COLIMITS OF MONADS

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Dedicated to the seventieth birthday of Manuela Sobral

ABSTRACT. The category of all monads over many-sorted sets (and over other “set-like” categories) is proved to have coequalizers. And a general diagram has a colimit whenever all the monads involved preserve monomorphisms and have arbitrarily large joint pre-fixpoints. In contrast, coequalizers fail to exist e.g. for monads over the (presheaf) category of graphs.

For more general categories we extend the results on coproducts of monads from [3]. We call a monad separated if, when restricted to monomorphisms, its unit has a complement. We prove that every collection of separated monads with arbitrarily large joint pre-fixpoints has a coproduct. And a concrete formula for these coproducts is presented.

1. Introduction

Whereas limits in the category Monad($\mathcal{A}$) of monads over a complete category $\mathcal{A}$ are easy, since the forgetful functor into the category of all endofunctors creates limits, colimits are more interesting. For example, a coproduct of two monads need not exist in Monad($\mathcal{A}$) – in fact, there are only four (trivial) types of monads over Set having a coproduct with every monad, as proved in [3], see also Theorem 4.4 below. In that paper a formula for coproducts of monads over Set was presented, and we extend it to coproducts of separated monads over general categories $\mathcal{A}$. Separatedness means that a complement of the unit of the monad exists if we restrict ourselves to the category $\mathcal{A}_m$ of objects and monomorphisms of $\mathcal{A}$. All consistent monads over Set are separated, see [3], where a monad is called consistent if its unit is monic. (The only inconsistent monads over Set are the terminal monad, constantly 1, and its submonad given by $\emptyset \mapsto \emptyset$.) For other base categories many interesting monads fail to be separated.

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Our main result is that in “set-like” categories, e.g., many-sorted sets, vector spaces, or sets and partial functions, the category Monad \((\mathcal{A})\) has (a) all coequalizers and (b) colimits of every diagram of monos-preserving monads with arbitrarily large joint pre-fixpoints. (An object \(X\) is a pre-fixpoint of a monad \(S\) if \(SX\) is a subobject of \(X\).) That last condition is proved to be weaker than assuming that the monads are accessible, i.e., preserve \(\lambda\)-filtered colimits for some infinite cardinal \(\lambda\). Moreover, arbitrarily large joint pre-fixpoints are sufficient for coproducts of

1. monos-preserving monads over set-like categories
2. separated monads over rather general categories.

And if \(\mathcal{A} = \text{Set}\), this condition is in case of coproducts of nontrivial monads also necessary, see Theorem 4.11 below. It is an open problem whether having arbitrarily large joint pre-fixpoints is sufficient for coproducts of general monads over “reasonably” general categories.

Colimits of monads were studied by Kelly [9] who proved, inter alia, that for locally presentable base categories \(\mathcal{A}\) every diagram of accessible monads has a colimit in Monad \((\mathcal{A})\). Kelly also proved a formula for the colimit. In case of coproducts of consistent monads over \(\text{Set}\) a much simpler formula was presented in [3], inspired by the work of Ghani and Ustalu [8]: let \(S\) and \(T\) be consistent \(\lambda\)-accessible monads with unit complements \(\bar{S}\) and \(\bar{T}\), respectively. Then the coproduct monad is given by

\[
A \mapsto A + \colim_{i<\lambda} X_i + \colim_{i<\lambda} Y_i
\]

Here \(X_i\) and \(Y_i\) are the \(\lambda\)-chains formed by colimits on limit ordinals, whereas the isolated steps are defined by the following mutual recursion:

\[
X_{i+1} = \bar{S}(Y_i + A) \quad \text{and} \quad Y_{i+1} = \bar{T}(X_i + A)
\]

We prove that, unsurprisingly, the same formula holds for coproducts of separated monads on general categories.

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2. The Category of Monads

In this section some basic properties of the category of monads and monad morphisms are collected.
Notation 2.1. (a) Given a category $\mathcal{A}$ we write $[\mathcal{A}, \mathcal{A}]$ for the category of endofunctors on $\mathcal{A}$ and natural transformations between them. And Monad $(\mathcal{A})$ denotes the category of monads and monad morphisms. The obvious forgetful functor is denoted by $V : \text{Monad } (\mathcal{A}) \to [\mathcal{A}, \mathcal{A}]$.

(b) We use $\cdot$ to denote the parallel (horizontal) composition of natural transformations: given $a : F \to F'$ and $b : G \to G'$, where all functors are endofunctor of $\mathcal{A}$, we have $a \cdot b : FG \to F'G'$ given by $a \cdot b = aG' \cdot Fb = F'b \cdot aG$.

Recall also the interchange law:

$$(c \cdot d) \cdot (a \cdot b) = (c \cdot a) \cdot (d \cdot b).$$

Proposition 2.2. The forgetful functor of $\text{Monad } (\mathcal{A})$ creates limits.

Remark. Recall that creation of limits means that for every diagram $D$ in $\text{Monad } (\mathcal{A})$ with a limit cone $p_d : T \to WDDd$ of the underlying diagram in $[\mathcal{A}, \mathcal{A}]$ there exists a unique structure of a monad on $T$ for which each $p_d$ is a monad morphism. Moreover, the resulting cone is a limit in $\text{Monad } (\mathcal{A})$. The following two (easy) proofs work, more generally, for the category of monads over an arbitrary monoidal category. We have not found a reference for them, we thus present those proofs.

Proof. For the given diagram

$$D : D \to \text{Monad } (\mathcal{A})$$

denote the objects by

$$Dd = (T_d, \mu_d, \eta_d) \quad (d \in \text{obj } D).$$

Given a limit cone $p_d : T \to T_d$, the unit of the monad on $T$ is, necessarily, the unique natural transformation

$$\eta^T : \text{Id} \to T \quad \text{with} \quad p_d \cdot \eta^T = \eta_d \quad (d \in \text{obj } D).$$

(Recall that $p_d$'s are required to preserve unit.) And the multiplication $\mu^T : T \cdot T \to T$ is, necessarily, the unique natural transformation for which the squares

\[
\begin{array}{ccc}
T \cdot T & \xrightarrow{\mu^T} & T \\
\downarrow_{p_d \cdot p_d} & & \downarrow_{p_d} \\
T_d \cdot T_d & \xrightarrow{\mu_d} & T_d
\end{array}
\]

commute for all $d \in \text{obj } D$. The verification of the monad axioms is easy. To verify that this is a limit cone, let $q_d : (S, \mu^S, \eta^S) \to (T_d, \mu_d, \eta_d)$ be a cone of $D$. There exists a unique natural transformation $q : S \to T$ with $q_d = p_d \cdot q(d \in \text{obj } D)$. 

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It is a monad morphism. Indeed, the axiom \( q \cdot \eta^S = q^T \) follows, since \((p_d)\) is a monocone, from

\[
p_d \cdot (q \cdot \eta^S) = q_d \cdot \eta^S = \eta_d = p_d \cdot \eta^T.
\]

Analogously, the axiom \( q \cdot \mu^S = \mu^T \cdot q \cdot q \) follows from

\[
p_d \cdot (\mu^T \cdot (q \cdot q)) = \mu_d \cdot (p_d \cdot p_d) \cdot (q \cdot q) = p_d \cdot \mu^T \cdot (q \cdot q)
\]

\[\square\]

**Corollary 2.3.** Limits of monads over a complete category \( A \) are computed object-wise (on the level of \( A \)).

**Proposition 2.4.** The forgetful functor of Monad \((A)\) creates absolute coequalizers.

**Remark.** Recall that this means that given a parallel pair of monad morphisms \( p, q : S \to T \) whose coequalizers \( c \) in \([A, A]\):

\[
\begin{array}{ccc}
S & \xrightarrow{p} & T \\
\downarrow{q} & & \downarrow{c} \\
& & C
\end{array}
\]

is absolute (that is, preserved by every functor with domain \([A, A]\)), there exists a unique monad structure on \( C \) making \( c \) a monad morphism. Moreover, \( c \) is a coequalizer of \( p \) and \( q \) in Monad \((A)\).

