map $[t] \mapsto \bar{t}$ is an isomorphism from $D/{\approx}$ to D_0.

($\Rightarrow$) If $t \approx u$ we prove that t and u are tree bisimilar. Then, by Remark 2.5.2, it follows that $\bar{t}$ and $\bar{u}$ are tree bisimilar, which implies that $\bar{t} = \bar{u}$ by Lemma 2.5.5.

Define $R \subseteq t \times u$ by relating nodes $x \in t$ and $y \in u$ iff they have the same depth n and for every $m \geq n$ the node of $\partial_m t$ corresponding to x equals the node of $\partial_m u = \partial_m t$ corresponding to y. Using that t and u are finitely branching, it is not difficult to prove that R is a tree bisimulation. We leave the details to the reader.

($\Leftarrow$) Conversely, suppose we have $\bar{t} = \bar{u}$. Then we have, by composing with the quotient maps, a tree bisimulation $R \subseteq t \times u$. By restricting R to all pairs of nodes of depth at most n we obtain a tree bisimulation R_n between the cuttings of t and u at level n. Then also the extensional quotients $\partial_n t$ and $\partial_n u$ are bisimilar. Since $\partial_n t$ and $\partial_n u$ are strongly extensional by Proposition 2.5.3, we see that $\partial_n t = \partial_n u$ by Lemma 2.5.5. Thus, $t \approx u$. $\square$

We conclude this section by mentioning three concrete descriptions of the limit $\mathcal{P}^{\omega} 1$. Two of them make use of strongly extensional trees.

Example 2.5.7. The limit $\mathcal{P}_{\mathrm{f}}^{\omega} 1$ can be described as

(i) the set of all compactly branching strongly extensional trees as we shall see in Example 2.9.5 below,

(ii) the set of all saturated extensional trees, see [AMMS], and

(iii) the set of maximal consistent theories of modal logic, see [AMMS].

2.6 Terminal coalgebras for finitary Set functors

As we have seen, for a finitary endofunctor of Set the limit of the terminal ω^{op}-chain need not be the terminal coalgebra. However, James Worrell [W2] provided a construction of the terminal coalgebra that works for every finitary endofunctor (see Section 2.4) of Set.

Actually, Worrell proved a more general result about *accessible* endofunctors of Set (cf. Theorem 4.2.10). Our proof presented here for *finitary* functors is simpler than the one given in [W2]. We recall from Lemma 2.4.12 that if $F : \mathsf{Set} \to \mathsf{Set}$ is finitary, then the canonical map $m : F(F^\omega 1) \to F^\omega 1$ is monic.

Construction 2.6.1. Recall the terminal ω^{op}-chain of F and let

$$F^\omega 1 = \lim_{n \in \omega^{op}} F^n 1 \qquad \text{with limit projections } \ell_n : F^\omega 1 \to F^n 1,\ n < \omega.$$

We write $m : FF^\omega 1 \to F^\omega 1$ for the factorizing morphism of (2.9).

We write $F^{\omega+n}1$ for $F^n(F^\omega 1)$. Notice that we have an ω^{op}-chain with the connecting morphisms

$$F^\omega 1 \xleftarrow{\ m\ } F^{\omega+1} \xleftarrow{\ Fm\ } F^{\omega+2}1 \xleftarrow{\ FFm\ } \cdots \tag{2.15}$$

Example 2.6.2 (Worrell [W2])**.** For the finite power-set functor $\mathcal{P}_\mathrm{f}$ we have that $\mathcal{P}_\mathrm{f}^\omega 1$ is the set of all compactly branching trees. Here $m : \mathcal{P}_\mathrm{f}^{\omega+1} 1 \hookrightarrow \mathcal{P}_\mathrm{f}^\omega 1$ is the subset of those trees in $\mathcal{P}_\mathrm{f}^\omega 1$ finitely branching at the root, and $\mathcal{P}_\mathrm{f} m : \mathcal{P}_\mathrm{f}^{\omega+2} 1 \hookrightarrow \mathcal{P}_\mathrm{f}^{\omega+1} 1$ is the subset of trees finitely branching at levels 0 and 1, etc.

Theorem 2.6.3 (Worrell [W2])**.** *The terminal coalgebra for F is given as the limit of the ω^{op}-chain from (2.15); in symbols:*

$$\nu F = \lim_{n \in \omega^{op}} F^{\omega+n} 1,$$

and this is an intersection of subobjects since all connecting morphisms are injective.

Remark 2.6.4. If νF denotes this limit with projections $p_n : \nu F \to F^{\omega+n}1$ then there exists a unique morphism

$$r : F(\nu F) \to \nu F \qquad \text{with } p_{n+1} \cdot r = F p_n \text{ for all } n < \omega. \tag{2.16}$$

It is our taks to prove that r is invertible, and its inverse yields the terminal coalgebra for F. We prove an auxiliary result first.

We need the notion of a *distinguished element* of F. This is an element contained in FX for every non-empty set X. More precisely, let C_{01} be the functor with $C_{01}\emptyset = \emptyset$ and $C_{01}X = 1$ for non-empty X. Then $x \in FX$ is distinguished if there is a natural transformation $\eta : C_{01} \to F$ with $\eta_X = x : 1 \to FX$. Notice that every element x of $F\emptyset$ gives rise to a distinguished element; more precisely, for the unique map $t : \emptyset \to X$, $X \neq \emptyset$, the element $Ft(x)$ is distinguished.

Lemma 2.6.5 (Trnková [Tr])**.** *Every set functor F preserves nonempty finite intersections and maps empty intersections to distinguished elements.*

More precisely, given two subsets $m_i : X_i \hookrightarrow X$, $i = 1, 2$, with $m_1 \cap m_2$ nonempty, F preserves the corresponding pullback. And if $m_1 \cap m_2$ is empty then all elements of the pullback of Fm_1 and Fm_2 are distinguished.

Lemma 2.6.6. *Every finitary endofunctor F of Set preserves all non-empty intersections.*

Proof. We can assume, without loss of generality, that F preserves non-empty inclusions, i. e., given a subset $\emptyset \neq A \subseteq B$ then $FA \subseteq FB$ and F maps the inclusion map $A \hookrightarrow B$ to the inclusion map $FA \hookrightarrow FB$. Indeed, every set functor is naturally isomorphic to one preserving non-empty inclusions, see [AT].

Now fix sets $X_i \hookrightarrow X$ for $i \in I$, and assume that $\bar{X} = \bigcap_{i \in I} X_i$ is non-empty. We are to show that $F\bar{X} = \bigcap_{i \in I} FX_i$. The inclusion "$\subseteq$" is obvious. For the reverse inclusion, assume that $x \in FX_i$ for all $i \in I$. Since F is finitary, we may choose a *minimal* finite $A \subseteq X$ such that $x \in FA$. If x is a distinguished element, then $x \in F\bar{X}$ holds since $\bar{X}$ is non-empty. Assume then that x is not distinguished. For all i we see from Lemma 2.6.5 that $FX_i \cap FA = F(X_i \cap A)$. Thus $x \in F(X_i \cap A)$. By minimality of A, $A \subseteq X_i \cap A$. So at this point we know that $A \subseteq X_i$ for all i. Thus, $A \subseteq \bar{X}$. Therefore $x \in FA \subseteq F\bar{X}$, as desired. □

Proof of Theorem 2.6.3. (1) For every coalgebra $\alpha : A \to FA$ we observe that the canonical cone $\alpha_n : A \to F^n 1$ of Construction 2.3.2 yields a canonical cone $\alpha_{\omega+n} : A \to F^{\omega+n}1$, $n < \omega$, of the ω^{op}-chain (2.15) as follows:

$$\alpha_\omega : A \to F^\omega 1$$

is the unique morphism such that $\ell_n \cdot \alpha_\omega = \alpha_n$ holds for all $n < \omega$, and given $\alpha_{\omega+n}$, put $\alpha_{\omega+n+1} = F\alpha_{\omega+n} \cdot \alpha$. For the limit in Remark 2.6.4 we get the unique

$$\overline{\alpha} : A \to \nu F \qquad \text{with } p_n \cdot \overline{\alpha} = \alpha_{\omega+n} \text{ for all } n < \omega. \tag{2.17}$$

(2) If $F1 = \emptyset$, then F is constant with value $\emptyset$ and the theorem is trivial. Assume $F1 \neq 1$. Then we have a coalgebra $\alpha : 1 \to F1$ and from $\overline{\alpha} : 1 \to \nu F$ above we conclude $\nu F \neq \emptyset$ (and also $F^{\omega+n}1 \neq \emptyset$ for every n). By Lemma 2.4.12 we conclude that m is a split monomorphism. Hence, (2.15) is a chain of subobjects. So its limit νF is an intersection of the ω^{op}-chain of subobjects. From $\nu F \neq \emptyset$ we conclude that F preserves this intersection, and it follows that r in (2.16) is invertible.

(3) The coalgebra $(\nu F, r^{-1})$ is a terminal coalgebra because for every coalgebra $\alpha : A \to FA$ the above morphism $\overline{\alpha} : A \to \nu F$ is a coalgebra homomorphism: we show, equivalently, that $r \cdot F\overline{\alpha} \cdot \alpha = \overline{\alpha} : A \to \nu F$. Indeed, for every n we have due to (2.16) and (2.17)

$$p_{n+1} \cdot (r \cdot F\overline{\alpha} \cdot \alpha) = F(p_n \cdot \overline{\alpha}) \cdot \alpha = F\alpha_{\omega+n} \cdot \alpha = \alpha_{\omega+n+1} = p_{n+1} \cdot \overline{\alpha}.$$

Given another coalgebra homomorphism $h : A \to \nu F$ then it is easy to prove $p_n \cdot h = \alpha_{\omega+n}$ for all $n < \omega$, thus from (2.17) we get $p_n \cdot h = p_n \cdot \overline{\alpha}$, proving $h = \overline{\alpha}$. □

Corollary 2.6.7. *For every finitary set functor F, the terminal coalgebra is the union of images of all coalgebra homomorphisms with codomain $F^\omega 1$.*

Proof. Indeed, let $\overline{m}$ turn $F^\omega 1$ into a weakly terminal coalgebra, see Proposition 2.4.13. Then the limit projection $p_0 : \nu F \to F^\omega 1$ is a homomorphism, i. e., $Fp_0 \cdot r^{-1} = \overline{m} \cdot p_0$: this follows from $m \cdot p_1 = p_0$ (which implies $p_1 = \overline{m} \cdot p_0$) and $p_1 \cdot r = Fp_0$. We know that p_0 is a monomorphism. Since νF is the terminal coalgebra for F, the union of images of all coalgebra homomorphisms with codomain $(F^\omega 1, \overline{m})$ is nothing else than $p_0 : \nu F \to F^\omega 1$. □

2.7 Terminal coalgebras as quotients

In this section we present a concrete description of the terminal coalgebras for finitary set functors due to [AM] and which is inspired by Barr's description of the terminal coalgebra for $\mathcal{P}_{\mathrm{f}}$ (cf. Corollary 2.4.23). In essence we shall provide a concrete description of the largest congruence on the coalgebra T_Σ of all Σ-trees coming from a presentation of a given finitary functor, cf. Example 2.4.18.

Assumption 2.7.1. For the rest of this section we assume that we are given a finitary endofunctor $F : \mathsf{Set} \to \mathsf{Set}$ and we fix a corresponding finitary signature Σ and epitransformation $\varepsilon : H_\Sigma \to F$, cf. Definition 2.4.3.

Our goal is to represent a terminal F-coalgebra in terms of a terminal H_Σ-coalgebra and certain data implicit in ε. The overall idea is that since H_Σ is a signature functor, its terminal coalgebra is much easier to understand than that of F, see Example 2.3.7.

Being surjective, each component of ε is fully described by its kernel equivalence. Indeed, ε may be described in terms of a set of *ε-equations* of the form

$$\sigma(x_1, \dots, x_n) \;=\; \rho(y_1, \dots, y_m), \tag{2.18}$$

where $\sigma \in \Sigma_n$ and $\rho \in \Sigma_m$. That is, we take a countable set X, and then the kernel equivalence of ε_X is a set E of pairs of elements of $H_\Sigma(X)$ which we write as in (2.18). The variables occurring on the left and right are taken from X; each side might well have repeated variables, and there might be variables common to both sides. The relationship between the epitransformation ε and the ε-equations is that for all sets A, the kernel of ε_A is the equivalence relation generated by

$$\{(H_\Sigma f(u), H_\Sigma f(v)) : (u = v) \text{ is an } \varepsilon\text{-equation and } f : X \to A\}. \tag{2.19}$$

Examples 2.7.2. (i) The functor $\mathcal{P}_3$ takes a set X to the set of subsets of size at most 2. It is a quotient of $H_\Sigma X = 1 + (X \times X)$, where Σ has one constant symbol c and one binary operation symbol $*$. We also take $\varepsilon : H_\Sigma X \to \mathcal{P}_3 X$ to be $c \mapsto \emptyset$, and $x * y \mapsto \{x, y\}$. (We use infix notation for the binary symbol, as usual.) There is just one ε-equation, $x * y = y * x$, expressing the commutativity of $*$.

(ii) Let $FX = \coprod_{\sigma \in \Sigma} X^k / G_\sigma$ be an analytic functor, where k is the arity of σ and G_σ is the given group of permutations on k. Then F is presented by the signature Σ and the ε-equations are

$$\sigma(x_1, \dots, x_k) = \sigma(x_{p(1)}, \dots, x_{p(k)}) \qquad \text{for all } \sigma \in \Sigma \text{ and all } p \in G_\sigma.$$

(iii) We have seen the Aczel-Mendler $(-)^3_2$ functor in Example 2.4.4(ii). This functor corresponds to the ε-equations

$$\sigma(x, x) = \tau(x, x) \qquad \text{and} \qquad \tau(x, x) = \rho(x, x).$$

Note that the set defined in (2.19) is not transitive in this example, and this is why our description of the kernel of ε_X takes the generated equivalence relation.

(iv) The finite power set functor $\mathcal{P}_{\mathrm{f}}$ is a quotient of $H_\Sigma(X) = 1 + X + X^2 + \cdots$. This corresponds to a signature with one n-ary operation symbol σ_n for each n (cf. Example 2.4.4(i)). The ε-equations are all equations between flat terms with the same set of variables appearing on both sides. For example,

$$\sigma_2(x, y) = \sigma_3(x, x, y)$$

would be one of the ε-equations.

The following proposition formalizes the well-known fact that the initial algebra of an equational class of Σ-algebras is the quotient of $F_\Sigma = \mu H_\Sigma$ (the algebra of finite Σ-trees) modulo the congruence generated by the given equations. A proof can be found e. g. in [AM].

Proposition 2.7.3. *Let $(\mu F, \varphi)$ be an initial F-algebra. Consider μF as a H_Σ-algebra:*

$$H_\Sigma(\mu F) \xrightarrow{\varepsilon_{\mu F}} F(\mu F) \xrightarrow{\varphi} \mu F$$

Let $\hat{\varepsilon} : \mu H_\Sigma \to \mu F$ be the unique H_Σ-algebra morphism. Then μF is a quotient of μH_Σ by $\hat{\varepsilon}$: this map is surjective, and its kernel is the smallest Σ-congruence on μH_Σ containing all instances of the ε-equations.

In other words

$$\mu F = F_\Sigma/{\sim},$$

where for two finite Σ-trees s, t we have

$$t \sim s \qquad \text{iff } s \text{ and } t \text{ can be equated by finitely many applications of the } \varepsilon\text{-equations.} \tag{2.20}$$

As shown in [AM], a similar description holds for the terminal coalgebra; we have

$$\nu F = T_\Sigma/{\approx},$$

where $\approx$ is the congruence of finite and *infinite* applications of the ε-equations. Of course, we have to make clear what is meant by "infinite application" of ε-equations.

Recall that for every Σ-tree t, the limit projection $\ell_n : T_\Sigma = \nu H_\Sigma \to H_\Sigma^n 1$ is the cutting of t at level n where all leaves of that level are relabelled by $\perp$. We define an equivalence relation $\approx$ on T_Σ as follows: for two Σ-trees s and t we put

$$s \approx t \qquad \text{iff for all } n < \omega \text{ we have } \ell_n s \sim \ell_n t. \tag{2.21}$$

Here $\sim$ is the congruence of (2.20) for the endofunctor $(H_\Sigma(-) + \{\perp\})$ induced by the quotient with components $\varepsilon_X + \{\perp\}$: $H_\Sigma X + \{\perp\} \to FX + \{\perp\}$ (observe that all trees of the form $\ell_n t$, lie in $\mu(H_{\Sigma(-)} + \{\perp\})$).