**Proof.** The unit of \( C \) is, necessarily,

\[\eta^C = c \cdot \eta^T.\]

To define the multiplication \( \mu^C : C \cdot C \to C \), use the endofunctor of \([A, A]\) defined by \( X \mapsto X \cdot X \) on objects and by \( f \mapsto f \cdot f \) on morphisms. Since \( c \cdot c \) is the coequalizer of \( p \cdot p \) and \( q \cdot q \), we have a unique \( \mu^C \) for which \( c \) preserves multiplication:

\[
\begin{array}{ccc}
S & \xrightarrow{p \cdot p} & T \cdot T \\
\downarrow{q \cdot q} & & \downarrow{c \cdot c} \\
& & C \cdot C
\end{array}
\]

\[
\begin{array}{ccc}
& & \mu^C \\
\downarrow{\mu^C} & & \downarrow{\mu^C} \\
S & \xrightarrow{p} & T \\
\downarrow{q} & & \downarrow{c} \\
& & C
\end{array}
\]

The verification that \((C, \eta^C, \mu^C)\) is a monad and \( c \) is a coequalizer in Monad \((A)\) is easy. \[\square\]

**Definition 2.5.** An object \( Z \) is a fixpoint of an endofunctor \( H \) if \( HZ \simeq Z \), and it is a pre-fixpoint of \( H \) if \( HZ \) is a subobject of \( Z \).

We say that \( H \) has arbitrarily large pre-fixpoints provided that for every object \( X \) there exists a pre-fixpoint \( Z \) of \( H \) with \( Z \simeq Z + X \).
Example 2.6. A monos-preserving endofunctor $H$ of the category $\text{Set}^S$ of many-sorted sets has arbitrarily large pre-fixpoints iff for every cardinal $\alpha$ there exists a pre-fixpoint of $H$ all components of which have at least $\alpha$ elements.

Lemma 2.7. Every accessible endofunctor of a cocomplete category with monic coproduct injections has arbitrarily large pre-fixpoints.

Proof. If $H$ is accessible, then every object $B$ generates a free $H$-algebra $\bar{B}$ and $\bar{B} = B + HB$, see [2]. Given an object $A$ let $B$ be an infinite copower of $A$. Then the equality $A + B \simeq B$ implies $A + B \simeq B$, and $B$ is a pre-fixpoint. □

Notation 2.8. (a) For an endofunctor $H$ of $\mathcal{A}$ an algebra is a pair $(A \hookrightarrow a)$ consisting of an object $A$ and a morphism $a : HA \to A$. Homomorphisms of algebras are defined by the usual commutative square. The resulting category is denoted by $\text{Alg}_H$.

(b) $\mu H$ denotes the initial algebra (if it exists). By Lambek's Lemma [10] its algebra structure is invertible, thus, $\mu H$ is a fixpoint of $H$.

(c) If $H$ has free algebras, i.e., the forgetful functor $\text{Alg} H \to \mathcal{A}$ has a left adjoint, then $F_H$ denotes the corresponding monad over $\mathcal{A}$. And $\hat{\eta} : \text{Id} \to F_H$ denotes its unit, whose components are the universal arrows of the free algebras.

Theorem 2.9 (Barr [6] and Kelly [9]). If an endofunctor $H$ has free algebras, then $F_H$ is a free monad on $H$. The converse holds whenever the base category is complete.

Example 2.10. The power-set functor $P$ has no fixpoint, hence, it does not generate a free monad.

Construction 2.11 (see [2]). For every object $X$ of $\mathcal{A}$ define the free-algebra chain $W : \text{Ord} \to \mathcal{A}$ (with objects $W_i$ and morphisms $w_{i,j} : W_i \to W_j$ for all ordinals $i \leq j$) uniquely up to natural isomorphism by the following transfinite induction:

The objects are given by

$W_0 = X$

$W_{i+1} = X + HW_i,$

and

$W_j = \text{colim}_{i < j} W_i$ for limit ordinals $j$.

The morphisms are as follows:

$w_{0,1} : X \to X + HX$, coproduct injection

$w_{i+1,j+1} = id_X + Hw_{i,j}$
and

\((w_{i,j})_{i < j}\) is a colimit cocone (for limit ordinals \(j\)).

Whenever this chain converges after \(i\) steps, i.e., all connecting maps \(w_{i,j}\) are isomorphisms, then as proved in [2],

\[ W_i = F_H X \]

is the free algebra on \(X\). More detailed, the two components of

\[(w_{i,i+1})^{-1} : X + HW_i \to W_i\]

are the universal arrow and the algebra structure of \(W_i\), respectively.

**Definition 2.12.** A cocomplete category is said to have *stable monomorphisms* if

(a) coproducts of parallel collections of monomorphisms are monic

and

(b) colimits of chains of monomorphisms consist of monics, and the factorizing map of every cocone of monics is monic.

**Example 2.13.** Sets, graphs, posets, many-sorted sets and almost all “usual” varieties of algebras have stable monomorphisms. All presheaf categories have stable monomorphisms.

Condition (b) implies that the unique morphism from 0 to any given object is monic (since 0 is the colimit of the empty chain). Thus rings are an example of a variety not having stable monomorphisms. Indeed, the initial ring is the ring \(\mathbb{Z}\) of integers, and not all ring homomorphisms with this domain are monic.

**Theorem 2.14** (See [13]). Let \(H\) be an endofunctor of a cocomplete category with stable monomorphisms. If \(H\) preserves monomorphisms, the following conditions are equivalent:

(1) \(H\) has free algebras

(2) for every object \(X\) the free-algebra chain converges

and

(3) for every object \(X\) there exists an object \(Z\) with

\[ HZ + X \text{ a subobject of } Z. \]

**Corollary 2.15.** Let \(\mathcal{A}\) be a cocomplete category with stable monomorphisms. Every monos-preserving endofunctor with arbitrarily large pre-fixpoints generates a free monad.

Indeed, we verify Condition (3) above: choose a pre-fixpoint \(Z\) with \(Z \simeq Z + X\) to get \(HZ + X\) as a subobject of \(Z + X \simeq Z\).
Remark 2.16. Under the assumptions of the above theorem the free monad $F_H$ preserves monomorphisms. Indeed, let $m : X \to X'$ be a monomorphism. Denote by $W'_i$ the free-algebra chain above for $X'$. It is easy to see that we get a natural transformation

$$m_i : W_i \to W'_i \quad (i \in \text{Ord})$$

by

$$m_0 = m : X \to X'$$

$$m_{i+1} = m + Hm_i : X + HW_i \to X' + HW'_i$$

and

$$m_j = \colim_{i<j} m_i$$

for limit ordinals $j$.

An easy transfinite induction shows that $m_i$ is monic for every $i$: in the isolated step use the preservation of monics by $H$.

We know from the above theorem that for some ordinal $i$ we have

$$F_H X = W_i$$

and

$$F_H X' = W'_i.$$ 

For this ordinal we then also have

$$F_H m = m_i$$

(which follows by an easy inspection of the proof of the above theorem). Thus, $F_H m$ is monic.

Remark 2.17. If free $H$-algebras exist, the free monad $F_H$ fulfils

$$F_H = H \cdot F_H + Id,$$

with $\hat{\eta}$ as the right-hand injection.

Indeed, for every object $X$ let $X$ the free algebra on $\hat{\eta}_X : X \to \bar{X}$ with the algebra structure $\varphi_X : H\bar{X} \to \bar{X}$. Then $\bar{X} = H\bar{X} + X$ since $[\varphi_X, \hat{\eta}_X] : H\bar{X} + X \to \bar{X}$ is an isomorphism. (This is Lambek’s Lemma applied to $H(\cdot) + X$.) Since $F_HX = \bar{X}$, we see that the natural transformations $\varphi : HF_H \to F_H$ and $\hat{\eta} : Id \to F_H$ form coproduct injections of $F_H = H \cdot F_H + Id$.

3. Set-Like Base Categories

For the base categories $\mathcal{A}$ such as

- $\text{Set}$ or $\text{Set}^a$ (many-sorted sets)
- K-Vec (vector spaces)
- $\text{Set}_*$ (sets and partial functions)
we prove that the category of monads has coequalizers. And it has colimits of every diagram of monos-preserving monads that possess arbitrarily large joint pre-fixpoints. In case of coproducts over \( \mathcal{A} = \text{Set} \) that last condition was proved to be “almost” necessary in [3]: a collection of nontrivial monads over \( \text{Set} \) has a coproduct if they possess arbitrarily large joint fixpoints. We explain this in more detail in the next section devoted to coproducts of separated monads.

**Assumptions 3.1.** Throughout this section \( \mathcal{A} \) denotes a category which has

(a) limits and colimits

(b) stable monomorphisms (see Definition 2.12)

and

(c) split epimorphisms.

**Remark 3.2.** (a) \( \mathcal{A} \) is cowellpowered: every object \( X \) has only a set of quotients because \( X \) has only a set of idempotent endomorphisms. Indeed, for every quotient \( e : X \to Y \) choose a splitting \( i : Y \to X \) and get an idempotent \( i.e., \) then two epimorphisms with the same idempotent yield the same quotient.

(b) \( \mathcal{A} \) has (strong epi, mono)-factorizations of morphisms since every cowellpowered, cocomplete category does, see [4], 15.17.

**Lemma 3.3.** Monad \( (\mathcal{A}) \) has (strong epi, mono)-factorization of morphisms, and every strong epimorphism has all components epic.

**Proof.** We prove that every monad morphism \( f : S \to R \) has a factorization \( f = m \cdot e \) in Monad \( (\mathcal{A}) \) where \( m \) has monic components and \( e \) has (split) epic ones. It follows easily from Proposition 2.2 that \( m \) is a monomorphism in Monad \( (\mathcal{A}) \) and \( e \) is a strong epimorphism.

Indeed, start with a factorization of every \( f_A \) in \( \mathcal{A} \) as \( SA \xrightarrow{e_A} RA \xrightarrow{m_A} TA \) with \( e_A \) split epic and \( m_A \) monic in \( \mathcal{A} \). Then the diagonal fill-in makes \( R \) an endofunctor with natural transformations \( e : S \to R \) and \( m : R \to T \). The monad unit of \( R \) is \( \mu^R = e \cdot \eta^S : Id \to R \). And the monad multiplication is given by the following diagonal fill-in:
This is well-defined because $e_A \ast e_A = e_{RA \cdot Re_A}$ is a epimorphism. To verify the unit axioms $\mu^R \cdot \eta^R = id$, consider the following diagram:

\[
\begin{array}{c}
SA \xrightarrow{e_A} RA \xrightarrow{m_A} TA \\
\eta_S \downarrow \quad \quad \eta_R \downarrow \quad \quad \eta_T \downarrow \\
SSA \xrightarrow{e_A \ast e_A} RRA \xrightarrow{m_A \ast m_A} TTA \\
\mu_S \downarrow \quad \mu_R \downarrow \quad \mu_T \downarrow \\
SA \xrightarrow{e_A} RA \xrightarrow{m_A} TA
\end{array}
\]

Its outward square commutes since $S$ and $T$ both satisfy the corresponding axiom. Naturality of $\eta^S$ implies that the upper left-hand square commutes:

\[
e_{RA} \cdot S e_A \cdot \eta^S_A = e_{RA} \cdot \eta^S_R \cdot e_A = \eta^R_A \cdot e_A.
\]

Analogously for the upper right-hand square. Consequently, the diagonal passage from $SA$ to $TA$ in the above diagram satisfies (due to $\mu^T_A \cdot \eta^T_TA = id$) the equality

\[
m_A \cdot (\mu^R_A \cdot \eta^R_RA) \cdot e_A = m_A \cdot e_A.
\]

Since $n_A$ is strongly monic and $e_A$ epic, this implies $\mu^R_A \cdot \eta^R_RA = id$.

The verification of the other unit axiom $\mu^R \cdot R\eta^R = id$ is analogous.

The proof of the associativity

\[
\mu^R \cdot R \mu^R = \mu^R \cdot R \mu^R
\]

follows from the following diagram:

\[
\begin{array}{c}
SSSA \xrightarrow{e_A \ast e_A \ast e_A} RRRRA \xrightarrow{m_A \ast m_A \ast m_A} TTTTA \\
\mu_S \downarrow \quad \mu_R \downarrow \quad \mu_T \downarrow \quad \mu_T \downarrow \\
SSA \xrightarrow{e_A \ast e_A} RRA \xrightarrow{m_A \ast m_A} TTA \\
\mu_A \downarrow \quad \mu_A \downarrow \quad \mu_A \downarrow \\
SA \xrightarrow{e_A} RA \xrightarrow{m_A} TA
\end{array}
\]
We only need to check that the epimorphism $e_A \ast e_A \ast e_A$ merges the above parallel pair. Since $m_A$ is a monomorphism and the outward square of the above diagram is the following commutative square

\[
\begin{array}{ccc}
SSSA & \xrightarrow{f_A \ast f_A \ast f_A} & TTTA \\
\mu^S_A \cdot \mu^S_A & & \mu^T_A \cdot \mu^T_A \\
SA & \xrightarrow{f_A} & TA
\end{array}
\]

the associativity of $\mu^S$ and $\mu^T$ clearly implies that of $\mu^R$. \qed

Recall from Definition 2.5 the concept of arbitrarily large pre-fixpoints of an endofunctor. Here is a “collective” version:

**Definition 3.4.** A collection $F_i$ ($i \in I$) of endofunctors is said to have **arbitrarily large joint pre-fixpoints** if for every object $A$ and every cardinal $\alpha > 0$ there exists a joint pre-fixpoint $X$ such that $X + A \simeq X \simeq \prod_\alpha X$.

**Example 3.5.** For categories $\textbf{Set}$ and $\textbf{K-Vec}$ or $\textbf{Set}^*$ this means that for every cardinal $\alpha$ there exists a joint pre-fixpoint of cardinality at least $\alpha$. (In $\textbf{K-Vec}$ use the fact that for infinite cardinals $\alpha \geq \text{card}K$ dimension $\alpha$ is equivalent to cardinality $\alpha$.)

For many-sorted sets, $\textbf{Set}^S$, this means that for every cardinal $\alpha$ there exists a joint pre-fixpoint whose components have cardinalities at least $\alpha$.