Theorem 2.7.4 ([AM]). *Consider the following F coalgebra*

$$T_\Sigma \xrightarrow{\tau_\Sigma} H_\Sigma T_\Sigma \xrightarrow{\varepsilon_{T_\Sigma}} FT_\Sigma \tag{2.22}$$

and let $\hat{\varepsilon} : T_\Sigma :\to \nu F$ be the corresponding unique F-coalgebra homomorphism. Then $\hat{\varepsilon}$ is epic and νF is a quotient of T_Σ via $\hat{\varepsilon}$. Moreover, the kernel equivalence of $\hat{\varepsilon}$ is the above equivalence $\approx$; in symbols:

$$\nu F = T_\Sigma/{\approx}.$$

Remark 2.7.5. There is an interesting connection of the last result and the congruence $\approx$ to the terminal ω^{op}-chains of H_Σ and F. Firstly, ε induced a natural transformation γ from the terminal chain of H_Σ to the terminal chain of F by induction: $\gamma_0 = id_1$ and

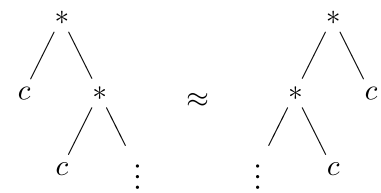
$$\gamma_{n+1} = (H_\Sigma H_\Sigma^n 1 \xrightarrow{\varepsilon_{H_\Sigma^n 1}} F H_\Sigma^n 1 \xrightarrow{F\gamma_n} F F^n 1).$$

It is not difficult to prove that $\gamma_n : H_\Sigma^n 1 \to F_\Sigma^n 1$ is an epimorphism with the kernel equivalence given by the congruence $\sim$ of finite applications of ε-equations from (2.20). Thus, for every $s, t \in T_\Sigma$ we have

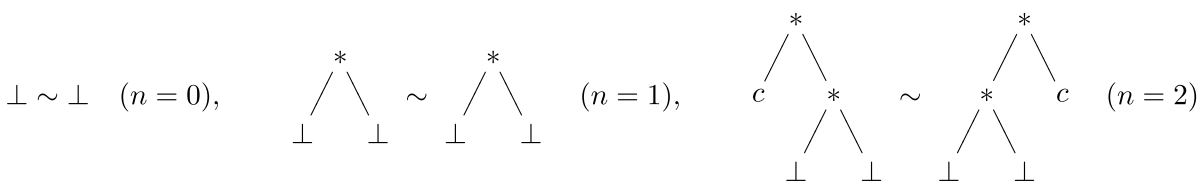
$$s \approx t \qquad \text{iff } \gamma_n \cdot \ell_n(s) = \gamma_n \cdot \ell_n(t) \text{ for all } n < \omega,$$

where ℓ_n is cutting trees at level n and γ_n is the quotient of finite application of ε-equations. It turns out that the maps $\gamma_n \cdot \ell_n : T_\Sigma \to F^n 1$ form the canonical cone of the coalgebra from (2.22).

Example 2.7.6. (i) We continue Example 2.7.2(ii) where $F = \mathcal{P}_3$. We have congruent trees

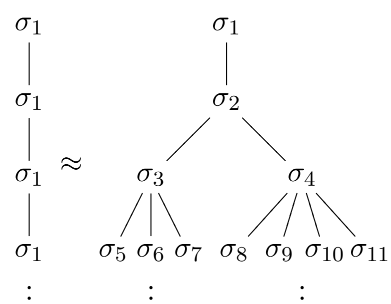


since we have



etc.

(ii) For the finite power-set functor $\mathcal{P}_\mathrm{f}$ recall Example 2.7.2(iv). Here we have



similarly as in (i) above. (Notice that these Σ-trees represent the extensional trees in (2.12).)

An isomorphic description of νH_Σ is as the coalgebra of all finitely branching ordered trees, and $\hat{\varepsilon}$ computes for every such tree the strongly extensional quotient: one forgets the order of children of a given tree t and then forms the quotient of t modulo the greatest tree bisimulation (cf. Section 2.5). Similarly, the maps γ_n, $n < \omega$, compute extensional quotients of finitely branching trees of height n. So $s \approx t$ iff for all $n < \omega$ we have: $\ell_n s$ and $\ell_n t$ have the same extensional quotient, cf. (2.13).

(iii) We mentioned analytic functors F on Set in Examples 2.7.2(ii). By Theorem 2.7.4 we have a direct description of νF: Let $F = \coprod_{\sigma\in\Sigma} X^k / G_\sigma$ be an analytic functor, where k is the arity of σ and G_σ is the given group of permutations on k. Then the terminal coalgebra is the quotient

$$\nu F = T_\Sigma / {\approx}$$

of the Σ-tree coalgebra modulo the equivalence $\approx$ analogous to $\sim$ of Example 2.2.13 but allowing infinitely many permutations of children of nodes, i. e., νF is the coalgebra of all Σ-trees modulo permutations of children of any σ-labelled node (using elements of the permutation group asssociated with σ).

In particular, for the bag functor $\mathcal{B}X = \coprod_{n\in\mathbb{N}} X^n/S_n$ (cf. Example 2.2.14) the terminal coalgebra $\nu\mathcal{B}$ is the coalgebra of all finitely branching non-ordered trees. In fact, the corresponding polynomial functor is the finite-list functor $FX = X^*$. We know that νF is the algebra of all finitely branching trees. And $\nu\mathcal{B}$ is the quotient of the coalgebra allowing arbitrary permutations of children. This means that the carrier consists of all unordered trees.

2.8 $\mathsf{CPO}_\perp$-enriched categories

The famous construction of a model of the untyped-λ-calculus presented by Dana Scott in [Sc] also applied ω^{op}-limits. The category he used was that of domains and embedding-projection maps, and we define these below. Later, Gordon Plotkin and Mike Smyth introduced the concept of a locally continuous endofunctor and noticed that in the category of domains, the finitary constructions of the initial algebra and the terminal coalgebra coincide for these endofunctors, yielding a *canonical* fixed point, see [SP]. Paul Taylor proved the same result in the (more "natural") category of domains and adjoint pairs [Ta], and this was generalized further in the unpublished Ph.D. thesis of Jiří Velebil [Ve]. Adámek [A87] derives a general form concerning the categories of domains and continuous functions. The details follow. In fact, in this subsection we decided to give the full details since we have the feeling that the material has not been presented in this comprehensive way before.

Definition 2.8.1. Let F be an endofunctor on a category $\mathcal{A}$ having both μF and νF. Since the corresponding structures of νF is an isomorphisms by Lambek's Lemma, we have a unique F-algebra homomorphism $i : \mu F \to \nu F$. We say that F has a *canonical* fixed point, if i is an isomorphism, so that $\mu F = \nu F$.

We have seen this phenomenon for $FX = X_\perp$ on $\mathsf{CPO}_\perp$ in Examples 2.2.3 and 2.3.5(iii). In this section we shall study canonical fixed points in categories of cpos, and in the next section we consider complete metric spaces.

Remark 2.8.2. Recall the category

$$\mathsf{CPO}_\perp$$

of cpos (see Definition 2.1.4) and strict continuous maps, i. e., continuous maps preserving the least element $\perp$.

Examples 2.8.3. (i) The poset $\mathsf{Pfn(X,Y)}$ of partial functions from X to Y ordered by (set-theoretical) inclusions is a strict cpo with $\perp$ the nowhere defined function.

(ii) For every set X the "flat" cpo $X + \{\perp\}$ is the strict cpo with all pairs in X incomparable.

(iii) A coproduct in $\mathsf{CPO}_\perp$ is the disjoint union with all $\perp$-elements merged to one.

(iv) A product in $\mathsf{CPO}_\perp$ is the cartesian product ordered componentwise.

Remark 2.8.4. The category $\mathsf{CPO}_\perp$ is cartesian closed: the internal hom-objects $[X, Y]$ are the hom-sets $\mathsf{CPO}_\perp(X, Y)$ ordered pointwise. We denote by $\perp_{X,Y}$ the least element (constant function of value $\perp_Y$) and by $\bigsqcup f_i$ the joins of ω-chains $f_i \colon X \to Y$.

The $\mathsf{CPO}_\perp$-*enriched categories* are thus categories $\mathcal{A}$ whose hom-set carry a cpo structure, and composition is strict and continuous.

Examples 2.8.5. (1) $\mathsf{CPO}_\perp$ is, of course, $\mathsf{CPO}_\perp$-enriched.

(2) The category Pfn of sets and partial functions is $\mathsf{CPO}_\perp$-enriched by inclusion: the least element of $\mathsf{Pfn(X,Y)}$ is the nowhere defined function, and $\bigsqcup f_i$ is the set-theoretic union.

(3) The category Rel of sets and relations is also $\mathsf{CPO}_\perp$-enriched by inclusion.

(4) If $\mathcal{A}$ is $\mathsf{CPO}_\perp$-enriched, then so is $\mathcal{A}^{op}$: consider the same ordering of morphisms; just the direction is reversed.

(5) A product of $\mathsf{CPO}_\perp$-enriched categories is $\mathsf{CPO}_\perp$-enriched (w. r. t. componentwise ordering).

Remark 2.8.6. Let $\mathcal{A}$ be a $\mathsf{CPO}_\perp$-enriched category.

(1) All limits in $\mathcal{A}$ are automatically enriched. In fact, given a limit cone $p_x\colon P \to D_x$ of a diagram D, then for every chain $(g_i)_{i<\omega}$ in $\mathcal{A}(X,P)$ we have

$$f = \bigsqcup_{i<\omega} g_i \qquad \text{iff} \qquad p_x \cdot f = \bigsqcup_{i<\omega} p_x \cdot g_i \quad \text{for every } x \text{ in } D.$$

This follows from the continuity of composition: $\bigsqcup_{i<\omega M} p_x \cdot g_i = p_x \cdot \bigsqcup_{i<\omega} g_i$.

(2) Dually: all existing colimits are enriched.

(3) An initial object 0, whenever it exists, is also terminal: by strictness of composition the unique morphism in $\mathcal{A}(X,0)$ is $\perp_{X,0}$.

(4) As observed by [Barr] whenever $\mathcal{A} \neq \emptyset$ has colimits of ω-chains, 0 exists: choose any object A, then the chain

$$A \xrightarrow{\perp} A \xrightarrow{\perp} A \xrightarrow{\perp} \dots$$

has colimit 0.

Definition 2.8.7 (D. Scott)**.** In a $\mathsf{CPO}_\perp$-enriched category a morphism $e\colon X \to Y$ is called an *embedding* if there exists a morphism $\hat{e}\colon Y \to X$ with

$$\hat{e} \cdot e = id_X \quad \text{and} \quad e \cdot \hat{e} \sqsubseteq id_Y. \tag{2.23}$$

The morphism $\hat{e}$ is clearly uniquely determined by (2.23) and is called the *projection* for e.

Examples 2.8.8. (1) In $\mathsf{CPO}_\perp$ the embeddings are precisely those monomorphisms $e\colon X \to Y$ such that for every $y \in Y$ there exists the largest $x \in X$ with $e(x) \sqsubseteq y$. In fact, this condition allows us to define $\hat{e}\colon Y \to X$ by choosing this largest x as $\hat{e}(y)$, then $\hat{e} \cdot e = id_X$ (since e is one-to-one) and $e \cdot \hat{e} \sqsubseteq id_Y$. The verification that the condition is also necessary is trivial.

(2) Every split monomorphism in Pfn is an embedding: here $\hat{e} = e^{op}$ (the opposite relation).

(3) Every split monomorphism in Rel is an embedding: split monomorphisms are precisely those relations $e\colon X \to Y$ which are injective maps and for these $\hat{e} = e^{op}$.

Observation 2.8.9. (1) Objects of $\mathcal{A}$ and embeddings form a category

$$\mathcal{A}^E$$

which is self-dual. In fact, a composite $e \cdot f$ of embeddings is an embedding with

$$\widehat{e \cdot f} = \hat{e} \cdot \hat{f}.$$

Thus, we have an isomorphism of categories

$$I \colon \mathcal{A}^E \longrightarrow (\mathcal{A}^E)^{op} \qquad IX = X \text{ and } Ie = \hat{e}.$$

(2) A general example of an embedding is a coproduct injection v_i of an arbitrary coproduct $X = \coprod_{i\in I} X_i$; indeed, $\hat{v}_i \colon X \to X_i$ has components id_{X_i} and $\perp_{X_i,X_j}$ for $j \neq i$. In particular, the morphisms with domain 0 are always embeddings.

(3) The name "projection" stems from the dual of (2): for every product $X = \prod_{i\in I} X_i$ the projection π_i has the form $\hat{e}_i$ where $e_i \colon X_i \to X$ has components id_{X_i} and $\perp_{X_i X_j}$.

Lemma 2.8.10 (Basic Lemma ([SP])). *Let $\mathcal{A}$ be a $\mathsf{CPO}_\perp$-enriched category with colimits of ω-chains. Then the category $\mathcal{A}^E$ of embeddings is closed under colimits of ω- chains in A. Moreover a cocone $c_i \colon E_i \to C$ of a ω-chain (E_i) of embeddings is a colimit cocone in $\mathcal{A}$ iff*

$$c_i \cdot \hat{c}_i \textit{ is a } \omega\textit{-chain in } \mathcal{A}(C, C) \textit{ with } \bigsqcup_{i<\omega} c_i \cdot \hat{c}_i = id_C. \tag{2.24}$$

Remark 2.8.11. Although formulated for ω-chains only, the Basic Lemma holds for λ-chains for all limit ordinals λ (in categories enriched so that hom-sets are posets with joins of all chains and composition preserves joins of chains). This is immediately seen from the proof below.

Proof of Lemma 2.8.10. (a) Given an ω-chain with embeddings as connecting maps $e_{ij} \colon E_i \to E_j$ we first prove that for every cocone of embeddings $c_i \colon E_i \to C$ with (2.24) we have $C = \operatorname{colim} E_i$. For every cocone $d_i \colon E_i \to D$ in $\mathcal{A}$, we first prove

$$d_i \cdot \hat{c}_i \sqsubseteq d_j \cdot \hat{c}_j \quad \text{for } i \leq j. \tag{2.25}$$

Indeed, since $(c_i)_{i<\omega}$ is a cocone, we have $\hat{c}_i = \hat{e}_{ij} \cdot \hat{c}_j$, thus

$$d_i \cdot \hat{c}_i = d_j \cdot e_{ij} \cdot \hat{e}_{ij} \cdot \hat{c}_i \sqsubseteq d_j \cdot \hat{c}_j.$$

Therefore there exists

$$d = \bigsqcup_{i<\omega} d_i \cdot \hat{c}_i \;:\; C \to D. \tag{2.26}$$

This is the desired factorization: to see the the equality

$$d_i = d \cdot c_i \tag{2.27}$$

observe that for each i we have

$$d \cdot c_i = \bigsqcup_{j<\omega} d_j \cdot \hat{c}_j \cdot c_i$$

and we can take the join over all $j \geq i$ (see (2.25)). Since for each such j we have

$$d_j \cdot \hat{c}_j \cdot c_i = d_j \cdot \hat{c}_j \cdot c_j \cdot e_{ij} = d_j \cdot e_{ij} = d_i,$$

we obtain (2.27). The factorization is unique: from $d \cdot c_i = d_i$ we obtain, using (2.24),

$$d = d \cdot \bigsqcup_{i<\omega} c_i \cdot \hat{c}_i = \bigsqcup_{i<\omega} d_i \cdot \hat{c}_i.$$

(b) Consider a ω-chain $(E_i)_{i<\omega}$. Let $c_i \colon E_i \to C$ be a colimit cocone in $\mathcal{A}$. We prove that c_i are embeddings satisfying (2.24), and for every cocone of embeddings $d_i \colon E_i \to D$ the unique factorization $d \colon C \to D$ is also an embedding. For every $i < \omega$ the shortened chain of all E_j with $i \le j < \omega$ has the colimit $(c_j)_{j \ge i}$, and the morphisms $\hat{e}_{ij} \colon E_j \to E_i$ form a cocone: we have a commutative diagram

$$\begin{tikzcd}
E_i \arrow[r, "e_{i,j}"] \arrow[rr, bend left, "e_{i,k}"] \arrow[dr, equal] & E_j \arrow[r, "e_{j,k}"] \arrow[d, "\hat{e}_{ij}"] & E_k \arrow[dl, "\hat{e}_{ik}"] \\
& E_i &
\end{tikzcd}$$

Indeed, from $e_{jk} \cdot e_{ij} = e_{ik}$ we obtain

$$\hat{e}_{ik} \cdot e_{jk} = \hat{e}_{ij} \cdot \hat{e}_{jk} \cdot e_{jk} = \hat{e}_{ij}.$$

Thus, there exists a unique factorization $\hat{c}_i : C \to E_i$ such that the following triangles commute:

$$\begin{tikzcd}
E_j \arrow[r, "c_j"] \arrow[dr, "\hat{e}_{ij}"'] & C \arrow[d, "\hat{c}_i"] \\
& E_i
\end{tikzcd} \qquad \text{for all } j \ge i. \tag{2.28}$$

In particular, $\hat{c}_i \cdot c_i = id$.

(b1) We prove (2.24). Due to $c_i = c_j \cdot e_{ij}$ we have, for $j \ge i$, $\hat{c}_i = \hat{e}_{ij} \cdot \hat{c}_j$, which yields

$$c_i \cdot \hat{c}_i = c_j \cdot e_{ij} \cdot \hat{e}_{ij} \cdot \hat{c}_j \sqsubseteq c_j \cdot \hat{c}_j \tag{2.29}$$

Thus the join in (2.24) exists, and we need to prove

$$\left(\bigsqcup_{j<\omega} c_j \cdot \hat{c}_j \right) \cdot c_i = c_i \qquad \text{for every } i < \omega.$$

Again we can restrict ourselves to $j \ge i$, and here $\hat{c}_j \cdot c_j = id$ (cf. (2.28)) implies

$$c_j \cdot \hat{c}_j \cdot c_i = c_j \cdot \hat{c}_j \cdot c_j \cdot e_{ij} = c_j \cdot e_{ij} = c_i.$$

(b2) c_i is an embedding with projection $\hat{c}_i$. Indeed, we have (2.28) and $c_i \cdot \hat{c}_i \sqsubseteq id_C$ follows from (2.24).

(b3) For every cocone of embeddings $d_i \colon E_i \to D$ the factorizing morphism $d \colon C \to D$ is an embedding. Since we already proved (2.24) we see from part (a) that (2.26) holds. Also, for $i \le j$ we have $\hat{d}_i = \hat{e}_{ij} \cdot \hat{d}_j$, thus

$$c_i \cdot \hat{d}_i = c_j \cdot e_{ij} \cdot \hat{e}_{ij} \cdot \hat{d}_j \sqsubseteq c_j \cdot \hat{d}_j \quad \text{for } i \le j$$

and we can define

$$\hat{d} = \bigsqcup_{i<\omega} c_i \cdot \hat{d}_i \colon D \to C. \tag{2.30}$$

We now prove that d is an embedding with projection $\hat{d}$. We have

$$d \cdot \hat{d} \sqsubseteq id_D$$

because the left-hand side is the join of $(d_i \cdot \hat{c}_i) \cdot (c_j \cdot d_j)$, and we can restrict ourselves to $j \geq i$ and get

$$\begin{array}{rcll} d_i \cdot \hat{c}_i \cdot c_j \cdot \hat{d}_j & = & d_i \cdot \hat{e}_{ij} \cdot \hat{c}_j \cdot c_j \cdot \hat{d}_j & \text{since } \hat{c}_i = \hat{e}_{ij} \cdot \hat{c}_j \\ & = & d_i \cdot \hat{e}_{ij} \cdot \hat{d}_j & \text{since } \hat{c}_j \cdot c_j = id \\ & = & d_i \cdot \hat{d}_i & \text{since } \hat{e}_{ij} \cdot \hat{d}_j = \hat{d}_i \\ & \sqsubseteq & id_D & \text{since } d_i \text{ is an embedding.} \end{array}$$

And

$$\hat{d} \cdot d = id$$

follows from $\hat{d} \cdot d \cdot c_i = c_i$ for all i: indeed, the left-hand side is, by (2.27) and (2.30), the join of $c_j \cdot \hat{d}_j \cdot d_i$, and restricting again to $j \geq i$ we have

$$c_j \cdot \hat{d}_j \cdot d_i = c_j \cdot \hat{d}_j \cdot d_j \cdot e_{ij} = c_j \cdot e_{ij} = c_i.$$

□

Definition 2.8.12. A functor $F\colon \mathcal{A} \to \mathcal{B}$ between $\mathsf{CPO}_\perp$-enriched categories is called *locally continuous* if it preserves ω-joins in hom-sets: given $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \ldots$ in $\mathcal{A}(X, Y)$, then $F(\bigsqcup f_n) = \bigsqcup F f_n$.

Examples 2.8.13. (1) Id is a locally continuous endofunctor.

(2) A composite, product or coproduct of locally continuous functors is locally continuous, cf. Examples 2.8.3.

(3) All polynomial functors are locally continuous.

(4) The functor $FX = X_\perp$ from Example 2.2.3 is locally continuous.

Theorem 2.8.14 (Smyth, Plotkin)**.** *Every locally continuous endofunctor of a* $\mathsf{CPO}_\perp$*-enriched category with colimits of* ω*-chains has a canonical fixed point* $\mu F = \nu F$.

Proof. If $F\colon \mathcal{A} \to \mathcal{A}$ is locally continuous, then F clearly preserves embeddings. Consequently, it yields an ω-chain of embeddings as the initial ω-chain (see (2.1)), and from the Basic Lemma 2.8.10 we conclude that F preserves its colimit $F^\omega 0 = \operatorname{colim}_{n<\omega} F^n 0$. Indeed, the colimit is characterized by the join $\coprod_{n<\omega} c_n \cdot \hat{c}_n = id$ in $\mathcal{A}(C, C)$ which F preserves. Thus, μF exists and is equal to $\operatorname{colim}_{n<\omega} F^n 0$. Using the self-duality of $\mathcal{A}^E$, see Observation 2.8.9, we conclude that this colimit is at the same time a limit of the ω^{op}-chain of projections—and this is the terminal ω^{op}-chain for F. Thus, $\mu F = \operatorname{colim} F^n 0 = \lim F^n 1 = \nu F$. □

Theorem 2.8.15. *Let* $\mathcal{A}$ *be a* $\mathsf{CPO}_\perp$*-enriched category with* ω*-colimits. Every locally continuous functor* $F\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$ *defines an endofunctor* F^E *of* $\mathcal{A}^E$ *by*

$$F^E X = F(X, X) \quad \textit{and} \quad F^E e = F(\hat{e}, e),$$

and F^E *has an initial algebra.*

Proof. The colimit $(F^E)^\omega 0 = \mathrm{colim}_{n<\omega}(F^E)^n 0$ is, by the Basic Lemma 2.8.10, characterized by (2.24). In $\mathcal{A}^{op} \times \mathcal{A}$ we have $\bigsqcup_{n<\omega}(\hat{c}_n \cdot c_n, c_n \cdot \hat{c}_n) = id$ which implies, since F is locally continuous,

$$\bigsqcup_{n<\omega} F(\hat{c}_n, c_n) \cdot F(c_n, \hat{c}_n) = id.$$

Since $F^E c_n = F(\hat{c}_n, c_n)$ is an embedding whose projection is $F(c_n, \hat{c}_n)$, the last join tells us that $(F^E c_n)$ is the colimit cocone in $\mathcal{A}^E$, thus, F^E preserves the colimit of its initial ω-chain. □

Corollary 2.8.16. *For every locally continuous endofunctor $F\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$ an object $X \cong F(X,X)$ exists.*

Example 2.8.17. Scott's model of untyped λ-calculus. The formulas t of λ-calculus have the form

$$t ::= k \mid x \mid tt \mid \lambda x.t$$

where k ranges through a set K of constants, and x through a countable set of variables. The meaning of $t_1 t_2$ is "application": we evaluate t_2 (a function) in t_1. The meaning of $\lambda x.t$ is "λ-abstraction": this function takes a value a and responds with $t[a/x]$, the term t in which x is substituted by a. Thus if D is the set of all closed terms, we obtain an isomorphism

$$D \cong K + D \times D + [D, D].$$

No such set D exists because $\mathsf{card}[D,D] > \mathsf{card}\, D$ whenever D is not a singleton set.

Dana Scott [Sc] decided to use a cartesian closed category with products and coproducts, and interpret the above equation in that category. He used originally continuous lattices, but Smyth and Plotkin [SP] made it clear that $\mathsf{CPO}_\perp$ is (simpler and) sufficient. In fact, consider the flat cpo

$$K_\perp = K + \{\perp\}$$

with all pairs in K incomparable, in place of the set K. Interpret $+$ and $\times$ as usual in $\mathsf{CPO}_\perp$ and recall from Example 2.8.13 that $D \mapsto D \times D$ is locally continuous. Finally, the function $D \mapsto [D, D]$ of internal hom-objects, which are the posets $\mathsf{CPO}_\perp(D, D)$ ordered pointwise, is a locally continuous functor from $\mathsf{CPO}_\perp^{op} \times \mathsf{CPO}_\perp$ to $\mathsf{CPO}_\perp$. In fact, recall that this functor is defined on morphisms $(f, g)\colon (X, A) \to (Y, B)$ of $\mathsf{CPO}_\perp^{op} \times \mathsf{CPO}_\perp$ as follows: the value of $[f, g] : [X, A] \to [Y, B]$ at $u\colon X \to A$ is

$$Y \xrightarrow{f} X \xrightarrow{u} A \xrightarrow{g} B.$$

This value is continuous in u because composition is continuous in $\mathsf{CPO}_\perp$. We thus obtain a locally continuous functor

$$F\colon \mathsf{CPO}_\perp^{op} \times \mathsf{CPO}_\perp \to \mathsf{CPO}_\perp \quad \text{with} \quad F(X, Y) = K_\perp + Y \times Y + [X, Y].$$

If D is the initial algebra for F^E, see Theorem 2.8.15, then

$$D \cong K_\perp + D + [D, D]$$

is a model of λ-calculus. (Observe that the "artificial" $\perp$ of K disappears in the formation of coproduct.)

Remark 2.8.18. Surprisingly, for a number of set functors F the terminal coalgebra also carries a structure of a cpo and it is obtained from the initial algebra as a free cpo completion. Moreover, $\lim_{n<\omega} F^n 1$ is a limit of an ω^{op}-chain of projections.

This concerns all endofunctors $F\colon \mathsf{Set} \to \mathsf{Set}$ which

(i) preserve colimits of ω^{op}-chains and

(ii) are grounded, i.e., $F\emptyset \neq \emptyset$.

For such a functor F we choose an element of $F\emptyset$

$$p\colon 1 \to F\emptyset.$$

Use the notation $w_{n,k}\colon F^n\emptyset \to F^k\emptyset$ and $v_{k,n}\colon F^k 1 \to F^n 1$ for the connecting morphisms of the initial and terminal ω-chain, and put $u\colon \emptyset \to 1$. We then obtain the morphisms

$$e_{k,n} \equiv F^n 1 \xrightarrow{F^n p} F^{n+1}\emptyset \xrightarrow{w_{n+1,k}} F^k\emptyset \xrightarrow{F^k u} F^k 1 \text{ for } k > n.$$

This enables us to define an ordering on $F^n 1$ as follows:

$$x \sqsubseteq y \text{ iff } x = y \text{ or } y = v_{k,n}(e_{k,n}(x)) \text{ for some } k > n.$$

Theorem 2.8.19. *(Adámek [Ad$_2$]) Each $F^n 1$ is a cpo with least element $e_{n,1} \cdot p\colon 1 \to F^n 1$, and each $e_{k,n}$ is an embedding with projection $v_{k,n}$. Moreover,*

νF is a limit of the projections $v_{k,n}$ in $\mathsf{CPO}_\perp$

and

μF is a subposet of νF whose free $\mathsf{CPO}_\perp$-completion is νF.

Example 2.8.20. A polynomial functor H_Σ is grounded iff Σ contains a nullary symbol p. Recall from Example 2.3.7 that set $H_\Sigma^n 1$ can, for $1 = \{*\}$, be identified with the set of all trees of depth at most n whose nodes of depth n are labelled by $*$ and other nodes with k successors are labelled by k-ary symbols in Σ. The above ordering is the least one for which $*$ is the smallest element and all operations are monotone. The connecting map $H_\Sigma^{n+1} 1 \to H_\Sigma^n 1$ cuts level $n+1$ and relabels level n with $*$. The limit of this ω^{op}-chain in $\mathsf{CPO}_\perp$ is the set of all Σ-trees with the above ordering.

The initial algebra is the set of all well-founded Σ-trees, see Example 2.2.11; its free $\mathsf{CPO}_\perp$-completion is νH_Σ because every Σ-tree t is a join of a sequence t_n in μH_Σ: take t_n to be the cutting of t at level n and relabelling level n by $*$.

2.8.1 Scott complete categories

A higher-order variation of Theorem 2.8.14 was established in Adámek [A87]. Cpo's are generalized as follows: A category $\mathcal{A}$ is called *Scott complete* if it has (i) filtered colimits, (ii) an initial object $\perp$, (iii) limits of diagrams with a cone, and (iv) a set of finitely presentable objects whose closure under filtered colimits is $\mathcal{A}$. We obtain a 2-category of Scott-complete categories, using as 1-cells the functors preserving filtered colimits and the terminal object, and as 2-cells the natural transformations.

Theorem 2.8.21 (Adámek [A87]). *Every locally continuous 2-endofunctor of the category of Scott complete categories has a canonical fixed point.*

This result was inspired by the related paper of Michael Barr [Barr] where the functor F was assumed to preserve both ω-colimits and ω^{op}-limits.

Definition 2.8.22. (Freyd [Fr2] A category is called *algebraically compact* if every endofunctor has a canonical fixed point.

Remark 2.8.23. According to Theorem 2.8.14 all $\mathsf{CPO}_\perp$-enriched categories are "algebraically compact w. r. t. locally continuous functors". However, no "reasonable" category is algebraically compact:

Corollary 2.8.24. *If an algebraically compact category has products (or coproducts), then it is equivalent to the terminal one-morphism category.*

This follows easily from Theorem 2.2.19.

2.9 CMS-enriched categories

Another structure, besides complete partial orderings, which leads to canonical fixed points, is *complete metric*. We consider the category CMS of complete metric spaces with distances bounded by 1, see Example 2.2.4. Every hom-set $\mathsf{CMS}(X, Y)$ is again a complete metric space with the pointwise metric given by

$$d_{X,Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) \,.$$

For example, the terminal coalgebra νH_Σ of all Σ-trees, see Example 2.2.11, is a complete metric space under the metric ρ, where $\rho(t, t) = 0$, and for $t \neq u$,

$$\rho(t, u) \quad = \quad 2^{-n}, \quad \text{where } n \text{ is least such that } t \text{ and } u \text{ differ on some point of level } n \tag{2.31}$$

This was first considered by Arnold and Nivat [AN].

Our study builds on the following classical result:

Theorem 2.9.1 (Banach Fixed Point Theorem). *Let $f : X \to X$ be a contracting function on a non-empty complete metric space. Then f has a unique fixed point.*

Definition 2.9.2 (America, Rutten [AmR]). A functor F on CMS is called *ε-contracting* for a constant $\varepsilon < 1$ provided that each derived function $\mathsf{CMS}(X, Y) \to \mathsf{CMS}(FX, FY)$ is an ε-contracting map; that is, $d_{FX,FY}(Ff, Fg) \leq \varepsilon \cdot d_{X,Y}(f, g)$ for all non-expanding maps $f, g : X \to Y$.

Example 2.9.3. (i) Any polynomial functor H_Σ on Set has a *contracting lifting* H'_Σ to CMS. This means that the following square

$$\begin{tikzcd} \mathsf{CMS} \arrow[r, "H'_\Sigma"] \arrow[d, "U"'] & \mathsf{CMS} \arrow[d, "U"] \\ \mathsf{Set} \arrow[r, "H_\Sigma"'] & \mathsf{Set} \end{tikzcd}$$

commutes, where U is the functor taking a metric space to its set of points. For a simple example, the functor $FX = X^n$, lifts to $F'(X, d) = (X^n, \frac{1}{2} d_{\max})$ (where $d_{\max}$ is the maximum metric) which is a contracting functor with $\varepsilon = \frac{1}{2}$. And coproducts of $\frac{1}{2}$-contracting liftings are $\frac{1}{2}$-contracting liftings of coproducts. The initial algebra as well as the terminal terminal coalgebra of H'_Σ on CMS is the set of all Σ-trees equipped with the metric of (2.31) as we will see below.

(ii) Lifting of $\mathcal{P}_{\mathrm{f}}$ to complete metric spaces. Let us first recall that in a metric space (X, d) the distance $d(a, A)$ of a point $a \in X$ to a set $A \subseteq X$ is the infimum of $d(a, x)$ for $x \in A$. The *Hausdorff distance* of sets $A, B \subseteq X$ is the join of distances of points of one of the sets to the other one. Notation:

$$d^*(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}. \tag{2.32}$$

The *Hausdorff functor* is the functor

$$\mathcal{H} : \mathsf{CMS} \to \mathsf{CMS}$$

assigning to every metric space (X, d) the space of all compact subsets of X equipped with the above Hausdorff metric d^*. This gives a functor on the category MS, and as shown in [Ku] (Lemma 3), for a complete metric space X the space $\mathcal{H}X$ is also complete. Also notice that $\mathcal{H}$ is indeed a lifting of $\mathcal{P}_{\mathrm{f}}$: for a discrete space X we have $\mathcal{H}X = \mathcal{P}_{\mathrm{f}}X$. The Hausdorff functor itself is not contracting, but the every functor $\varepsilon\mathcal{H}$ obtained from it by scaling all distances by $\varepsilon < 1$ is.