**Proposition 3.6.** Every collection of accessible endofunctors has arbitrarily large joint pre-fixpoints.

**Proof.** If $H_r$ ($r \in R$) are accessible endofunctors, then so is $H = \prod_{r \in R} H_r$. And every pre-fixpoint of $H$ is a joint pre-fixpoint of all $H_r$. Thus our task is for a given object $A$ and an infinite cardinal $\alpha$, to find a pre-fixpoint $X$ of $H$ with $X \simeq A + X \simeq \alpha \cdot X$. The copower $\alpha \cdot H$ of $\alpha$ copies of $H$ is accessible, thus, it has a free algebra on $B = \alpha \cdot A$. As in the proof of Lemma 2.7 this free algebra $\tilde{B}$ fulfils

$$\tilde{B} = \alpha \cdot A + \alpha \cdot H\tilde{B} = \alpha \cdot (A + H\tilde{B})$$

Obviously, $H\tilde{B}$ is a subobject of $\alpha \cdot H\tilde{B}$, hence, a pre-fixpoint of $H$. And $\tilde{B} \simeq A + \tilde{B} \simeq \alpha \cdot \tilde{B}$. \qed

**Theorem 3.7.** Every small collection of monos-preserving monads with arbitrarily large joint pre-fixpoints has a coproduct in Monad $(A)$.
Proof. Let \( S_i = (S_i, \mu_i, \eta_i), i \in I \) be such a collection. Then the endo-functor \( S = \coprod_{i \in I} S_i \) preserves monomorphisms, and it has arbitrarily large pre-fixpoints: given an object \( A \) find \( X \) with \( S_iX \to X \) for all \( i \in I \) and \( X \cong X + A \cong \coprod_i X \) to get

\[
SX = \coprod_{i \in I} S_iX \to \coprod_{i \in I} X \cong X.
\]

By Corollary 2.15 the functor \( S = \coprod_{i \in I} S_i \) generates a free monad \( F_S \) with the universal arrow \( \hat{\eta} : S \to FS \); the coproduct injections are denoted by

\[
v_i : S_i \to S \quad (i \in I)
\]

The forgetful functor \( \text{Monad}(\mathcal{A}) \to [\mathcal{A}, \mathcal{A}] \) creates limits, see Proposition 2.2, and we conclude that for the slice category \( F_S / \text{Monad}(\mathcal{A}) \) the the corresponding forgetful functor

\[
U : F_S / \text{Monad}(\mathcal{A}) \to F_S / [A, A]
\]

also creates limits. Now consider an arbitrarily cocone \( f = (f_i) \) consisting of monad morphisms \( f_i : S_i \to T_f(i \in I) \). The functor \( [f_i] : S \to T_f \) generates a unique monad morphism \( \bar{f} : F_S \to T_f \) with \( \bar{f} \cdot \hat{\eta} = [f_i] \) that we factorize as in Lemma 3.3.

We get a (possibly large) collection of objects \((e_f, R_f)\) of the slice category \( F_S / \text{Monad}(\mathcal{A}) \). This collection has a product in \( F_S / [A, A] \). Indeed, recall from Remark 3.2 that \( \mathcal{A} \) is cowellpowered, thus, for every object \( A \) all quotients of the object \( F_SA \) form a complete lattice. Form the meet \( e_A : F_SA \to RA \) of all quotients \((e_f)_A : F_SA \to R_fA\) ranging through all cocones \( f \). For every cocone \( f \) above we have a morphism

\[
q^A_f : RA \to R_fA \text{ with } (e_f)_A = q^A_f \cdot e_A.
\]

The resulting functor \( R \) and natural transformations \( q_f : R \to R_f \) clearly form a product of all \( e_f \) in \( F_S / [A, A] \). Consequently, there exists a product \((e, R)\) of the objects \((e_f, R_f)\) in \( F_S / \text{Monad}(\mathcal{A}) \) as \( f \) ranges through all cocones; see Proposition 2.2. For the projections \( q_f : R \to R_f \) define

\[
p_f = m_f \cdot q_f : R \to T_f.
\]
Then $\bar{f} = m_f \cdot e_f = m_f \cdot q_f \cdot e = p_f \cdot e$ implies $f = \bar{f} \cdot \eta = p_f \cdot e \cdot \eta$, therefore

$$f_i = f \cdot v_i = p_f \cdot e \cdot \eta \cdot v_i.$$ 

We claim that $R$ is the coproduct of $S_i (i \in I)$ in Monad $(\mathcal{A})$ with respect to

$$u_i = e \cdot \eta \cdot m_i : S_i \rightarrow R \quad (i \in I).$$

\begin{equation}
\begin{array}{c}
\xymatrix{
S_i 
\ar[dd]^-{v_i} 
\ar[rr]^-{u_i} 
& & S 
\ar[dd]^-{\bar{\eta}} 
\ar[rr]^-{\eta} 
& & \mathbb{R} \\
| 
\ar[rr]^-{\eta} 
& & \mathbb{R} \\
S_i 
\ar[rr]^-{f_i} 
& & \mathcal{T}_f
}
\end{array}
\end{equation}

(a) Each $u_i$ is a monad morphism. This follows from the fact that $(p_f)$ is a collectively monic cone in $[\mathcal{A}, \mathcal{A}]$ and each $f_i$ is a monad morphism. Indeed, the condition $u_i \cdot \eta_i = \eta^R$ follows from

$$p_f \cdot (u_i \cdot \eta_i) = f_i \cdot \eta_i = \eta^R$$

see (1)

$$= f_i \text{ a monad morphism}$$

$$= p_f \cdot \eta^R \quad p_f \text{ a monad morphism}.$$ 

The verification of the condition

$$\mu_i \cdot u_i = \mu^R \cdot u_i = \mu^R \cdot R u_i \cdot u_i S_i.$$
follows from the following diagram

All the inner parts but the upper one (to be proved commutative) commute: recall $f_i = p_f \cdot u_i$, use the fact that $p_f$ is a monad morphism for the lower square, and use the naturality of $p_f$ for $p_f R \cdot R u_i = T_f u_i \cdot p_f S_i$. Since $f_i$ is a monad morphism, the outward square also commutes. This, together with the collective monicity of all $p_f$’s, proves that the upper square commutes.

For every cocone $f = (f_i)_{i \in I}$ the monad morphism $p_f$ is the desired factorization: $f_i = p_f \cdot u_i$, see (1). This is unique since whenever $r : R \to T_f$ is a monad morphism with $f_i = r \cdot u_i$ for all $i$, then $f = r \cdot [u_i]$. From (1) we see that $[u_i] = e \cdot \hat{\eta}$, thus $r \cdot e \cdot \hat{\eta} = f = p_f \cdot e \cdot \hat{\eta}$ which implies $r \cdot e = p_f \cdot e$ by the universal property of $\hat{\eta}$; hence $r = p_f$ since $e$ is epic.

\[\Box\]

Remark 3.8. (a) Kelly described colimits of monads, see [9], Section 27 as follows:

Let $D$ be a diagram in Monad $(\mathcal{A})$ with objects $T_i = (T_i, \mu_i, \eta_i)$ for $i \in I$.

Form the category $\mathcal{C}_D$ of all pairs $(A, (a_i)_{i \in I})$ where $A$ is an object of $\mathcal{A}$ and $a_i : T_i A \to A$ is an Eilenberg-Moore algebra for $T_i (i \in I)$ such that for every connecting morphism $f : i \to j$ of the indexing category the
triangle

\begin{equation}
\begin{tikzcd}
T_i A \arrow{r}{a_i} \arrow{rd}{(Df)_A} & A \\
& T_j A \arrow{u}{a_j}
\end{tikzcd}
\end{equation}

commutes. The morphisms of \( \mathcal{C}_D \) are the morphisms of \( \mathcal{A} \) which are algebra homomorphisms for every \( T_i \). We have the obvious forgetful functor

\[ U_D : \mathcal{C}_D \to \mathcal{A}. \]

Kelly proved that if \( U_D \) has a left adjoint, then the corresponding monad on \( \mathcal{A} \) is a colimit of \( \mathcal{D} \) in Monad (\( \mathcal{A} \)). The converse also holds if \( \mathcal{A} \) is a complete category.