Theorem 2.9.4. *Every contracting endofunctor F of CMS has a terminal coalgebra*

$$\nu F = \lim_{n \in \omega^{op}} F^n 1.$$

The proof is analogous to that of Theorem 2.9.4 below. Moreover, whenever $F\emptyset = \emptyset$ the functor has a canonical (in fact unique) fixed point.

The next example is due to J. Worrell [W2]; although he used complete ultrametric spaces, his result also holds for the category CMS, see [AMMS].

Example 2.9.5. We denote by

$$\mathcal{P}_{\mathrm{f}}^* : \mathsf{CMS} \to \mathsf{CMS}$$

the lifting of $\mathcal{P}_{\mathrm{f}}$ obtained by scaling the Hausdorff functor by $\varepsilon = \frac{1}{2}$. More detailed, for a complete metric space (X, d) we have

$$\mathcal{P}_{\mathrm{f}}^*(X, d) = \text{ all compact subsets of } X \text{ with metric } \frac{1}{2}d^*.$$

This scaling makes $\mathcal{P}_{\mathrm{f}}^*$ obviously contracting, therefore

$$\nu\mathcal{P}_{\mathrm{f}}^* = \lim_{n \in \omega^{op}} \mathcal{P}_{\mathrm{f}}^* 1$$

is the terminal coalgebra, see Theorem 2.9.4. In fact, James Worell described it as

$$\nu\mathcal{P}_{\mathrm{f}}^* = \text{ all compactly branching strongly extensional trees.}$$

Here a strongly extensional tree (see Definition 2.5.1) is called *compactly branching* if for any vertex the set of all maximal subtrees in compact in the metric (2.31) on the set of unordered trees. The argument used in [W2] is: for this metric space it is easy to verify that it is (the unique) fixed point of $\mathcal{P}_{\mathrm{f}}^*$.

In order to obtain a terminal coalgebra using Theorem 2.9.4 above we needed to scale the Hausdorff functor by $\frac{1}{2}$. Later, in Theorem 4.2.12, we shall obtain a result that allows us to obtain a terminal coalgebra for the class of functors formed by coproducts, products and compositions of polynomial functors and the Hausdorff functor without any scaling, see Example 4.2.14.

A negative result If X is discrete, every subset is closed, whereas the compact subsets of X are exactly the finite ones. Let $\mathcal{P}_c : \mathsf{MS} \to \mathsf{MS}$ be the functor taking a space (X, d) to the set of its closed subsets, again with the Hausdorff metric d^* of (2.32). As shown by van Breugel [vB], there is no terminal coalgebras for $\mathcal{P}_c : \mathsf{MS} \to \mathsf{MS}$.

Definition 2.9.6. A category $\mathcal{A}$ is CMS-*enriched* if its hom-sets come equipped with a complete metric, and composition is non-expanding in both variables.

Not surprisingly, CMS is CMS-enriched. Complete metric spaces were first applied by de Bakker and Zucker [dBZ], and their method was further developed by America and Rutten [AmR]. In both of these papers, the morphisms are a little different from the non-expansive maps: they are the *projection-embedding pairs* consisting of an isometric embedding $e : X \to Y$ together with a non-expanding map $e^* : Y \to X$, such that $e^* \cdot e = id_X$. Later, the result of [AmR] was extended to CMS-enriched categories by Adámek and Reiterman [ARe]. Their work introduced the following notion related to the concept of hom-contractive functor in [AmR].

Definition 2.9.7. An endofunctor F of a CMS-enriched category is *weakly contracting* if there is a number $0 < \varepsilon < 1$ such that for every endomorphism $f : X \to X$, we have

$$d_{FX}(Ff, id_{FX}) \leq \varepsilon \cdot d_X(f, id_X).$$

Remark 2.9.8. The following result is a variation on a theorem proved by P. America and J. Rutten in [AmR] where they use the category of complete metric spaces and embeddings (split monos):

Theorem 2.9.9 (Adámek and Reiterman [ARe])**.** *Let $\mathcal{A}$ be a CMS-enriched category with a zero-object $0 = 1$. Then every weakly contracting endofunctor F has a canonical fixed point*

$$\nu F = \mu F = \lim_{n \in \omega^{op}} F^n 1.$$

If F is contracting, then it has a unique fixed point (up to isomorphism).

Example 2.9.10. Let CMS^p be the category of *pointed* complete metric spaces and non-expanding maps (preserving the distinguished point). Then CMS^p is CMS-enriched, and the one-point space is a zero object. Denote by $\mathbf{2}$ the two-element object with distance 1. The functor $F : \mathsf{CMS}^p \to \mathsf{CMS}^p$ given by $(X, d) \mapsto (X, \frac{1}{2}d) + \mathbf{2}$ has a canonical fixed point: Its initial algebra $\mathbb{N} \cup \{\infty\}$, cf. Theorem 2.1.9, is also the terminal coalgebra.

3 Transfinite Iteration

3.1 Initial Chain

In this section, we pursue the *transfinite* iteration of the initial and terminal chains. We begin with a famous result on fixed points of monotone maps on directed complete partial orders (dcpos): A *dcpo* is a poset P with the property that all *directed subsets*[3] have least upper bounds, and with a least element 0. Let $f : P \to P$ be monotone. Then f generates an ordinal-indexed sequence in P:

$$\begin{aligned} f^0(0) &= 0, \\ f^{j+1}(0) &= f(f^j(0)) \text{ for all ordinals } j, \end{aligned}$$

and

$$f^j(0) = \bigsqcup_{i<j} f^i(0) \text{ for all limit ordinals } j.$$

Theorem 3.1.1 (Zermelo, Tarski, Knaster)**.** *Every monotone endofunction of a dcpo has a least fixed point μF. Moreover,*

$$\mu f = f^j(0)$$

for some ordinal κ.

Proof. The central point is that given a dcpo D of cardinality less than κ there must be some $j < \kappa$ such that $f^j(0) = f^{j+1}(0)$; if not, then $\{f_j : j < \kappa\}$ is a subset of D of size κ, and this is a contradiction. Thus, $f^\kappa(0)$ is a fixed point of f. Also, an easy induction shows that every fixed point of f must be at least as large as each iterate $f^j(0)$. □

We attribute this theorem to Zermelo, since the mathematical content of the result appears in his 1904 paper proving the Wellordering Theorem. Although it is common to refer to it as the Tarski-Knaster Theorem, their result dealt with a slightly different situation, see [K].

A category-theoretic generalization of Theorem 3.1.1 was formulated by Adámek [A74]. It was applied there to the functor $F(-)+A$; in other words, the free F-algebra on an object A was considered instead of the initial F-algebra.

Definition 3.1.2. Let $\mathcal{A}$ be a category with an initial object 0 and with colimits of chains. For every endofunctor F the *initial chain* is the chain in $\mathcal{A}$ indexed by Ord, the ordered class of all ordinals j, and having objects $F^j 0$ defined by

$$\begin{aligned} &F^0 0 &&= 0, \\ &F^{j+1} 0 &&= F(F^j 0) \quad \text{for all ordinals } j, \end{aligned}$$

and

$$F^j 0 = \operatorname*{colim}_{i<j} F^i 0 \text{ for all limit ordinals } j.$$

Its connecting morphisms $w_{j,k} \colon F^j 0 \to F^k 0$ for $j \leq k$ are uniquely determined by

$$\begin{aligned} &w_{0,1} \colon 0 \to F0 \text{ is unique },\\ &w_{j+1,k+1} = Fw_{j,k} \colon F(F^j 0) \to F(F^k 0), \\ &w_{i,k}\ (i < k) \text{ is the colimit cocone for limit ordinals } j. \end{aligned}$$

Definition 3.2 is correct: there exists a chain $W \colon \mathsf{Ord} \to \mathcal{A}$ unique up to natural isomorphism whose values $w_{ij} = W(i \to j)$ are those given above. For example, $w_{\omega,\omega+1} \colon F^\omega 0 \to F(F^\omega 0)$ need not be specified: since W preserves composition, we have $w_{\omega,\omega+1} \cdot w_{\omega,n+1} = w_{\omega+1,n+1} = Fw_{\omega,n}$ for all $n < \omega$ and this determines $w_{\omega,\omega+1}$ uniquely.

[3] A poset is directed if it is non-empty and every pair of elements has an upper bound.

Definition 3.1.3. We say that the initial chain of a functor F *converges in λ steps* if $w_{\lambda,\lambda+1}$ is an isomorphism.

Theorem 3.1.4. *[A74] Let $\mathcal{A}$ be a category with an initial object 0 and with colimits of chains. If the initial chain of a functor F converges in λ steps, then $F^\lambda 0$ is the initial algebra w.r.t. $w^{-1}_{\lambda,\lambda+1}\colon F(F^\lambda 0) \to F^\lambda 0$.*

The proof is completely analogous to that of 2.1.9. Given an F-algebra (A, α), we obtain a unique cocone $\alpha_j\colon F^j 0 \to A$ with $\alpha_{j+1} = \alpha \cdot F\alpha_j$ for all ordinals j. And α_λ is then proved to be the unique homomorphism from $(F^\lambda 0, w^{-1}_{\lambda,\lambda+1})$ into (A, α).

Corollary 3.1.5. *Let $\mathcal{A}$ be a category with an initial object 0 and with colimits of chains. Then the initial chain of a functor F preserving colimits of λ-chains converges in λ steps, hence, $\mu F = F^\lambda 0$.*

Example 3.1.6. Consider the countable power set functor $\mathcal{P}_c$ on Set. For $\lambda = \aleph_1$ (the first uncountable cardinal), $\mathcal{P}_c$ preserves colimits of λ-chains. Then $\mathcal{P}_c^\lambda \emptyset$ is an initial algebra. It is often called HC, the set of *hereditarily countable sets*, cf. Example 2.2.12.

Example 3.1.7. For polynomial functors H_Σ, see Example 2.2.11, we show that the initial chain yields the algebra

$$\mu H_\Sigma = \text{ well-founded } \Sigma\text{-trees.}$$

We introduced Σ-trees in Example 2.2.11; *well-founded* means that every path in the tree is finite. Before analyzing the initial chain for H_Σ, we introduce the concept of depth which generalizes the traditional one (for finite trees) to an ordinal depth for infinite trees. Recall that for a finite tree t the depth $d(t)$ is always equal to the join of $d(t_i) + 1$ for all children t_i of t:

(a) if t has no children, its depth is $0 = \sup \emptyset$, and

(b) if t has children $t_1, \dots, t_k$ then $d(t) = \sup_{i=1,\dots,k}(d(t_i) + 1)$.

Recall also the ordinal sum $\lambda + 1$: this is the ordinal successor of λ.

Definition 3.1.8. The *depth* of a tree t is defined to be

(a) the ordinal $d(t) = \sup(d(t_i) + 1)$ where the join ranges over all children t_i of t, in case each $d(t_i)$ is an ordinal, or

(b) ∞ in case $d(t_i) = \infty$ for some child t_i of t.

Example 3.1.9. Every ordinal is the depth of some tree. For a precise proof see Theorem 3.1.12 below. Let us illustrate this on some examples:

If σ is a ω-ary operation and τ is unary, then

$$\begin{array}{ll}
t_0 = \sigma(x, \tau x, \tau^2 x, \dots) & \text{has depth } \omega \\
t_1 = \tau\sigma(x, \tau x, \tau^2 x, \dots) & \text{has depth } \omega + 1 \\
t_2 = \tau^2\sigma(x, \tau x, \tau^2 x, \dots) & \text{has depth } \omega + 2 \\
\vdots & \\
\sigma(t_0, t_1, t_2, \dots) & \text{has depth } \omega + \omega.
\end{array}$$

An example of a tree of depth ∞ is the infinite path tree $\tau(\tau(\tau \dots$

Lemma 3.1.10. *A tree has depth less than ∞ iff it is well-founded.*

Proof. If t has depth $i \in \mathsf{Ord}$ then it is well-founded: this is a trivial transfinite induction on i. Conversely, if t is well-founded and has depth ∞, we find a contradiction: there exists a child t_{i_1} of t of depth ∞ and a child t_{i_2} of t_{i_1} of depth ∞ etc., and these indices $i_1, i_2, \dots$ yield an infinite path in t. $\square$

Example 3.1.11. Let us analyze the initial chain of H_Σ: we can represent elements of

$$H_\Sigma^1 \emptyset = \Sigma_0$$

as the Σ-trees of depth 0 (that is, singleton trees labelled by nullary symbols). Then the elements of

$$H_\Sigma^2 \emptyset = \coprod \Sigma_k \times \Sigma_0^k$$

are represented by precisely all Σ-trees of depths 0 or 1. In general, for every ordinal j we have

$$H_\Sigma^j \emptyset = \text{ all } \Sigma\text{-trees of depth less than } j.$$

This is easy to see by transfinite induction: the case $j = 0$ is clear (no Σ-tree has depth less than 0), the isolated induction step follows from $H_\Sigma^{j+1}\emptyset = \coprod \Sigma_k \times (H_\Sigma^j \emptyset)^k$, and in the limit step j we see that the colimit of the chain of inclusions is the union:

$$H_\Sigma^j \emptyset = \bigcup_{m<j} H_\Sigma^m \emptyset = \text{ all } \Sigma\text{-trees of depths less than } j.$$

Theorem 3.1.12. *For every signature Σ, the initial algebra of H_Σ is the algebra of all well-founded Σ-trees. The initial chain converges in 0 steps if $\Sigma_0 = \emptyset$, in one step if $\Sigma = \Sigma_0 \neq \emptyset$, and otherwise in precisely α steps for the first regular cardinal*[4] *α larger than all arities.*

Proof. Assume $\Sigma_0 \neq \emptyset$ and $\Sigma \neq \Sigma_0$.

(1) The initial chain converges in α steps and yields $H_\Sigma^\alpha \emptyset =$ all well-founded Σ-trees. This follows from 3.1.10 and the fact that if a tree of depth $< \infty$ is α-branching, then it has depth less than α (an easy proof by transfinite induction).

(2) The initial chain does not converge in less than α steps. This follows from the fact that every ordinal $i < \alpha$ is the depth of some Σ-tree. We prove this in two steps:

(a) Assume that α is the supremum of arities in Σ. For $i = 0$ take any member σ of Σ_0, for $i + 1$ let t_0 have depth i and form $t_1 = \tau(t_0, t_0, \dots)$ where the root is labelled by any member $\tau \in \Sigma - \Sigma_0$. If $i < \alpha$ is a limit ordinal, take $\sigma \in \Sigma$ of arity $k > i$ and for every $j < i$ take a Σ-tree t_j of depth j (induction hypothesis). Extend this notation for all $i \leq j < k$ by $t_j := t_0$. Then the Σ-tree $\sigma(t_j)_{j<k}$ has depth i.

(b) Let α be larger than the supremum β of all arities of Σ. Then either β is finite and $\alpha = \omega$. In this case it is easy to see that every natural number is the depth of some Σ-tree (recall $\Sigma_0 \neq \emptyset \neq \Sigma - \Sigma_0$). Or β is infinite and has cofinality $\beta_0 < \beta$, in which case α is the cardinal successor of β. The proof that all $i < \beta$ are depths of Σ-trees is as in (a) above, it remains to verify that there is a Σ-tree of depth β. Choose an operation $\sigma \in \Sigma$ of arity β_0. Since β_0 is the cofinality of β, we have ordinals $\gamma_i < \beta$ indexed by all $i < \beta_0$ with $\beta = \sup\{\gamma_i : i < \beta_0\}$. For every i choose a Σ-tree t_i of depth γ_i, and for $i \geq \beta_0$ put $t_i = t_0$. Then $\sigma(t_i)_{i<\gamma}$ has depth β. $\square$

[4] A cardinal is *regular* if it is infinite and does not have a cofinal smaller than itself. (The smallest non-regular infinite cardinal is $\aleph_\omega$.)

Remark 3.1.13. We see from Theorem 3.1.12 that for every regular cardinal α there is a set functor whose convergence of the initial chain takes precisely α steps. How about other ordinals? In Theorem 3.1.12 we saw that 0 or 1 step are possible for H_Σ. For the functor $C_{2,1}$ constantly equal to 1 except $C_{2,1}\emptyset = 2$ the initial chain takes 2 steps. One can construct (a bit more technically) a set functor that needs 3 steps. And this is all:

Theorem 3.1.14 ([AT09]). *For every endofunctor of* Set *with an initial algebra, the initial chain converges either in at most 3 steps or in λ steps for some regular cardinal λ.*

Are all initial algebras obtainable by iteration? As it happens, the answer is negative: consider the category $\mathsf{Ord}^\top$, the ordinals with a new element $\top$ added "on top." Let $F : \mathsf{Ord}^\top \to \mathsf{Ord}^\top$ given by $F(\lambda) = \lambda + 1$, and $F(\top) = \top$. Clearly, $\top$ is the only fixed point, and this is not $F^\lambda 1$ for any λ.

However, as we mention shortly, this does not happen in "reasonable" categories, at least not for functors preserving monomorphisms.

Definition 3.1.15 ([TAKR]). A class $\mathcal{M}$ of monomorphisms in a category $\mathcal{A}$ is called *constructive* provided that it is closed under composition, and for every chain of monomorphisms in $\mathcal{M}$, (i) a colimit exists and is formed by monomorphisms in $\mathcal{M}$, and (ii) the factorization morphism of every cocone of monomorphisms in $\mathcal{M}$ is again a monomorphism in $\mathcal{M}$.

Remark 3.1.16. In particular $\mathcal{A}$ has an initial object 0 and all morphisms with domain 0 lie in $\mathcal{M}$.

Examples 3.1.17. (i) The categories of sets, graphs, posets, and semigroups all have the constructive class of all monomorphisms.

For a non-example, we consider the category BiP of Example 2.2.15. The point is that the unique morphism $0 \to X$ is not always monic.

(ii) In $\mathsf{CPO}_\perp$-enriched categories the class of all embeddings is constructive: see Basic Lemma 2.8.10 and Observation 2.8.9 (2).

Theorem 3.1.18 ([TAKR]). *Let $\mathcal{A}$ have a constructive class $\mathcal{M}$ of monomorphisms and be $\mathcal{M}$-wellpowered. Let $F : \mathcal{A} \to \mathcal{A}$ preserve monomorphisms in $\mathcal{M}$. The following are equivalent:*

(i) F has a fixed point, i.e., an object $A \cong FA$.

(ii) μF exists,

and

(iii) the initial chain converges.

Proof. Lambek's Lemma tells us that (ii) implies (i). To see that (i)⇒(iii), we argue as in the proof of Theorem 3.1.4: for every ordinal j the morphism $\alpha_j \colon F^j 0 \to A$ from the induced cocone of the algebra A is a member of $\mathcal{M}$ (an easy proof by transfinite induction). If the initial failed to converge, the fixed point object A would have a proper class of subobjects in $\mathcal{M}$. For (iii) ⇒ (ii) see Theorem 3.1.4. □

Example 3.1.19. (i) For every endofunctor of Set the three conditions of Theorem 3.1.18 are equivalent.

This follows from Theorem 3.1.18 in the case that F preserves monomorphisms. For general F, see [AKP]. However, we present here a substantially shorter proof:

Assume $F\emptyset \neq \emptyset$ (else the initial chain converges in zero steps) and let $\alpha\colon FA \xrightarrow{\cong} A$ be a fixed point with the corresponding cocone $\alpha_j\colon F^j 0 \to A$, see Theorem 3.1.4. We prove that α_ω is a monomorphism. Choose a morphism $u_0\colon A \to F\emptyset$ and observe that $! = u_0 \cdot \alpha_0 \colon \emptyset \to F\emptyset$. Then the morphisms $u_n\colon A \to F^{n+1}0$ defined by $u_{n+1} = Fu_n \cdot \alpha^{-1}$ clearly fulfil

$$w_{n,n+1} = u_n \cdot \alpha_n \colon F^n 0 \to F^{n+1} 0 \quad \text{for all } n < \omega.$$

Given elements $x, y \in F^\omega 0$ merged by α_ω, we prove $x = y$. Choose $n < \omega$ and elements x', y' of $F^n 0$ mapped by $w_{n,\omega}$ to the given pair. Then $\alpha_n = \alpha_\omega \cdot w_{n,\omega}$ merges x' and y', thus, these elements are also merged by

$$(w_{n+1,\omega} \cdot u_n) \cdot \alpha_n = w_{n+1,\omega} \cdot w_{n,n+1} = w_{n,\omega},$$

which proves $x = y$.

Since α_ω is a monomorphism with nonempty domain, it splits, thus, $F\alpha_\omega$ is also a monomorphism. This proves that $\alpha_{\omega+1} = \alpha \cdot F\alpha_\omega$ is a monomorphism. In this manner we see that all α_j with $j \geq \omega$ are monomorphisms, and then we argue as in Theorem 3.1.18.

(ii) The category Pfn of sets and partial functions has the property that split monomorphisms (which are precisely the monomorphisms of Set) form a constructive class. Thus, also here the three conditions are equivalent for every endofunctor.

(iii) Also in the category Rel of sets and relations the three conditions are equivalent for all endofunctors (for the same reason).

Example 3.1.20 ([AT09])**.** Unfortunately, Theorem 3.1.18 does not generalize to settings such as many-sorted sets. For example, consider two-sorted sets. Let $F : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}$ be given by

$$F(X, Y) \quad = \quad \begin{cases} (1,1) & \text{if } X \neq \emptyset \\ (\emptyset, \mathcal{P}Y) & \text{if } X = \emptyset \end{cases}$$

Although $(1,1)$ is a fixed point, F has no initial algebra: the F-algebras $(\emptyset, \mathcal{P}Y) \to (\emptyset, Y)$ cannot be initial by Lambek's Lemma, and from those algebras of the form $(1,1) \to (X,Y)$ there exist no homomorphisms to those of the form $(\emptyset, \mathcal{P}Y) \to (\emptyset, Y)$.

Example 3.1.21 ([AT])**.** Another unfortunate fact: the collection of all set functors possessing initial algebras is not well behaved. There are set functors F_1, F_2 having initial algebras such that neither $F_1 + F_2$ nor $F_1 \times F_2$ has one. In fact, given a set γ of cardinals, let $\mathcal{P}^\gamma$ be the following modification of the power-set functor:

$$\mathcal{P}^\gamma X = \{M \subseteq X : \text{card } M \notin \gamma\}$$

defined on morphism $f\colon X \to Y$ by

$$\mathcal{P}^\gamma_f(M) \quad = \quad \begin{cases} f[M] & \text{if card } M \notin \gamma \text{ and } f|_M \text{ is a monomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

It is easy to see that every infinite set X with card $X \in \gamma$ is a fixed point of $\mathcal{P}^\gamma$, thus, $\mathcal{P}^\gamma$ has an initial algebra.

Take any disjoint pair of non-empty classes with $\gamma_1 \cup \gamma_2 = \mathsf{card}$. Then the functors $F_i = \mathcal{P}^{\gamma_i}$ have initial algebras, but $F_1 + F_2$ and $F_1 \times F_2$ have no fixed points.

Theorem 3.1.22 ([AT09])**.** *For the category Set^S of many sorted sets whenever an endofunctor has an initial algebra, then the initial chain converges.*

3.2 Terminal Chain

This is nothing else than the dual of the initial chain of Definition 3.1.2. This was formulated explicitly by M. Barr [Barr]:

Definition 3.2.1. Let $\mathcal{A}$ be a category with a terminal object 1 and with limits of (co)chains. For every endofunctor F the *terminal chain* is the chain in $\mathcal{A}$ indexed by Ord^{op}, the dual of the ordered class of ordinals ^{j}, having objects $F^j 1$ defined by

$$\begin{aligned} F^0 1 &= 1 \\ F^{j+1} 1 &= F(F^j 1) \quad \text{for all ordinals } j \end{aligned}$$

and

$$F^j 1 = \lim_{i<j} F^i 1 \quad \text{for all limit ordinals } j.$$

We write $v_{k,j} : F^k 1 \to F^j 1$, $j \geq k$, for the uniquely determined connecting morphisms, and we say that the terminal chain *converges* in λ steps if $v_{\lambda+1,\lambda}$ is an isomorphism.

For finitary set functors the chain (2.15) is precisely the part of the above chain from ω to $\omega + \omega$. For example $m = v_{\omega+1,\omega}$.

Theorem 3.2.2. *Let $\mathcal{A}$ be a category, let λ be a cardinal, and assume that (i) $\mathcal{A}$ has limits of λ^{op}-chains of length at most λ, and (ii) F preserves limits of λ^{op}-chains. t Then F has the terminal coalgebra*

$$\nu F = F^\lambda 1.$$

This is just the dual of Corollary 3.1.5. As an example, recall that for every (not necessarily finitary) signature Σ, the polynomial functor H_Σ preserves ω^{op}-limits. Thus, the transfinite chain does not bring anything new. In contrast, for the finite power-set functor $\mathcal{P}_\mathrm{f}$ the terminal chain converges in $\omega + \omega$ steps, see Example 2.6.2.

Example 3.2.3. In contrast to Example 3.1.19 (i), a set functor with a fixed point need not have a terminal coalgebra. Here is an example from [AK95]. For $\gamma = \{\omega\}$ the set functor

$$\mathcal{P}^{\{\omega\}} \qquad \text{(see Example 3.1.21)}$$

has an initial algebra but not a terminal coalgebra. Recall from 3.1.21 that $\mathcal{P}^{\{\omega\}}X$ is the set of all subsets of X of cardinality different from ω. Obviously, $\mathbb{N}$ is a fixed point of $\mathcal{P}^{\{\omega\}}$: the set $\mathcal{P}^{\{\omega\}}\mathbb{N}$ of all finite subsets is countable.

The reason why $\mathcal{P}^{\{\omega\}}$ does not have a terminal coalgebra follows from the fact that the terminal ω-chain is the same for $\mathcal{P}^{\{\omega\}}$ and $\mathcal{P}_\mathrm{f}$. It is not difficult to derive from the description in Example 2.5.7 that the limit $\mathcal{P}_\mathrm{f}^\omega 1$ is uncountable. From this it follows that the λth iteration of $\mathcal{P}^{\{\omega\}}$ at 1 has power $card(\mathcal{P}^{\{\omega\}})^\lambda(1) > \lambda$ for all infinite cardinals λ.

Open Question 3.2.4. For which ordinals α does there exist a set functor whose convergence of the terminal chain requires precisely α steps? We have

$\alpha = 0$	whenever $F1 = 1$,
$\alpha = 1$	for example for constant functors $C_M, M \not\cong 1$,
$\alpha = \omega$	for H_Σ,
$\alpha = \omega + \omega$	for $\mathcal{P}_\mathrm{f}$,

and not much more seems to be known. However, whenever a terminal coalgebra exists, it can be constructed by the terminal chain. This holds, more generally, for many-sorted sets:

Theorem 3.2.5. *[AT09] Whenever an endofunctor of* Set^S *has a terminal coalgebra, then the terminal chain converges.*

Remark 3.2.6. This generalizes the previous result on endofunctors of Set in Adámek and Koubek [AK95]. Both proofs heavily depend on the theory of algebraized chains developed by Jan Reiterman in his PhD thesis and summarized in Koubek and Reiterman [KR].

The expected generalization of Theorem 3.2.5 to, say, all presheaf categories does not hold. In [AT09] an endofunctor of the category of graphs is constructed that has a terminal coalgebra although the coalgebra chain does not converge.

We now turn to the category of dcpos and consider sufficient conditions for endofunctors with a fixed point to have a canonical one.

Notation 3.2.7. We write

$$\Delta\mathsf{CPO}_\perp$$

for the category of dcpos (posets with directed joins and therefore a least element) and functions which are *continuous* (they preserve directed joins) and *strict* (preserving the least element. Analogously to Remark 2.8.4 a category is $\Delta\mathsf{CPO}_\perp$-enriched if its hom-sets carry a dcpo structure and composition is strict and continuous.

Definition 3.2.8. A functor F between $\mathsf{CPO}_\perp$-enriched categories is called *stable* if for every idempotent endomorphism $f\colon X \to X$, $f = f \cdot f$, we have:

$$f \sqsubseteq id_X \quad \text{implies } Ff \sqsubseteq id_{FX}.$$

Examples 3.2.9. (i) Every locally continuous functor (see Definition 2.8.12), is stable.

(ii) Id is always stable. A composite, a product or coproduct of stable endofunctors of $\Delta\mathsf{CPO}_\perp$ is stable.

(iii) For every stable functor $F\colon \Delta\mathsf{CPO}_\perp \to \Delta\mathsf{CPO}_\perp$ the *lifting* given by $F_\perp X = FX \cup \{\perp\}$ ($\perp$ a new bottom element) is stable.

Remark 3.2.10. The H_Σ-algebras in $\Delta\mathsf{CPO}_\perp$ are the *strict continuous* Σ*-algebras*: they are given by a cpo, A, which is a Σ-algebra such that every operation $\sigma\colon A^n \to A$ is continuous and strict: $\sigma(\perp,\perp,\perp,\dots) = \perp$.

The non-strict continuous Σ-algebras are the algebras for the endofunctor

$$H'_\Sigma X = \coprod_{\sigma\in\Sigma} X^n_\perp \qquad n = \mathrm{ar}(\sigma),$$

where each summand is lifted. We know that H'_Σ is also stable.

Observation 3.2.11. (i) Stable functors F preserve embeddings. In fact, $F\hat{e}$ is the projection for Fe.

(ii) The converse holds in categories with *split idempotents*, i.e., given $g \cdot g = g$ there exists a factorization $g = e \cdot u$ with $u \cdot e = id$. (This weak condition holds in $\Delta\mathsf{CPO}_\perp$, Pfn (the category of sets and partial functions, Rel (the category of sets and relations) and all categories having equalizers.)

That is: for endofunctors of categories with split idempotents we have:

$$F \text{ stable} \Leftrightarrow F \text{ preserves embedding-projection pairs.}$$

Indeed, let F preserve these pairs and let $g \cdot g = g \sqsubseteq id_X$. For the above factorization we have $e \cdot u = g \sqsubseteq id_X$, thus, e is an embedding with $u = \hat{e}$, therefore $Fu = \widehat{Fe}$ and we conclude

$$Fg = Fe \cdot \widehat{Fe} \sqsubseteq id_{FX}.$$

(iii) If F is a stable endofunctor, then the initial chain

$$0 \xrightarrow{u} F0 \xrightarrow{Fu} FF0 \xrightarrow{FFu} \dots$$

is given by embeddings, and the corresponding projections form the terminal chain for F. In fact, $0 = 1$ by (3) in Remark 2.8.6 and so $\hat{u}\colon F0 \to 0$ is unique. Thus, up to the ordinal ω we get the embedding-projection pairs

$$\begin{tikzcd} 0 \arrow[r, shift left, "u"] & F0 \arrow[l, shift left, "\hat{u}"] \arrow[r, shift left, "Fu"] & FF0 \arrow[l, shift left, "F\hat{u}"] \arrow[r, shift left, "FFu"] & \cdots \arrow[l, shift left, "FF\hat{u}"] \end{tikzcd}$$

For step ω (and all limit steps) we know from Basic Lemma 2.8.10 that a colimit of a chain of embeddings, formed in $\mathcal{A}$, stays as a colimit in $\mathcal{A}^E$.

Corollary 3.2.12. *Let $\mathcal{A}$ be a $\Delta\mathsf{CPO}_\perp$-enriched category with colimits of chains. Then every stable endofunctor F with a fixed point has a canonical fixed point $\mu F = \nu F$.*

This follows from Theorem 3.1.18 applied to the class $\mathcal{M}$ of all embeddings. This is a constructive class, see Lemma 2.8.10 and Remark 2.8.11. Due to Observation 3.2.11 F preserves embeddings, and wellpoweredness w.r.t. embeddings follows from the observation that the number of split subobjects of an object $\mathcal{A}$ is bounded by the number of endomorphisms of A.

Remark 3.2.13. The above Corollary 3.2.12 stems from [Barr], where the slightly stronger assumption that F be locally monotone was made, and the proof is somewhat more technical.

Example 3.2.14. A stable endofunctor of $\Delta\mathsf{CPO}_\perp$ without fixed points. We denote by $C(X)$ the free $\Delta\mathsf{CPO}_\perp$-completion of X, that is, C is the composite (= monad) of the forgetful functor $\Delta\mathsf{CPO}_\perp \to \mathsf{Pos}$ and its left adjoint. This can be described as

$$C(X) = \text{all directed, down-closed subsets of } X \text{ including } \emptyset$$

ordered by inclusion. The directed joins in $C(X)$ are unions, and the universal arrow $X \to C(X)$ takes x to $\downarrow x = \{y : y \leq x\}$. The endofunctor C is stable: if $e \sqsubseteq id_X$, then the set

$$\{a \in C(X) : Ce(a) \sqsubseteq a\}$$

contains all $a = \downarrow x$ and $a = \emptyset$ and is closed under directed joins – thus, this is all of $C(X)$.