(b) Given a discrete diagram \( \mathcal{D} \) of monads \( T_i \) \((i \in I)\) the category \( \mathcal{C}_D \) has as objects multi-algebras

\[ (A, (a_i)_{i \in I}) \text{ where } a_i : T_i A \to A \text{ lies in } \mathcal{A}^{T_i}, \]

and morphisms are those maps in \( \mathcal{A} \) that are homomorphisms for each of \( T_i \) simultaneously. A coproduct of the monads \( T_i \) exists in Monad (\( \mathcal{A} \)) whenever every object of \( \mathcal{A} \) generates a free multi-algebra.

**Theorem 3.9.** Every diagram with a weakly terminal object has a colimit in Monad (\( \mathcal{A} \)). In particular, Monad (\( \mathcal{A} \)) has coequalizers.

**Proof.** Let \( D : \mathcal{D} \to \text{Monad } (\mathcal{A}) \) be a diagram with objects \( T_i = (T_i, \mu_i, \eta_i) \) for \( i \in I \), and let \( T_j \) be weakly terminal, i.e., for every \( i \in I \) there exists a connecting morphism \( f : T_i \to T_j \) in \( D \).

(a) Form the full subcategory \( \mathcal{C} \) of \( \mathcal{A}^{T_j} \) of all algebras \( a : T_j A \to A \) for \( T_j \) such that for every pair \( f, g : T_i \to T_j \) of connecting morphisms of \( D \) \((i \in I)\) we have

\[ a \cdot f_A = a \cdot g_A \]

This category is closed in \( \mathcal{A}^{T_j} \) under products, which easily follows from the forgetful functor \( U^{T_j} \) creating limits. It is also closed under subalgebras. More precisely, let \( m : (A, a) \to (B, b) \) be a homomorphism in \( \mathcal{A}^{T_j} \) with \( m \)
monic in $\mathcal{A}$. If $(B, b)$ lies in $C$, then so does $(A, a)$:

Since the forgetful functor $U^{T_j}$ creates limits, the category $\mathcal{A}^{T_j}$ is complete and wellpowered. Let us prove that it is also cowellpowered. Given a factorization of a homomorphism $h : (A, a) \rightarrow (B, b)$ in $\mathcal{A}^{T_j}$ as a strong epimorphism $e : C \rightarrow B$ followed by a monomorphism $m : C \rightarrow B$ in $\mathcal{A}$, the diagonal fill-in makes $e$ and $m$ homomorphisms:

Thus, if $h$ is a strong epimorphism in $\mathcal{A}^{T_j}$ then $m$ is an isomorphism (recall that $U^{T_j}$ creates limits, thus, reflects isomorphisms), consequently, $h$ is an epimorphism in $\mathcal{A}$. Since $\mathcal{A}$ is cowellpowered (see Remark 3.2) we conclude that $\mathcal{A}^{T_j}$ is cowellpowered.

(b) Every full subcategory of $\mathcal{A}^{T_j}$ closed under products and subobjects is reflective, see [4], 16.9. Thus, the obvious forgetful functor $U : C \rightarrow \mathcal{A}$ has a left adjoint.

The theorem now follows from Remark 3.8 and the fact that there exists an isomorphism $E$ of categories such that the triangle

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{U} & C \\
\downarrow{U_D} & & \downarrow{E} \\
\mathcal{C} & \xrightarrow{U} & C
\end{array}
$$
commutes. Indeed, $E$ is the “projection to $j$”

$$E(A, (d_i)_{i \in I}) = (A, d_j).$$

From the triangles (2) we deduce that $(A, d_j)$ satisfies (3). Thus, $E$ is a well-defined, faithful functor. It is surjective on objects: for every algebra $(A, a)$ in $C$ define, given $i \in I$,

$$a_i = a \cdot f_A : T_i A \to A$$

for any connecting morphism $f : T_i \to T_j$.

Then $a_i$ is well-defined due to (3) and, since $f$ is a monad morphism, $(A, a_i)$ is an Eilenberg-Moore algebra for $T_i$. Finally, to prove that $E$ is an isomorphism, we verify that it is full. Let

$$k : (A, a) \to (B, b)$$

be a homomorphism in $C$. Then we need to prove that for every $i \in I$ this is a homomorphism from $(A, a_i)$ to $(B, b_i)$, where again $b_i = b \cdot f_B$. Use the following diagram

$$
\begin{array}{c}
\begin{array}{c}
T_i A \\ \downarrow T_i k
\end{array}
\xrightarrow{f_A}
\begin{array}{c}
T_j A \\ \downarrow T_j k
\end{array}
\xrightarrow{a}
A \\
\begin{array}{c}
T_i B \\ \downarrow T_i k
\end{array}
\xrightarrow{f_B}
\begin{array}{c}
T_j B \\ \downarrow T_j k
\end{array}
\xrightarrow{b}
B
\end{array}
$$

\[a_i\]

**Corollary 3.10.** Every diagram of monos-preserving monads with arbitrarily large joint pre-fixpoints has a colimit in Monad $(A)$.

Indeed, apply the usual construction of colimits as a coequalizer of a parallel pair between coproducts; see [11]. Given a diagram $\mathcal{D}$ in Monad $(A)$ with monos-preserving objects $S_i = (S, \mu_i, \eta_i)$ for $i \in I$ having arbitrarily large joint pre-fixpoints, then also every collection of monads indexed by $I \times J$, where $J$ is an arbitrary set and $S_i = S_{(i,j)}$ for all $(i, j) \in I \times J$, has arbitrarily large joint pre-fixpoint. (Indeed, for every object $A$ and every cardinal $\alpha$ put $\alpha' = \alpha + \text{card} J$. By applying Definition 3.4 to $A$ and $\alpha'$ for the former collection indexed by $I$, we get the required condition for the new collection.) Thus, the two coproducts needed to construct $\text{colim} \mathcal{D}$ as a coequalizer in Monad $(A)$ exist.
Remark 3.11. Monad \((A)\) also has strong cointersections. That is, wide push-outs of strong epimorphisms \(e_i : T \to S_i (i \in I)\). The proof is analogous to that of Theorem 3.9. Let \(C\) be the full subcategory of \(A\) on all algebras \(a : TA \to A\) for which \(a\) factorizes though each \((e_i)_A : TA \to S_i A\) factorizes though \(a\). This subcategory is easily seen to be closed under products and subalgebras. And it is isomorphic to the category \(C_D\) of Remark 3.8. (Here we use the fact established in Lemma 3.3 that strong epimorphisms in Monad \((A)\) have epic components.) Thus, the cointersection of \(c_i\) exists in Monad \((A)\).

Example 3.12. For the base category of graphs \(Gra = \mathsf{Set}^{\Delta}\) we present a parallel pair of monad morphisms having no coequalizer in Monad \((Gra)\).

For every graph \(X = (V, E, s, t)\) with source and target maps \(s, t : E \to V\) we denote by \(X_e\) the set of all loops, i.e., the equalizer of \(s\) and \(t\). We construct two endofunctors \(H, K : Gra \to Gra\) and two natural transformations \(\sigma, \tau : H \to K\) such that for the coequalizer

\[ H \xrightarrow{\sigma} K \xrightarrow{\rho} L \quad \text{in } [Gra, Gra] \]

\(L\) does not generate a free monad, but \(H\) and \(K\) do. It follows immediately that the monad morphisms

\[ \bar{\sigma}, \bar{\tau} : F_H \to F_K \]

corresponding to \(\sigma\) and \(\tau\) (see Notation 2.8) do not have a coequalizer in Monad \((Gra)\): if \(S\) were the codomain of such a coequalizer, then since \(F(-)\) is a left adjoint, \(S\) would clearly be a free monad on \(L\).

Let \(\mathcal{P}\) denote the power-set functor. The endofunctor \(H\) is defined on objects \(X\) as follows:

\[ H(X) \text{ has vertices } \mathcal{P}(X_e) \text{ and no edges.} \]

The definition of \(H\) on morphisms \(g : X \to X'\) is as expected: \(H(g)\) is the domain-codomain restriction of the edge function of \(g\) to all loops. Analogously define \(K\):

\[ K(X) \text{ has vertices } \mathcal{P}(X_e) + \mathcal{P}(X_e) \text{ and edges } \mathcal{P}(X_e) \]

\[ s, t : \mathcal{P}(X_e) \to \mathcal{P}(X_e) + \mathcal{P}(X_e) \text{ are the coproduct injections} \]

That is, \(K(X)\) is the disjoint union of arrows indexed by \(\mathcal{P}(X_e)\). The definition on morphisms is again as expected. Let

\[ \sigma, \tau : H \to K \]
be the natural transformations corresponding to $s$ and $t$: for every $M \subseteq X_e$, $\sigma_X(M)$ is the source of the arrow labelled by $M$ and $\tau_X(M)$ is its target. The coequalizer $L$ of $\sigma$ and $\tau$ in $[\text{Gra}, \text{Gra}]$ is obvious: it assigns to every graph $X$ the graph on $\mathcal{P}(X_e)$ consisting of loops:

$$L(X) \text{ has vertices } = \text{ edges } = \mathcal{P}(X_e) \text{ and } s = t$$

The functor $H$ generates a free monad, since in Construction 2.11 we have

$$W_2 = X + H(X + HX) = H + HX = W_1.$$

Thus the construction converges in one step. The same is true about $K$.

It remains to prove that $L$ does not generate a free monad. By Theorem 2.9 it is sufficient to prove that $L$ does not have an initial algebra. Indeed, we prove that if

$$a : LA \to A$$

is an initial algebra, then $\mathcal{P}$ has an initial algebra (compare Example 2.10). Let $m : A_0 \to A$ be the subgraph of $A$ whose vertices are precisely the loops of $A$ and whose edges are just all the loops. Then $LA = LA_0$, and we obviously have a codomain restriction $a_0 : LA_0 \to A_0$ of $a$. And $m : (A_0, a_0) \to (A, a)$ is a homomorphism of algebras for $L$. The unique homomorphism $h : (A, a) \to (A_0, a_0)$ thus yields an endomorphism $m \cdot h$ of the initial algebra; hence $m \cdot h = \text{id}$. This proves $A = A_0$. That is, $A$ is the set $A_v$ of vertices endowed with all loops. But then $a : \mathcal{P}A_v \to A_v$ as an algebra for $\mathcal{P}$ is initial: given any algebra $b : \mathcal{P}B \to B$, form the graph $\tilde{B}$ of all loops in $B$ and obtain an obvious structure $\tilde{b} : \mathcal{L}\tilde{B} \to \tilde{B}$ of an $L$-algebra. Then $\mathcal{P}$-algebra homomorphisms from $(A, a)$ to $(\tilde{B}, b)$ are precisely the $L$-algebra homomorphisms from $(A_v, a)$ to $\tilde{B}$. This is the desired contradiction.

**Example 3.13.** The category Monad $(\text{Gra})$ also fails to have cointersections of split epimorphisms. The argument is completely analogous: the following split epimorphisms

$$\sigma_0 = [\sigma, \sigma, \tau, \text{id}] : H + H + H + K \to K$$

and

$$\tau_0 = [\tau, \sigma, \tau, \text{id}] : H + H + H + K \to K$$

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of \([\text{Gra}, \text{Gra}]\) have the cointersection as follows:

\[
\begin{array}{c}
\hat{H} = H + H + H + K \\
\sigma_0 \downarrow \quad \tau_0 \downarrow \\
K \quad \quad \quad \quad K \\
\rho \downarrow \quad \rho \downarrow \\
L
\end{array}
\]

Since \(L\) does not generate a free monad, the split epimorphisms \(\overline{\sigma}_0, \overline{\tau}_0 : F_H \to F_K\) do not have a cointersection in \(\text{Monad} (\text{Gra})\).

4. COPRODUCTS OF SEPARATED MONADS

Ghani and Ustalu presented in [8] an interesting formula for coproducts of ideal monads, see Example 4.6(4), which was, in case of monads over \(\text{Set}\), generalized in [3]. The present section is based on the ideas of the latter paper, extending the formula to separated monads over abstract categories. Separation means that the monad unit has a complement – not over the given category \(\mathcal{A}\) but over the category \(\mathcal{A}_m\) of all objects and all monomorphisms.

**Assumption 4.1.** Throughout this section \(\mathcal{A}\) denotes a cocomplete category in which a coproduct of parallel monomorphisms is always monic.

We denote by

\[
\mathcal{A}_m
\]

the category of all objects and all monomorphisms of \(\mathcal{A}\).

Every monos-preserving endofunctor \(F\) of \(\mathcal{A}\) defines an endofunctor of \(\mathcal{A}_m\) by restriction, we denote it by \(F\) again.

The coproduct + of \(\mathcal{A}\) is a monoid structure on \(\mathcal{A}_m\) (not having the universal property of coproducts, of course).

**Example 4.2.** (1) The exception monad \(M_E\) defined by \(X \mapsto X + E\) has coproduct with all monads \(S\): the coproduct is given by \(X \mapsto S(X + E)\).

(2) The terminal monad \(1\) given by \(X \mapsto 1\), also has all coproducts, the result is always \(1\).

(3) For monads over \(\text{Set}\), let \(M^0_E\) be the modification of \(M_E\) with \(\emptyset \mapsto \emptyset\) and \(X \mapsto X + E\) for all \(X \neq \emptyset\). Analogously, let \(1^0\) be given by \(\emptyset \mapsto \emptyset\) and \(X \mapsto 1\) for all \(X \neq \emptyset\). It is easy to see that Monad (\(\text{Set}\)) has all coproducts with \(M^0_E\) or with \(1^0\).

**Definition 4.3.** We call a monad over \(\text{Set}\) trivial if it is isomorphic to \(M_E\), \(M^0_E\), \(1\), or \(1^0\). These are precisely the monads corresponding to varieties of algebras with no operation of arity at least 1.
Theorem 4.4 (See [3]). A monad over Set has coproducts with all monads iff it is trivial.

Moreover, all monads over Set except 1 and 1° are consistent, i.e., the components of the monad unit are monic.

Definition 4.5. A monad \((S, \mu, \eta)\) is called separated if its unit has a complement in the following sense:

(i) \(S\) preserves monomorphisms

and

(ii) there exists an endofunctor \(\bar{S}\) of \(\mathcal{A}_m\) such that the restriction of \(S\) to \(\mathcal{A}_m\) fulfills

\[
S = \text{Id} + \bar{S}
\]

with the unit \(\eta\) as the left-hand injection.

Examples 4.6. (1) The exception monad \(M_E\) is separated; here \(\bar{M}_E\) is the constant functor of value \(E\).