The functor C has no fixed points. In fact, assuming the contrary, we have an isomorphism

$$\alpha\colon C(A) \to A$$

and we derive a contradiction. Define a chain

$$a_i \in C(A) \text{ for } i \in \mathsf{Ord}$$

by transfinite induction

$$\begin{aligned} a_0 &= \emptyset \\ a_{i+1} &= \downarrow \alpha(a_i) \end{aligned}$$

and for all limit ordinals j

$$a_j = \bigcup_{i<j} a_i.$$

It iqs easy to see that a_i is indeed a chain in $C(A)$: assuming $a_i \sqsubseteq a_{i+1}$ we have, since $\downarrow \alpha(-)$ is monotone, $a_{i+1} \sqsubseteq a_{i+2}$. Thus, there exists an ordinal i_0 with

$$a_{i_0} = a_{i_0+1}.$$

However, the least such ordinal i_0 cannot be 0 because $\alpha(\emptyset) = \perp$ implies

$$a_1 = \{\perp\} \neq a_0.$$

It also cannot be a successor ordinal, since from $a_{i_0-1} \neq a_{i_0}$ we derive $\downarrow \alpha(a_{i_0-1}) \neq \downarrow \alpha(a_{i_0})$, that is, $a_{i_0} \neq a_{i_0+1}$. And i_0 cannot be a limit ordinal: from

$$a_{i_0} = \bigcup_{i<i_0} a_i$$

we have, by continuity of α, that

$$x = \bigsqcup_{i<i_0} \alpha(a_i) \quad \text{for } x = \alpha(a_{i_0}).$$

Now $x \in \downarrow x = a_{i_0+1}$, however, $x \notin a_{i_0}$ because for every $i < i_0$ we have

$$x \notin a_i.$$

Indeed, this follows from the fact that since α is a monotone monomorphism, $\alpha(a_i)$ is a strictly increasing i_0-chain with join x. Thus, $\alpha(a_i) \neq x$, which implies $x \notin \downarrow \alpha(a_i) = a_{i+1}$. Therefore, we get a contradiction to $a_{i_0} = a_{i_0+1}$.

Remark 3.2.15. The concept of a locally continuous endofunctor F can be generalized to that of a locally λ-continuous one, where λ is a given infinite cardinal: This means that for every λ-chain $(f_i)_{i<\lambda}$ in $\mathcal{A}(X,Y)$ we have

$$F(\bigsqcup_{i<\lambda} f_i) = \bigsqcup_{i<\lambda} Ff_i \text{ in } \mathcal{A}(FX, FY).$$

For these endofunctors we have, whenever $\mathcal{A}$ is $\Delta\mathsf{CPO}_\perp$-enriched and has colimits of chains of cofinality at most λ, the following:

$$F \text{ has a canonical fixed point } \mu F = \nu F = \lim_{n<\lambda} F^n 1.$$

This is completely analogous to Theorem 2.8.14.

4 Sufficient Conditions for Terminal Coalgebras

4.1 Using the Adjoint Functor Theorems

Recall that for every endofunctor F of $\mathcal{A}$ we have a forgetful functor

$$U\colon \mathsf{Coalg}F \to \mathcal{A} \qquad\qquad (A,\alpha) \mapsto A.$$

When $\mathcal{A}$ has (any type of) colimits, the same type exists in $\mathsf{Coalg}F$ and is preserved by U. (This is why initial coalgebras are not interesting ....) Consequently, if $\mathcal{A}$ is cocomplete and cowellpowered (i. e., every object has only a set of quotients), then so is $\mathsf{Coalg}F$. Indeed, quotients are represented by epimorphisms, and $e\colon A \to B$ is an epimorphism iff the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "id"] \\ B \arrow[r, "id"'] & B \end{tikzcd}$$

is a pushout.

Recall the following form of Freyd's Adjoint Functor Theorem (see e. g. [ML]): if $\mathcal{K}$ is a cocomplete and cowellpowered category with a *weakly terminal set* $\mathcal{A}$ of objects (meaning that every object of $\mathcal{K}$ has a morphism into an object of $\mathcal{A}$), then $\mathcal{K}$ has a terminal object. Applying this to $\mathsf{Coalg}F$ we get

Theorem 4.1.1 ([Barr]). *Let $\mathcal{A}$ be a cocomplete and cowellpowered category, and let $F : \mathcal{A} \to \mathcal{A}$ have a weakly terminal set of coalgebras. Then F has a terminal coalgebra.*

Remark 4.1.2. The proof of Freyd's Adjoint Functor Theorem yields the following construction og the terminal coalgebra: let $\mathcal{S}$ be a weakly terminal set of coalgebras. Take the coproduct $\gamma : C \to FC$ of all coalgebras in $\mathcal{S}$. Now form the coequalizer of all the F-coalgebra homomorphisms from (C,γ) to itself; one readily shows that this coequalizer is the terminal F-coalgebra.

Observe that the coproduct (C,γ) is a weakly terminal coalgebra and the coequalizer of all its endomorphisms is the largest congruence on (C,γ). So Theorem 4.1.1 is a generalization of Theorem 2.4.16 above.

Remark 4.1.3. A related criterion concerns the concept of a *generating set*, which is a set $\mathcal{G}$ of objects such that whenever parallel morphisms $p, q\colon X \to Y$ are distinct, then there exists a morphism $g\colon G \to X$ with $G \in \mathcal{G}$ and $p \cdot g \neq q \cdot g$. Generalizing the notion of *bounded set functor* of Kawahara and Mori [KM], we can define

Definition 4.1.4. A functor $F\colon \mathcal{A} \to \mathcal{A}$ is called *bounded* by a set $\mathcal{G}$ of objects if for every coalgebra $\alpha\colon A \to FA$ and every morphism $g\colon G \to A$ with $G \in \mathcal{G}$ there exists a coalgebra $\alpha' : A' \to FA'$ with $A' \in \mathcal{G}$ and a coalgebra homomorphism $m\colon (A',\alpha') \to (A,\alpha)$ through which g factorizes:

$$\begin{tikzcd} & A' \arrow[r, "\alpha'"] \arrow[d, "m"] & FA' \arrow[d, "Fm"] \\ G \arrow[ur, dashed, "\exists"] \arrow[r, "g"'] & A \arrow[r, "\alpha"'] & FA \end{tikzcd}$$

In [KM] this was defined for $\mathcal{A} = \mathsf{Set}$ and $\mathcal{G}$ the (essentially small) collection of all sets of cardinality at most λ.Then F is called bounded by λ. Observe that in this case it is sufficient to request that for every element $x \in A$ there exists a subcoalgebra $A' \hookrightarrow A$ containing x and with card $A' \leq \lambda$. In fact, the general case follows from the observation that a union of subcoalgebras is always a subcoalgebra. The following is an easy generalization of the result of [KM] for sets.

Theorem 4.1.5. *Let $\mathcal{A}$ be a cocomplete and cowellpowered category with a terminal object. Every endofunctor bounded by a generating set (4.1.3) has a terminal coalgebra.*

Proof. This uses Freyd's Special Adjoint Functor Theorem: we prove that $U\colon \mathsf{Coalg}F \to \mathcal{A}$ has a right adjoint $U \dashv R$ (then $R1$ is terminal in $\mathsf{Coalg}F$). For that we need, since $\mathsf{Coalg}F$ is cocomplete and cowellpowered and U preserves colimits, just a generating set in $\mathsf{Coalg}F$. In fact, let $\mathcal{G}$ be a generating set in $\mathcal{A}$ such that F is bounded by it. Then the set of all coalgebras $\alpha\colon A \to FA$ with $A \in \mathcal{G}$ is generating. In fact, given homomorphisms $p, q\colon (B, \beta) \to (\bar{B}, \bar{\beta})$ if $p \neq q$ there exists $g\colon G \to A$ with $G \in \mathcal{G}$ and $p \cdot g \neq q \cdot g$. Let $m\colon (B', \beta') \to (B, \beta)$ be a subcoalgebra with $B' \in \bar{\mathcal{G}}$ such that g factorizes through m, then clearly $p \cdot m \neq q \cdot m$. $\square$

The following theorem due to Gumm and Schröder [GS] shows the existence of terminal coalgebras of quotients of functors with terminal coalgebras. In the special case of finitary endofunctors of Set we have already considered quotients of functors in Section 2.7.

Definition 4.1.6. A *quotient functor* of a functor $F\colon \mathcal{A} \to \mathcal{A}$ is represented by a functor $G\colon \mathcal{A} \to \mathcal{A}$ and a natural transformation $e\colon F \to G$ with epimorphic components.

Another such transformation $e'\colon F \to G'$ represents the same quotient iff some natural isomorphism $h\colon G \to G'$ fulfils $e' = h \cdot e$.

Theorem 4.1.7. *Let $\mathcal{A}$ be a cocomplete and cowellpowered category in which epimorphisms split. Let $\varepsilon : H \to F$ be a quotient functor. Then if H has a terminal coalgebra, then so has F and νF is a quotient coalgebra of νH via the unique coalgebra homomorphism $\hat{\varepsilon}$ from the coalgebra*

$$\nu H \xrightarrow{\tau} H(\nu H) \xrightarrow{\varepsilon_{\nu H}} F(\nu H). \tag{4.1}$$

to νF.

Proof. (1) Let $\tau' : \nu H \to F(\nu H)$ be the coalgebra in (4.1). Then the same argument as in Lemma 2.4.6 shows tat $(\nu H, \tau')$ is a weakly terminal F-coalgebra. Thus, by Theorem 4.1.1, the terminal F-coalgebra $\sigma : \nu F \to F(\nu F)$ exist, and therefore we have the unique F-coalgebra homomorphism $\hat{\varepsilon}$ from $(\nu H, \tau')$ to $(\nu F, \sigma)$.

(2) It remains to show that $\hat{\varepsilon}$ is a (split) epimorphism. By part (1) of our proof we have an F-coalgebra homomorphism s from $(\nu F, \sigma)$ to $(\nu H, \tau')$. By definition $\hat{\varepsilon}$ is the unique F-coalgebra homomorphism from $(\nu H, \tau')$ to the terminal F-coalgebra $(\nu F, \sigma)$. Thus, $\hat{\varepsilon} \cdot s$ is an F-coalgebra homomorphism from νF to itself, whence $\hat{\varepsilon} \cdot s = id$ so that $\hat{\varepsilon}$ is indeed a split epimorphism. $\square$

4.2 Finitary and Accessible Functors

We proved in Section 2.4 that finitary endofunctors of Set have a terminal coalgebra. We now generalize this to accessible endofunctors of Set and also of more general categories. P. Gabriel and F. Ulmer used in [GU] this concept to generalize "finiteness":

Definition 4.2.1. An object A of a category $\mathcal{A}$ is called *finitely presentable* if the hom-functor $\mathcal{A}(A, -): \mathcal{A} \to \mathsf{Set}$ is finitary (i.e., preserves filtered colimits).

Examples 4.2.2. In sets, posets, graphs and metric spaces "finitely presentable" means finite. In $\mathsf{CPO}_\perp$ no nontrivial object is finitely presentable.

More generally, a λ-*filtered colimit* for an infinite regular cardinal λ is a colimit of a diagram $D: \mathcal{D} \to \mathcal{A}$ such that (i) for less than λ objects there always exists a cocone in $\mathcal{D}$ and (ii) for less than λ parallel morphisms $f_i: A \to B$ ($i \in I$) there exists a coequalizing morphism $g: B \to C$ (that is, $g \cdot f_i: A \to C$ is independent of $i \in I$). An object is called λ-*presentable* if its hom-functor preserves λ-filtered colimits. In sets, posets, graphs and metric spaces this means precisely that the cardinality is less than λ. In $\mathsf{CPO}_\perp$ for any $\lambda > \omega$ again λ-presentable objects are precisely those of cardinality less than λ.

Definition 4.2.3. A functor F is called *accessible* if it preserves λ-filtered colimits for some cardinal λ. (Thus, F is called *finitary* in case $\lambda = \omega$.)

Examples 4.2.4. (i) The set functor $X \to X^n$ is finitary for every natural number n. It is accessible for every cardinal n.

(ii) A coproduct of accessible functors is accessible. Thus, all polynomial functors H_Σ are accessible. They are finitary iff the signature Σ is finitary.

(iii) The functor $\mathcal{P}_3$ and $\mathcal{P}_f$ are finitary, $\mathcal{P}_{\omega_1}$ is accessible (choose $\lambda = \omega_1$ in Definition 4.2.3) and $\mathcal{P}$ is not accessible.

Theorem 4.2.5 ([AP01], [GS]). *For a set functor F the following conditions are equivalent:*

(i) F preserves λ-filtered colimits;

(ii) every element of FA, for any set A, lies in the image of Fb for some subset $b: B \hookrightarrow A$ of less than λ elements.

Moreover, they imply that F is bounded by sets of cardinality $\leq \lambda$.

Remark 4.2.6. Thus, every accessible set functor is bounded. The converse is also true: every bounded functor is accessible (but in general not for the same λ).

Proof of Theorem 4.2.5. Indeed, (i) $\to$ (ii) follows from F preserving the λ-directed union of all subsets $b: B \hookrightarrow A$ of less than λ elements. The other implications are slightly more involved. Let us present just one of them:

(ii) $\to$ (iii) Given a coalgebra $a: A \to FA$ and a subset $b_0: B_0 \hookrightarrow A$ of less than λ elements, we define a λ-chain of subsets $b_i: B_i \hookrightarrow A$ ($i < \lambda$) of less than λ elements for which the union

$$B = \bigcup_{i<\lambda} B_i$$

turns out to be the desired subcoalgebra containing B_0. In the induction step, given $b_i: B_i \hookrightarrow A$ use the fact that F preserves the λ-directed union (= colimit) of all subsets of less than λ elements containing B_i. Consequently, our subobjects b_i factorize through one of F-images of these subsets. That is, there exists a morphism f_i as follows:

$$\begin{tikzcd}
B_i \arrow[r, "f_i"] \arrow[d, "b_i"'] & FB_{i+1} \arrow[d, "Fb_{i+1}"] \\
A \arrow[r, "a"'] & FA
\end{tikzcd}$$

where $b_{i+1}\colon B_{i+1} \hookrightarrow A$ is a subset containing B_i and having less than λ elements. For limit ordinals: put $B_i = \bigcup_{j<i} B_j$. Clearly, the union $b\colon B \to A$ of this λ-chain is a subcoalgebra (use the above maps f_i) containing B_0. $\square$

Proposition 4.2.7 ((Adámek, Porst [AP04], Gumm, Schröder [GS]))**.** *For every set functor F the following conditions are equivalent:*

(i) F is accessible (= bounded),

(ii) F is a quotient of a polynomial endofunctor H_Σ,

(iii) F is a quotient of the endofunctor

$$G_{CM}\colon X \mapsto C \times X^M \quad \textit{for some sets } C \textit{ and } M.$$

Observe that coalgebras for $G_{C,M}$ are deterministic automata with output set C and input set M: every coalgebra $a\colon A \to C \times A^M$ yields an output map $A \to C$ and a next-state map in curried from $A \to A^M$:

$$\frac{A \times M \to A}{A \to A^M}.$$

Corollary 4.2.8. *The collection of accessible set functors is closed under products, coproducts, subfunctors and quotient functors.*

Just subfunctors need explanations: use Definition 4.2.3 and Example 4.2.4 (ii). The next result follows from Theorem 4.1.5.

Corollary 4.2.9 ([KM])**.** *Every accessible set functor has a terminal coalgebra.*

Makkai and Paré [MP] proved a stronger result: the categories of coalgebras of accessible functors are complete. For accessible set endofunctors Worrell provided an upper bound for the terminal chain to converge:

Theorem 4.2.10. *(Worrell [W2]) For every set endofunctor F preserving λ-filtered colimits , the terminal coalgebra is obtained by taking at most $\lambda + \lambda$ steps in the terminal chain:*

$$\nu F = F^{\lambda+\lambda}1.$$

The proof of this theorem is similar to the proof of Theorem 2.6.3. Moreover, if F preserves countable intersections, then

$$\nu F = F^{\lambda+\omega}1.$$

We can generalize Corollary 4.2.9 to an important class of categories:

Definition 4.2.11. (Gabriel, Ulmer [GU]) Let λ be an infinite cardinal. A category $\mathcal{A}$ is called *locally λ-presentable* if it is cocomplete and it has a set $\mathcal{A}_\lambda$ of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathcal{A}$. And $\mathcal{A}$ is called *locally presentable* if it is locally λ-presentable for some λ.

Examples of locally finitely presentable categories ($\lambda = \omega$) are sets, posets, graphs, groups (in fact, all varieties of algebras, also many-sorted). The categories $\mathsf{CPO}_\perp$, MS and CMS are locally ω_1-presentable.