(2) Every free monad \(F_H\) which preserves monomorphisms is separated. (In particular, if \(\mathcal{A}\) has stable monomorphisms, all free monads on monomorphisms-preserving functor are separated.) Here \(\bar{F}_H = H \cdot F_H\); use Remarks 2.16 and 2.17.

(3) All consistent monads on Set (i.e., all except 1 and 1°) are separated. See [3], Proposition IV.5.

(4) Ideal monads of Elgot [7] are separated if they preserve monomorphisms. Recall that an ideal monad \(S = (S, \mu, \eta)\) is one for which an endofunctor \(\bar{S}\) of \(\mathcal{A}\) exists such that (i) \(S = \text{Id} + \bar{S}\) in \([\mathcal{A}, \mathcal{A}]\) with the left-hand injection \(\eta\) and (ii) \(\mu\) restricts to a natural transformation \(\bar{\mu} : \bar{S}S \to \bar{S}\).

(5) In particular, the free completely iterative monad \(S\) on an endofunctor \(H\) given by the greatest fixpoint

\[
S\mathcal{A} = \nu X \cdot (A + HX)
\]

is separated, with \(\bar{S} = H \cdot S\), whenever it preserves monomorphisms, see [1].

Notation 4.7. Let \(S_i (i \in I)\) be separated monads. For every object \(A\) of \(\mathcal{A}\) define an endofunctor \(H_A\) of \(\mathcal{A}_m^I\) as follows:

\[
H_A(X_i)_{i \in I} = (\bar{S}_i Y_i)_{i \in I} \text{ where } Y_i = A + \coprod_{j \in I, j \neq i} X_j
\]

If \(H_A\) has an initial algebra, we denote its components by \(S^*_i A\):

\[
\mu H_A = (S^*_i A)_{i \in I}
\]
**Remark 4.8.** Let \((X_i)\) be a fixed point of \(H_A:\)

\[ X_i \simeq \bar{S}_i Y_i {\text{ for all } i \in I} \]

Then the coproduct \(A + \coprod_{i \in I} X_i\) carries a canonical structure of a multi-algebra: the algebra structure for \(\bar{S}_i\) is the free algebra on \(Y_i\). Indeed, the usual free algebra is \((S_i Y_i, \mu_{Y_i}^i)\). And the above coproduct is isomorphic to \(S_i Y_i : \)

\[
A + \coprod_{i \in I} X_i \cong Y_i + X_i \\
\simeq Y_i + \bar{S}_i Y_i \\
= S_i Y_i
\]

In particular: if the initial algebra \(\mu H_A = (S_i^*)_{i \in I}\) exists, then the coproduct \(A + \coprod_{i \in I} S_i^* A\) is a multi-algebra. We prove that it is free on \(A\) w.r.t. the right-hand coproduct injection \(\text{inl} : A \to A + \coprod_{i \in I} S_i^* A:\)

**Theorem 4.9.** *A coproduct of separated monads \(S_i (i \in I)\) exists whenever the initial algebra \(\mu H_A = (S_i^*)\) exists for every object \(A\). It is defined by*

\[
A \mapsto A + \coprod_{i \in I} S_i^* A
\]

**Remark 4.10.** The monad unit \(\eta_A\) is the right-hand coproduct injection. The multiplication follows from \(A + \coprod_{i \in I} S_i^* A\) being the free multi-algebra on \(A\).

**Proof.** Let \(S_i = (S_i, \mu^i, \eta^i)\) be the given monads. Following Remark 3.8(b) all we need proving is that the multi-algebra \(\bar{A} = A + \coprod_{i \in I} S_i^* A\) is free.

(1) Let us describe its algebra structure explicitly for every \(S_i\). The initial-algebra structure of \(\mu H_A\) is by Remark 4.8 given by isomorphisms

\[
\varphi_i : \bar{S}_i Y_i \to X_i
\]

where \(X_i = S_i^* A\) and \(Y_i = A + \coprod_{j \neq i} X_j\). This defines isomorphisms

\[
\varphi_i \equiv \bar{A} = Y_i + X_i \xrightarrow{Y_i + \varphi_i^{-1}} Y_i + \bar{S}_i Y_i = S_i Y_i
\]
And the algebra structure \( \sigma_i \) of \( \bar{A} \) for \( S_i \) is transported by this isomorphism from the free-algebra structure \( \mu^i_{Y_i} \):

\[
\begin{array}{ccc}
S_iA & \xrightarrow{\sigma_i} & \bar{A} \\
\downarrow \phi_i & & \downarrow \bar{\phi}_i^{-1} \\
S_iS_iY_i & \xrightarrow{\mu^i_{Y_i}} & S_iY_i \\
\end{array}
\]

(2) For every multi-algebra

\[
\beta_i : S_iB \to B \quad (i \in I)
\]

and every morphism \( f : A \to B \) we prove that a unique multi-algebra homomorphism

\[
\tilde{f} : \bar{A} \to B \text{ with } f = \tilde{f} \cdot \text{inl}
\]

exists. The object \( \nabla B = (B, B, B \ldots) \) is an algebra for \( H_A \) w.r.t. \( (b_i)_{i \in I} : H_A(\nabla B) \to \nabla B \) given as follows:

\[
b_i = S_i(A + \coprod_{j \neq i} B) \xrightarrow{S_i[f, \psi]} \bar{S}_iB \xrightarrow{S_iB} S_iB \xrightarrow{\beta_i} B
\]

The middle subobject is the right-hand coproduct injection of \( S_iB = B + \bar{S}_iB \). We have a unique homomorphism from the initial algebra \( \mu H_A \):

\[
(h_i)_{i \in I} : (X_i)_{i \in I} \to \nabla B
\]

which means that the square

\[
\begin{array}{ccc}
\bar{S}_iY_i = \bar{S}_i(A + \coprod_{j \neq i} X_j) & \xrightarrow{\varphi_i} & X_i \\
\downarrow \bar{S}_i(A + \coprod_{j \neq i} B) & & \downarrow h_i \\
\bar{S}_i(A + \coprod_{j \neq i} B) & \xrightarrow{b_i} & B
\end{array}
\]

(5)

commutes for every \( i \). Put

\[
\bar{h}_i = [h_j]_{j \neq i} : \coprod_{j \in I, j \neq i} X_j \to B
\]

Then (5) is equivalent to the commutativity of the following square:

\[
\begin{array}{ccc}
\bar{S}_iY_i & \xrightarrow{\varphi_i} & X_i \\
\downarrow \bar{S}_i[f, \bar{h}_i] & & \downarrow h_i \\
\bar{S}_iB & \xrightarrow{\beta_i} & B
\end{array}
\]

(6)
We are going to prove that the desired extension of $f$ is

$$\tilde{f} = [f, \bar{h}] : A + \coprod_{j \in I} X_j \to B$$

where $\bar{h} = [h_j]_{j \in I}$.

That is, we first need to prove that $\tilde{f}$ is a homomorphism for $H_A$. Thus for every $i \in I$ we must prove that the following diagram commutes:

(7)

(The upper line is the algebra structure $\sigma_i$ of $\tilde{A}$.) The middle square is the naturality of $\mu^i$, the lower-one is a monad-algebra axiom for $(B, \beta_i)$. We only need to prove that the right-hand square commutes: the left-hand one is its image under $S_i$. Using $S_iY_i = Y_i + S_iX_i$ we get the following presentation of the right-hand square, recalling (4):

$$S_iY_i = Y_i + S_iX_i \xrightarrow{Y_i + \phi_i} A + \coprod_{j \neq i} X_j$$

The left-hand component with domain $Y_i$ clearly commutes: recall that $\eta^i_B = \text{inl} : B \to S_iB$, thus $\beta_i \cdot \text{inl} = \text{id}$ due to the monad axioms for $(B, \beta_i)$. The right-hand component forms the square (6).