Theorem 4.2.12. *Every accessible endofunctor if a locally presentable category has a terminal coalgebra.*

This result follows from a more general result of M. Makkai and R. Paré [MP] which shows that $U\colon \mathsf{Coalg}F \to \mathcal{A}$ has a right adjoint. But we present a full proof here for the convenience of the reader. A direct proof for set functors is presented by Barr [Barr].

Proof of Theorem 4.2.12. Since $\mathcal{A}$ is locally presentable, it is cocomplete and cowellpowered and the set $\mathcal{A}_\lambda$ is generating, see [AR] or [GU] for these facts. Moreover, we can find λ for which $\mathcal{A}$ is locally λ-presentable and F is λ-accessible. For the set $\mathcal{G} = \mathcal{A}_\lambda$ we form the closure $\bar{\mathcal{G}}$ under colimits of λ-chains and verify the remark in Theorem 4.1.5. Let $\alpha\colon A \to FA$ be a coalgebra and $g\colon G \to A$ a morphism where G is λ-presentable. We define a λ-chain B, $i < \lambda$, of λ-presentable objects together with a cocone $b_i\colon B_i \to A$ analogously to Theorem 4.1.5: first

$$b_0 = g \qquad \text{and} \qquad B_0 = G.$$

Given $b_i\colon B_i \to A$ express A as a λ-filtered colimit of λ-presentable objects G_t and with the colimit cocone $g_t\colon G_t \to A$. Since B_i is λ-presentable, $\mathcal{A}(B_i, -)$ preserves this colimit, thus, b_i factorizes through g_t for some t, and we put $B_{i+1} = G_t$ and $b_{i+1} = g_t$

$$\begin{tikzcd} & G_t = B_{i+1} \arrow[d, "g_t"] \\ B_i \arrow[ur, dashed] \arrow[r, "b_i"'] & A \end{tikzcd}$$

Also $FA = \operatorname{colim} FG_t$ is a λ-filtered colimit (since F is λ-accessible), so $a \cdot b_i\colon A \to FA$ factorizes through some Fg_t, and without loss of generality this is the same t as above:

$$\begin{tikzcd} B_i \arrow[r, "f_i"] \arrow[d, "b_i"'] & FB_{i+1} \arrow[d, "Fb_{i+1}"] \\ A \arrow[r, "\alpha"'] & FA \end{tikzcd}$$

For limit ordinals put $B_i = \operatorname{colim} B_j$ and obtain the induced morphism $b_i\colon B_i \longrightarrow A$. Also for $i = \lambda$ we form a colimit $\operatorname{colim}_{i<j} B_j$

$$B = \operatorname*{colim}_{i<\lambda} B_i \in \bar{\mathcal{G}}$$

which is λ-filtered, thus, preserved by F. The above morphisms f_i yield a coalgebra structure

$$\operatorname*{colim}_{i<\lambda} f_i\colon B \to FB$$

such that

$$b = \operatorname{colim}\ b_i \colon B \to A$$

is a coalgebra homomorphism. We have factorization of g through b: recall $g = b_0$ and use the first colimit morphism of $B = \operatorname{colim} B_i$. $\square$

Theorem 4.2.13 ([AT10])**.** *Let F be an accessible endofunctor of a locally presentable category preserving monomorphisms. Then the terminal chain converges:*

$$\nu F = F^j 1 \qquad \textit{for some ordinal number } j.$$

Example 4.2.14. All endofunctors of CMS formed by products, coproducts and compositions of polynomial functors and the Hausdorff functor have a terminal coalgebra. In fact, the follows from Theorem 4.2.12: all these functors are accessible. For $\mathcal{H}$ this was proved in [vB]; in fact, $\mathcal{H}$ is even a finitary functor: see [AMMS].

5 Terminal Coalgebras as Algebras

In this section we are going to provide an equivalent characterization of terminal coalgebras. Recall that by (the dual of) Lambek's Lemma 2.1.1 the structure morphism of a terminal coalgebra νF for a functor F is an isomorphism. Thus, νF can be considered as an F-algebra. The equivalent characterization of νF arises in connection with an important property of this algebra: it is a *completely iterative algebra*, i. e., it allows for the unique solution of recursive equations. An object of $\mathcal{A}$ carries a terminal coalgebra for F iff it carries an initial cia, see Theorem 5.1.12.

The second part of this section deals with a third interesting fixed point of a functor besides its initial algebra and its terminal coalgebra. For finitary endofunctors of Set this *rational fixed point* R, as we call it here, has two equivalent characterizations: as an F-algebra R is the initial iterative algebra for F, where an algebra is *iterative* if it allows for the unique solution of finite systems of recursive equations (the finiteness being the difference to completely iterative algebras). As a coalgebra, R is the terminal locally finite coalgebra for F, i. e., for every F coalgebra with a finite carrier there exists a unique coalgebra homomorphism into R. So R precisely collects all the behaviors of finite systems considered as F-coalgebras. Accordingly, examples of rational fixed points are: regular languages, rational streams, rational (or regular) trees for a signature etc. We discuss rational fixed points in Subsection 5.2.

5.1 Completely Iterative Algebras

The idea of algebras with unique solutions of recursive equations stems from work in general algebra by Evelyn Nelson [N] und Jerzy Tiuryn [T]. Nelson introduced iterative algebras for a signature as algebras with unique solutions of finite recursive systems of equations. Dropping the finiteness assumption one arrives at the notion of a completely iterative algebra. The latter notion was introduced in [M_1] and this paper also investigates the connection to terminal coalgebras. One of the results is that terminal coalgebras are equivalently characterized as initial completely iterative algebras. But before we make this more precise let us explain the notion of completely iterative algebras with a concrete example.

Consider a polynomial functor H_Σ on Set arising from a signature Σ. The general algebras for the signature Σ are precisely the algebras for the functor H_Σ, see Example 2.2.11. A Σ-algebra A is called *completely iterative* if every system of mutually recursive equations

$$x_i \approx t_i, \qquad i \in I, \tag{5.1}$$

where I is some (possibly) infinite set, $X = \{ x_i \mid i \in I \}$ is a set of variables and each t_i is a term over $X + A$ none of which is just a single variable, has a unique solution in A. By a solution we mean a set $\{ {x_i}^\dagger \mid i \in I \}$ of elements of A such that the formal equations in the given system (5.1) become actual identities in A.

Example 5.1.1. For example, the functor $FX = X \times X + 1$ is the polynomial functor associated to the signature Σ with a binary operation symbol $*$ and a constant symbol c. The terminal coalgebra νF consists of all binary trees (see Example 2.3.7); let us consider inner nodes of binary trees as being labelled by $*$ and leaves as labelled by c. Now for any binary tree $t \in \nu F$ consider the following systems of equations:

$$x_1 \approx x_2 * t \qquad x_1 \approx x * c$$

Then the solution of this equation in νF is formed by the two infinite trees

$$ {x_1}^\dagger \approx (\cdots(\cdots * c) * t) * c) * t) \qquad \text{and} \qquad {x_2}^\dagger \approx (\cdots(\cdots * t) * c) * t) * c) \tag{5.2}$$

written as infinite Σ-terms.

Notice that in general, it is sufficient to allow as the right-hand sides of equation systems (5.1) only *flat-terms*, i. e., terms which are either of the form $\sigma(x_1, \dots, x_n)$ for some operation symbol σ from Σ or are elements $a \in A$. In fact, every arbitrary system can be "flattened" by introducing enough fresh auxiliary variables to represent subterms of non-flat terms. For example, the above system (5.2) can be flattened as follows:

$$\begin{array}{lcl@{\qquad}lcl} x_1 & \approx & x_2 * z_1 & x_2 & \approx & x_1 * z_2 \\ z_1 & \approx & t & z_2 & \approx & c \end{array}$$

Each flat system of equations can be presented as a function $e : X \to H_\Sigma X + A$, and solutions in an H_Σ-algebra $\alpha : H_\Sigma A \to A$ are presented as functions $e^\dagger : X \to A$ such that

$$e^\dagger = [\alpha, id_A] \cdot (H_\Sigma e^\dagger + id_A) \cdot e.$$

Again, this equation just expresses the fact that the formal equations are turned into identities in A. This motivates Definition 5.1.3 below.

Assumption 5.1.2. Throughout the rest of this subsection we assume that $\mathcal{A}$ denotes a category with binary coproducts.

Definition 5.1.3. [M_1] Let F be an endofunctor on $\mathcal{A}$. By a *flat equation morphism* in an object A we mean a morphism $e : X \to FX + A$. For an algebra $\alpha : FA \to A$ we call $e^\dagger : X \to A$ a *solution* of e in A if the square below commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX + A \arrow[r, "Fe^\dagger + id_A"'] & FA + A \arrow[u, "{[\alpha, id_A]}"'] \end{tikzcd} \tag{5.3}$$

Finally, an algebra (A, α) for F is called *completely iterative* (or, cia for short) provided that every flat equation morphism in A has a unique solution.

Examples 5.1.4.

(i) The algebra of addition on the extended natural numbers $\widetilde{\mathbb{N}} = \{1, 2, 3, \dots\} \cup \{\infty\}$ is a cia for the functor $FX = X \times X$, see [AMV$_1$].

(ii) Infinite binary trees form a cia for $FX = X \times X + 1$. All finitely branching strongly extensional trees form a cia for $\mathcal{P}_{\mathrm{f}}$. All unordered finitely branching trees form a terminal coalgebra for the bag functor $\mathcal{B}$. These are instances of the more general fact in Proposition 5.1.5 below.

(iii) Unary algebras over Set are precisely the algebras for the identity functor $F = Id$ on Set. A unary algebra $\alpha : A \to A$ is a cia if and only if

(a) there exists a unique fixed point $a_0 \in A$ of α and

(b) there exists a well-founded strict order on $A \setminus \{a_0\}$ such that α is increasing, i. e., $\alpha(a) > a$ for all $a \in A, a \neq a_0$.

(iv) Classical algebras are seldom cias. A group is a cia for the functor $HX = X \times X$ expressing the binary operation iff its unique element is the unit 1, since the recursive equation $x \approx x \cdot 1$ has a unique solution. A lattice is a cia for H iff it has a unique element; consider the equation $x \approx x \vee x$.

Proposition 5.1.5. *The terminal coalgebra νF is a cia.*

More precisely, let $t : \nu F \to F(\nu F)$ be a terminal F-coalgebra. By Lambek's Lemma [L], t is invertible, and then $t^{-1} : F(\nu F) \to \nu F$ is the structure of a cia.

Proof. Let $e : X \to FX + \nu F$ be a flat equation morphism. From e we form the following F-coalgebra

$$\overline{e} \equiv X + \nu F \xrightarrow{[e,\mathsf{inr}]} FX + \nu F \xrightarrow{FX+t} FX + F(\nu F) \xrightarrow{\mathrm{can}} F(X + \nu F)\,,$$

where $\mathsf{can} = [F\mathsf{inl}, F\mathsf{inr}]$. Let $h : X + \nu F \to \nu F$ be the corresponding unique coalgebra homomorphism and define

$$e^\dagger = (X \xrightarrow{\mathsf{inl}} X + \nu F \xrightarrow{h} \nu F).$$

One readily shows that $h \cdot \mathsf{inr}$ is a coalgebra homomorphism from νF to itself, whence $h \cdot \mathsf{inr} = id$. Now it is not difficult to prove that $e^\dagger$ is a solution of e iff $[e^\dagger, id]$ is a coalgebra homomorphism from $(X + \nu F, \overline{e})$ to the terminal coalgebra $(\nu F, t)$. Since the latter exists uniquely, so does the former. For further details details see [M$_1$], Example 2.5. $\square$

Next we prove that algebras over complete metric spaces are cias. Recall that CMS is the category of complete metric spaces, where distances are measured in the interval $[0, 1]$, and non expanding maps. We shall consider a *contracting* endofunctor F on CMS, see Definition 2.9.2.

Proposition 5.1.6 ([AMV$_3$])**.** *Let $F : \mathsf{CMS} \to \mathsf{CMS}$ be a contracting endofunctor. Then every non-empty F-algebra is completely iterative.*

Proof. Suppose that F is contracting with a contraction factor $\varepsilon < 1$. Let $\alpha : FA \to A$ be a non-empty F-algebra and let $e : X \to FX + A$ be a flat equation morphism. Recall that the hom-set $\mathsf{CMS}(X, A)$ is a complete metric space with the supremum metric and this space is non-empty as A is non-empty. Definition 5.1.3 of a solution of a flat equation morphism $e : X \to HX + A$ states that $e^\dagger$ is a fixed point of the function Φ on $\mathsf{CMS}(X, A)$ given by

$$\Phi : (s : X \to A) \mapsto ([\alpha, A] \cdot (Fs + A) \cdot e).$$

We shall now prove that this function is a contraction on $\mathsf{CMS}(X, A)$. Indeed, for two nonexpanding maps $s, t : X \to A$ we have

$$\begin{array}{rcll} d_{X,A}(\Phi s, \Phi t) & = & d_{X,A}([\alpha, A] \cdot (Fs + A) \cdot e, [\alpha, A] \cdot (Ft + A) \cdot e) & \\ & & & \text{(by the definition of } \Phi) \\ & \leq & d_{FX,FA}(Fs + A, Ft + A) & \text{(since composition is nonexpanding)} \\ & = & d_{FX,FA}(Fs, Ft) & \\ & \leq & \varepsilon d_{X,A}(s, t) & \text{(since } F \text{ is } \varepsilon\text{-contracting).} \end{array}$$

By Banach's Fixed Point Theorem 2.9.1, there exists a unique fixed point of Φ, viz. a unique solution of e. □

Example 5.1.7. Completely metrizable algebras are cias. Many set functors F have a lifting to contracting endofunctors F' of CMS. We have seen this for polynomial endofunctors in Example 2.9.3. We call an F-algebra $\alpha : FA \to A$ *completely metrizable* if A if there exists a complete metric d on A such that the algebra structure is a non-expanding map $\alpha : F'(A, d) \to (A, d)$.

Every non-empty completely metrizable F-algebra is a cia. Indeed, to every equation morphism $e : X \to FX + A$ its unique solution is the unique solution of $e : (X, d_0) \to F'(X, d_0) + (A, d)$ in the where d_0 is the discrete metric.

Cias constitute a full subcategory of the category of all F-algebras. That the choice of all homomorphisms of F-algebras is appropriate for cias follows from the fact, established in [M_1], Proposition 2.3, that these morphisms correspond precisely to the morphisms preserving solutions of flat equations in the obvious sense. We omit the details in this survey.

We now turn to constructions of cias in our general category $\mathcal{A}$, which will then lead to the proof that initial cias and terminal coalgebras are the same.

Lemma 5.1.8. *For every endofunctor $F : \mathcal{A} \to \mathcal{A}$, if (A, α) is a cia for F, then so is $(FA, F\alpha)$.*

Proof. Let $e : X \to FX + FA$ be a flat equation morphism in FA. Form the equation morphism

$$\overline{e} = (X \xrightarrow{e} FX + FA \xrightarrow{id+\alpha} FX + A)$$

and define

$$e^\dagger = (X \xrightarrow{e} FX + FA \xrightarrow{[\overline{e}^\dagger, id]} FA.)$$

It is not difficult to check that $e^\dagger$ is a unique solution of e in FA. For the details see [M_1], Proposition 2.6 □

Proposition 5.1.9. *The category of cias for an endofunctor F of $\mathcal{A}$ is closed under all limits that exist $\mathcal{A}$.*

Example 5.1.10. The trivial terminal algebra $F1 \to 1$ is trivially a cia.

The proof of Proposition 5.1.9 is analogous to the proof of a similar result (Proposition 2.20) in [AMV$_1$] concerning the weaker notion of iterative algebras, and so we omit a proof. From this result we see that all objects in the terminal coalgebra chain are cias (even if the terminal coalgebra itself does not exist).

Corollary 5.1.11. *Let $\mathcal{A}$ be a complete category. Then in the terminal chain of F all algebras $F^i! : F(F^i 1) = F^{i+1}1 \to F^i 1$ are cias.*

Proof. This is easy to see by transfinite induction. In fact, $! : F1 \to 1$ is trivially a cia. For successor steps use the fact that for any cia (A, α) the F-algebra $(FA, F\alpha)$ is a cia, too, see Lemma 5.1.8. Finally, for the limit step apply Proposition 5.1.9. $\square$

Theorem 5.1.12. *The initial cia for F is precisely the same as the terminal F-coalgebra.*

More precisely, we shall prove that for an endofunctor $F : \mathcal{A} \to \mathcal{A}$:

(i) If (T, τ) is an initial cia for F, the inverse τ^{-1} is the structure of a terminal F-coalgebra.

(ii) If $(\nu F, \tau)$ exists, then the inverse τ^{-1} is the structure of an initial cia for F.

Proof. For a coalgebra $\gamma : C \to FC$ and an algebra $\alpha : FA \to A$ consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "\gamma"] \arrow[dd, bend right=60, "e"'] & A \\
FC \arrow[r, "Fh"'] \arrow[d, "\mathsf{inl}"] & FA \arrow[u, "\alpha"'] \arrow[d, "\mathsf{inl}"'] \\
FC + A \arrow[r, "Fh+id"'] & FA + A \arrow[uu, bend right=60, "{[\alpha,id]}"']
\end{tikzcd} \tag{5.4}$$

Let us define $e = \mathsf{inl} \cdot \gamma$ so that the left-hand part of this diagram commutes. Notice also that the right-hand part and the lower square of the diagram obviously commute. Now the outside of the diagram commutes iff the upper square does; in other words, h is a coalgebra-to-algebra homomorphism from (C, γ) to (A, α) iff it is a solution of the flat equation morphism e in the algebra A. We are now ready to prove our theorem.

Ad (i), suppose that $\tau : FT \to T$ is an initial cia. Since $(FT, , F\tau)$ is a cia by Lemma 5.1.8 an argument similar to Lambek's Lemma 2.1.1 shows that τ is an isomorphism. So we have the coalgebra $\tau^{-1} : T \to FT$, and we need to verify that it terminal. Indeed, for every coalgebra (C, γ) take (A, α) in diagram (5.4) to be the algebra (T, τ). Then since this algebra is a cia we have a unique coalgebra-to-algebra homomorphism, i. e., a unique coalgebra homomorphism from (C, γ) to (T, τ^{-1}).

Ad (ii), suppose that the terminal coalgebra $\tau : \nu F \to F(\nu F)$ exists. Then it is a cia by Proposition 5.1.5, and it remains to verify its initiality. So let (A, α) be any cia and let (C, γ) in diagram (5.4) be $(\nu F, \tau)$. Since A is a cia we have a unique solution of e in A, equivalently a unique coalgebra-to-algebra homomorphism from $(\nu F, \tau)$ to (A, α), i. e., a unique F-algebra homomorphism from the cia νF to the cia A. $\square$

Remark 5.1.13. (i) An analogous result to Theorem 5.1.12 can be stated for free cias: TX is a free cia on the object X iff it is a terminal coalgebra for the functor $F(-) + X$. Assuming the existence of free cias TX for a functor, it turns out that T is the object assigment of a monad on $\mathcal{A}$. Furthermore this monad is characterized by a universal property: it is the free completely iterative monad on F. These results appear in [M$_1$, AAMV].

(ii) In many settings, one is less interested in obtaining unique solutions (or fixed points) than in canonical ones, e. g., least fixed points in complete partial orders. It is the idea of Bloom's and Ésik's iteration theories [BÉ] to study the equational properties that characterize least fixed points in complete partial orders. It turns out that similar ideas are important in connection with cias, and they lead to the notion of a *complete Elgot algebra*. A complete Elgot algebra for a functor F is a triple $(A, a, (\_)^\dagger)$ where $a : FA \to A$ is an algebra for F and $(\_)^\dagger$ is an operation taking a flat equation morphism $e : X \to FX + A$ to a solution $e^\dagger : X \to A$ such that two simple and well-motivated properties are satisfied. Complete Elgot algebras were introduced in [AMV$_3$] and the main result of that paper is that they form precisely the category of Eilenberg-Moore algebras for the monad T of item (i) above.

Every cia is, of course, a complete Elgot algebra. Further examples are continuous algebras (i. e., algebras for locally continuous functors on CPO) or complete lattices, which are complete Elgot algebras for $FX = X \times X$ on Set.

(iii) Completely iterative algebras and complete Elgot algebras as in item (ii) play a significant rôle in the category-theoretic semantics of recursive program schemes as presented in [MM]. This approach to the semantics of recursive function definitions is based on terminal coalgebras for functors in lieu of concepts from general algebra like signatures and infinite trees. In the category theoretic setting completely iterative algebras serve as those classes of algebras in which one interprets recursive program schemes—in those algebras one can uniquely solve recursive program schemes. The article [MMS] discusses various applications of completely iterative algebras for the semantics of recursive specifications: Milner's CCS, stream circuits, non-well founded sets and formal languages.

The interval $[0, 1]$ as a terminal coalgebra. In the remainder of this section, we study a category $\mathcal{A}$ and an endofunctor $F : \mathcal{A} \to \mathcal{A}$ whose terminal coalgebra is carried by the unit interval $[0, 1]$. The category is BiP, the category of of bipointed sets in Example 2.2.15, and we also mentioned the functor $F : \mathsf{BiP} \to \mathsf{BiP}$, where $FX = X \oplus X$. We discuss its terminal coalgebra. We consider the F-coalgebra $I = ([0, 1], 0, 1)$ with structure $a : I \to FI$ given by $a(x) = \mathsf{inl}(2x)$ for $x \in [0, \frac{1}{2})$, $a(x) = \mathsf{inr}(2x - 1)$ for $x \in (\frac{1}{2}, 1]$, and $a(\frac{1}{2}) = m$, where m is the "midpoint" of $X \oplus X$. Notice that a is an isomorphism.

Theorem 5.1.14 (Freyd [Fr])**.** *(I, a) is a terminal coalgebra of F on* BiP.

Proof. Let $e : X \to X \oplus X$ be a morphism in BiP. Regard X as a complete ultrametric space in the trivial way, with distance 1 between distinct points. Also, recall the lifting of the coproduct operation $+$ to CMS: it takes two spaces, scales the distance in each by a factor of $\frac{1}{2}$ and then sets the distance between the copies to 1. Consider also

$$X \xrightarrow{e} X \oplus X \xrightarrow{f} (X + X) + I$$

Here f takes the "midpoint" m of $X \oplus X$ to $\frac{1}{2} \in I$. Otherwise, f is the obvious injection into $X + X$.

The functor $X \mapsto X + X$ is $\frac{1}{2}$-contracting on CMS, and thus I is a cia for it with the structure $b : I + I \to I$ with $\mathsf{inl}(x) \mapsto \frac{x}{2}$, and $\mathsf{inr}(x) \mapsto \frac{x+1}{2}$; I is also a cia for $X \mapsto X + X$ on Set, see Example 5.1.7. Now consider the diagram below:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "\varphi"'] & X \oplus X \arrow[r, "f"] \arrow[d, "F\varphi"] & (X+X)+I \arrow[dd, "(\varphi+\varphi)+I"] \\
I \arrow[r, "a"] & I \oplus I & \\
 & & (I+I)+I \arrow[ul, "c"'] \arrow[ull, bend left, "{[b,I]}"]
\end{tikzcd}$$

We must c, but we should first explain that this diagram is in Set: we use CMS to get a solution φ, but then we forget the CMS structure. As for c, on the right summand I it is a, and on the left summand $I + I$ it takes $\mathsf{inl}(1)$ and $\mathsf{inr}(0)$ to m, and otherwise is the obvious map.

Note that $b : I + I \to I$ is a non-expanding map. By the cia property, there is a unique $\varphi : X \to I$ such that the outside of the figure above commutes. Clearly, φ is a morphism of BiP.

Note also that the region on the bottom commutes, as does the region on the right. So the square in the upper-left commutes, showing that φ is an F-coalgebra homomorphism. For the uniqueness, any F-coalgebra morphism ψ determines also $(\psi + \psi) + I$ on the right side of the figure. Using that a is an isomorphism this shows ψ to be a solution to the flat equation morphism $f \cdot e$. By the uniqueness of solutions, $\varphi = \psi$. $\square$

We should mention that the work here is only partially successful. As Freyd points out, "To this date, no one has found a functor whose terminal coalgebra is usefully the reals."

5.2 Iterative Algebras and the Rational Fixed Point

We have mentioned the concept of an *iterative* algebra. It is weaker than the notion of a completely iterative algebra in that one only requires finite systems (5.1) to have unique solutions.

Iterative algebras for a signature were introduced by Nelson [N] (see Tiuryn [T] for a related concept) as an easy approach to iterative theories of Elgot [El]. In [AMV$_1$, AMV$_2$] this was generalized from Set to arbitrary locally finitely presentable categories (cf. Definition 4.2.1). In this subsection we mention the most important results on iterative algebras. Most importantly, an initial iterative algebra for a Set endofunctor is, equivalently, a terminal locally finite coalgebra.

Assumption 5.2.1. Throughout the rest of this subsection we assume that $\mathcal{A}$ is a locally finitely presentable category and the $F : \mathcal{A} \to \mathcal{A}$ is a finitary endofunctor.

Definition 5.2.2. A flat equation morphism $e : X \to FX + A$ is called *finitary* if X is a finitely presentable object of $\mathcal{A}$.

An H-algebra $\alpha : FA \to A$ is called *iterative* if every finitary flat equation morphism $e : X \to FX + A$ has a unique solution in A, i. e., there exists a unique morphism $e^\dagger : X \to A$ such that the square (5.3) commutes.

Let us remark that iterative algebras constitute a full subcategory of the category of all F-algebras: every homomorphism of F-algebras preserves solutions of flat equation morphisms in the expected sense, see [AMV$_2$].

Examples 5.2.3. We list a number of examples of iterative algebras in various categories. All but the first three examples are in fact initial iterative algebras.

(i) Every completely iterative algebra for F is, of course, an iterative algebra for the same functor.

(ii) A unary algebra $\alpha : A \to A$ is iterative if and only if α has a unique fixed point and no other cycles in A; cf. Example 5.1.4(iii).

(iii) The algebra of addition on the extended real numbers $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{ \infty \}$ is an iterative algebra for the functor $FX = X \times X$, see [AMV$_2$]. Notice that this is not a completely iterative algebra: the system of equations $x_0 \approx x_1 + 1, x_1 \approx x_2 + 1, \ldots$ has more than one solution (e. g. ${x_n}^\dagger = \infty$ or ${x_n}^\dagger = -n$).

(iv) Consider $FX = X \times X + 1$ on Set. Its initial cia is the algebra of all binary trees, see Example 5.1.4(ii). The subalgebra formed by all *rational* binary trees, i. e., those trees having (up to isomorphism) only finitely many different subtrees, is an iterative algebra for F. For example, the trees in (5.2) are rational, and the tree represented by

$$(\cdots(\cdots * x_2) * x_1) * x_0)$$

is not.

(v) Similarly, all rational finitely branching strongly extensional trees form an iterative algebra for $\mathcal{P}_\mathrm{f}$ (see [AMV$_2$]), and all rational unordered finitely branching trees form an iterative algebra for the bag functor $\mathcal{B}$, cf. Theorem 2.5.6 and Example 2.3.10.

(vi) Let $FX = X^A \times 2$ be the functor whose coalgebras are deterministic automata. Recall from Example 2.3.8 that the initial cia νF consists of all formal languages. Its subalgebra formed by all regular languages is the initial iterative algebra for F, see [AMV$_2$].

(vii) Let $FX = \mathbb{R} \times X$ with $\nu F = \mathbb{R}^\omega$ the streams of reals, i. e., sequences $\sigma = (\sigma(0), \sigma(1), \sigma(2), \ldots)$ of real numbers, cf. Example 2.3.6. Streams have been studied in a coalgebraic setting by Jan Rutten [Ru$_2$, Ru$_3$]. The convolution product of two streams is given by

$$(\sigma \times \tau)(n) = \sum_{i=0}^{n} \sigma(i) \cdot \tau(n-i)$$

and for streams σ with $\sigma(0) \neq 0$ there exists an inverse σ^{-1}, i. e., $\sigma \times \sigma^{-1} = (1, 0, 0, 0, \ldots)$. A *rational* stream is a stream of the form $\sigma \times \tau^{-1}$, where σ and τ have finitely many non-zero entries and $\tau(0) \neq 0$. When F is considered as an endofunctor on the category of real vector spaces then rational streams form an initial iterative algebra for F. However, when we consider F as an endofunctor of Set the initial iterative algebra for F is given by streams σ that are eventually periodic, i. e., those streams $\sigma = u\overline{v}$, where u and v are finite words on $\mathbb{R}$. See [M$_2$].

(viii) Consider the category of “sets in context”, i. e., the presheaf category $[\mathcal{F}, \mathsf{Set}]$ where $\mathcal{F}$ is the category of finite sets and all maps between them. Objects Γ of $\mathcal{F}$ are regarded as contexts of variables and we think of a presheaf $X : \mathcal{F} \to \mathsf{Set}$ as assigning to every context Γ a set $X(\Gamma)$ of “terms in context Γ”. This setting has been considered by Fiore, Plotkin and Turi [FPT] to capture variable binding and in particular λ-terms categorically. Indeed, let $FX = V + X \times X + \delta(X)$, where $V : \mathcal{F} \to \mathsf{Set}$ is the inclusion and $\delta(X)(\Gamma) = X(\Gamma + 1)$. Then μF is the presheaf of λ-terms up to α-equivalence. The initial cia νF is formed by all (finite and infinite) λ-trees, and the initial iterative algebra by all rational ones (up to α-equivalence). See [AMV$_4$].

Constructions of iterative algebras can be performed on the level of underlying objects and this implies that free iterative algebras exist:

Proposition 5.2.4 ([AMV$_2$])**.** *The category of iterative algebras for F is closed under limits and filtered colimits in the category of F-algebras. Thus, limits and filterted colimits are constructed on the level of the base category $\mathcal{A}$.*

Corollary 5.2.5. *Every object of $\mathcal{A}$ generates a free iterative algebra.*

Indeed, full subcategories of an locally finitely presentable category closed under limits and filtered colimits are reflective, see [AR]. It follows that the reflection of μF in the category of iterative F-algebras is the initial iterative F-algebra.

In [AMV$_2$] we also gave a coalgebraic construction of free iterative algebras. We mention this here only for the special case of initial iterative algebras.

Construction 5.2.6. Let $\mathcal{D}$ be the full subcategory of $\mathsf{Coalg}\, F$ given by all coalgebras $X \to FX$ with a finitely presentable carrier X. Let $D : \mathcal{D} \to \mathcal{A}$ be the restriction of the forgetful functor $U : \mathsf{Coalg}\, F \to \mathcal{A}$. Then $\mathcal{D}$ is an essentially small, filtered diagram, and we define

$$R = \operatorname{colim} D.$$

There exists a unique coalgebra structure $\varrho : R \to FR$ such that for every $X \to FX$ in $\mathcal{D}$ the corresponding colimit injection $e^\sharp : X \to R$ is an F-coalgebra homomorphism.

Theorem 5.2.7 ([AMV$_2$])**.**

(i) R is a fixed point of F; more precisely, $\varrho : R \to FR$ is an isomorphism,

(ii) $\varrho^{-1} : FR \to R$ is an initial iterative algebra for F.

Definition 5.2.8. We call R the *rational fixed point* of F.

Example 5.2.9. As we have seen in Examples 5.2.3, for the set functor $FX = X^A \times 2$ with deterministic automata as F-coalgebras we have

$$R = \text{all regular languages over alphabet } A.$$

For $FX = \mathbb{R} \times X$ the rational fixed point in the category of vector spaces

$$R = \text{all rational streams of real numbers.}$$

Furthermore, all examples in 5.2.3(iv)–(vii) provide rational fixed points.

Remark 5.2.10. Similarly as in Construction 5.2.6 we obtain for every object Y of $\mathcal{A}$ a *free* iterative algebra RY on Y. The corresponding monad R on $\mathcal{A}$ is then characterized as the free iterative monad on the endofunctor F. This generalizes and extends classical work on iterative theories by Elgot [El] and on iterative algebras for a signature by Nelson [N] and Tiuryn [T].

For more details on the topic of iterative algebras and (free) iterative monads we refer the reader to [AMV$_2$].

We now turn to an equivalent coalgebraic characterization of the rational fixed point R of our finitary endofunctor F. This is a result from [M$_2$]. We explain this here only for the special case $\mathcal{A} = \mathsf{Set}$.

Definition 5.2.11. For $F : \mathsf{Set} \to \mathsf{Set}$ a coalgebra $X \to FX$ is called *locally finite* if every finite subset of X is contained in a finite subcoalgebra of X.

Theorem 5.2.12. *The rational fixed point $\varrho : R \to FR$ is a terminal locally finite coalgebra for F.*

Indeed, this follows from the fact established in [M$_2$] that the locally finite coalgebras are precisely the filtered colimits of diagrams of coalgebras with a finite carrier.

Remark 5.2.13. Theorem 5.2.12 generalizes to arbitrary locally finitely presentable categories $\mathcal{A}$ (for a suitably generalized notion of local finiteness). For example, the coalgebra of rational streams mentioned in Example 5.2.3(vii) is the terminal locally finite dimensional coalgebra for $FX = \mathbb{R} \times X$ on the category of real vector spaces.

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