(3) To prove uniqueness, let $f : \tilde{A} \to B$ be a multi-algebra homomorphism with $f \cdot \text{inl} = \tilde{f}$. Define $h_i : X_i \to B$ to be the $i$-th component of $f$, thus, $f = [f, [h_i]]$. It is only needed to prove that the squares (5) commute: then $h_i$'s are determined uniquely, since $(X_i)$ is the initial algebra of $H_A$. Since $\tilde{f}$ is a multi-algebra homomorphism, (6) commutes. This clearly implies that (5) does.
Theorem 4.11 (See [3]). For nontrivial monads over Set the above sufficient condition for coproducts is also necessary: a coproduct exists iff for every set $A$ the initial algebra of $H_A$ exists.

Corollary 4.12. Let $A$ have stable monomorphisms. Every collection of separated monads with arbitrarily large joint pre-fixpoints has a coproduct in Monad $(A)$.

Indeed, assuming $S_i, i \in I$, have arbitrarily large joint pre-fixpoints, we prove that the endofunctor $H_A$ has an initial algebra. By Corollary 2.15 we only need to find, for every object $X = (X_i)$ of $A^I$, a prefixed point $Z$ of $H_A$ with $Z \simeq Z + X$.

The functor $S = \coprod_i \coprod_i S_i$ has arbitrarily large pre-fixpoints: given an object $Y$ of $A$, let $V$ be a joint pre-fixpoint of all $S_i$ with $Y + V \simeq V \simeq \coprod_{i+1} V$, then $V$ is a pre-fixpoint of $S$ due to

$$SV = \coprod_i \coprod_i S_i V \twoheadrightarrow \coprod_i \coprod_i V \simeq V.$$ 

By Corollary 2.15, $S$ has a free algebra on

$$Y = A + \coprod_{i \in I} X_i$$

(for the above object $X$ of $A^I$). Put $Y^* = F_S Y$. Remark 2.17 yields

$$Y^* = SY^* + Y = SY^* + A + \coprod_{i \in I} X_i.$$

The desired object $Z$ of $A^I$ is $Z = (Y, Y, Y, \ldots)$. Obviously $Y_i \cong Y_i + X_i$, thus, $Z \simeq Z + X$. And $H_A Z = (A + \coprod_{j \neq i} S_j Y^*)_{i \in I}$ is a subobject of $Z$ due to the following monomorphism:

$$A + \coprod_{j \neq i} S_j Y^* \hookrightarrow Y + \coprod_i SY^* \simeq Y + SY^* \simeq Y^*.$$

Corollary 4.13. Let $A$ have stable monomorphisms. A coproduct of accessible separated monads $S$ and $T$ is given by

$$A \mapsto A + \operatorname{colim} X_k + \operatorname{colim} Y_k$$

for the transfinite chains

$$X_k : 0 \rightarrow \tilde{S}A \rightarrow \tilde{S}(A + TA) \rightarrow \ldots$$

and

$$Y_k : 0 \rightarrow \tilde{T}A \rightarrow \tilde{T}(A + \tilde{S}A) \rightarrow \ldots$$
More precisely, there chains are defined by the mutual recursion
\[ X_{k+1} = \bar{S}(A + Y)_k \text{ and } Y_{k+1} = \bar{T}(A + X_k) \] (8)
on isolated steps, and by colimits on limit steps.

To see this, let \( \lambda \) be an infinite cardinal such that \( S \) and \( T \) preserve \( \lambda \)-filtered colimits. Then \( \bar{S} \) and \( \bar{T} \) also preserve \( \lambda \)-filtered colimits. (Indeed, given a \( \lambda \)-filtered colimit \( b_j : B_j \to B, j \in J \), we know that \( Sb_j = b_j + \bar{S}b_j \) is also a colimit cocone. For every cocone \( c_j : SB_j \to C \) consider the cocone \( b_j + c_j : SB_j \to B + C \). Since this factorizes uniquely through \( Sb_j \), it follows that \( c_j \) factorizes uniquely though \( \bar{S}b_j \). Thus \( \bar{S} \) preserves \( \lambda \)-filtered colimits, analogously \( \bar{T} \). Consequently, in (2) we see that the functor \( H_A(V,W) = (\bar{S}(W + A),\bar{T}(V + A)) \) preserves \( \lambda \)-filtered colimits. This implies, as proved in [A], that \( \mu H_A \) is the colimit of the \( \lambda \)-chain \( (X_i, Y_i) \) which is the free-algebra chain \( H_A \) and the initial object \( X \) of \( A^I_m \), see Construction 2.11. The recursion \( W_{k+1} = H_A W_k \) is precisely (8) above.

**Remark 4.14.** More generally, a coproduct of accessible separated monads \( S_i (i \in I) \) is given by
\[ A \mapsto A + \coprod_{i \in I} X^i_k \]
for the transfinite chains \( (X^i_k)_{k \in \text{Ord}} \) given on isolated steps by
\[ X^i_{k+1} = \bar{S}_i(A + \coprod_{j \neq i} X^j_k) \]
and on limit steps by colimits.

**Notation 4.15.** For a separated monad \( S \) define endofunctors \( \bar{S}_A \) of \( A_m \) by
\[ \bar{S}_A X = \bar{S}(A + X). \]
Thus, the above formula simplifies to \( X^i_{k+1} = (\bar{S}_i)_A \coprod_{j \neq i} X^j_k \). For two monads we also have a more compact formula:

**Corollary 4.16.** Let \( A \) have stable monomorphisms. The coproduct of a pair \( S, T \) of separated monads with arbitrarily large joint pre-fixpoints is given by
\[ A \mapsto A + \mu \bar{S}_A \bar{T}_A + \mu \bar{T}_A \bar{S}_A. \]

Indeed, the coproduct is given by \( A + S^*A + T^*A \), so all we need proving is that the endofunctor \( H_A \) has the initial algebra carried by \( (\mu \bar{S}_A \bar{T}_A, \mu \bar{T}_A \bar{S}_A) \).

We prove a more general statement:

**Lemma 4.17.** Given endofunctors \( F \) and \( G \) of \( A \) define an endofunctor \( H \) of \( A^2 \) by \( H(V,W) = (FW, GV) \). If \( (X,Y) \) is an initial algebra of \( H \), then \( X = \mu FG \) and \( Y = \mu GF \).
Proof. Let the algebra structure of $\mu H = (X,Y)$ be given by

$$ x : FY \rightarrow X \text{ and } y : GY \rightarrow X. $$

Then we prove that $GF$ has the initial algebra

$$ GF \xrightarrow{Gx} GX \xrightarrow{y} X, $$

by symmetry $\mu FG = X$.

For every algebra $\beta : GFB \rightarrow B$ of $GF$ form the algebra for $H$ on $(FB,B)$ with the following structure

$$ \text{id} : FB \rightarrow FB \text{ and } \beta : GFB \rightarrow B. $$

Given the unique homomorphism of $H$-algebras

$$ (a,b) : (X,Y) \rightarrow (FB,B) $$

it is easy to verify that $b : (X,y,Gx) \rightarrow (B,\beta)$ is a homomorphism for $GF$. Conversely, if $b : (X,y,Gx) \rightarrow (B,\beta)$ is a homomorphism for $GF$, then put $a = Fb.x^{-1} : X \rightarrow FB$. Then $(a,b) : (X,Y) \rightarrow (FB,B)$ is a homomorphism for $H$. Thus, $b$ is the unique homomorphism for $GF$, proving $\mu GH = X$. □

References


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